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SOME REMARKS ON THURSTON METRIC AND HYPERBOLIC METRIC

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Abstract. In this paper, we study the relations between the Thurston metric and the hyperbolic metric on a closed surface of genus $g \geq 2$. We show a rigidity result which says if there is an inequality between the marked length spectra of these two metrics, then they are isotopic. We obtain some inequalities on length comparisons between these metrics. Besides, we show certain distance distortions under conformal graftings, with respect to the Teichmüller metric, the length spectrum metric and Thurston's asymmetric metrics.

1. Introduction

Throughout the paper, S is a closed surface of genus $g \geq 2$. A complex projective structure on S is [6, 15] given by a coordinate system whose transition maps are restrictions of projective maps. Each complex projective structure induces a underlying complex structure, and correspondingly (by the uniformization theorem) a hyperbolic metric. Thurston (cf. [6]) discovered a connection between the complex projective structures and hyperbolic geometry. He showed that the space of complex projective structures on S, $P(S)$, is parametrized via the grafting homeomorphism $Gr : T(S) \times MF(S) \to P(S)$, where $T(S)$ is the Teichmüller space and $MF(S)$ is the space of measured foliations. In accordance with this parametrization, Thurston defined a Finsler metric, the Thurston metric, which characterizes the corresponding complex projective structure.

In this paper, we are mainly concerned with the relations between the Thurston metric and the hyperbolic metric, which also reflect the relations between the complex projective structure and complex structure. For results in this direction, see [4, 9, 15, 21], etc.

In Theorem 3.1 in Section 3, we show a rigidity result for the Thurston and hyperbolic metric. Generally, let ρ_1 and ρ_2 be two (isotopy classes of) metrics on S. Denote by $\mathrm{MLS}(\rho_i)$ the marked length spectrum of ρ_i, $i = 1, 2$. Write

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MLS(ρ_1) $\leq$ MLS(ρ_2) (resp. MLS(ρ_1) = MLS(ρ_2)) if $l_{\rho_1}(\alpha) \leq l_{\rho_2}(\alpha)$ (resp. $l_{\rho_1}(\alpha) = l_{\rho_2}(\alpha)$) for every homotopy class of essential closed curve $\alpha \subset S$, where $l_{\rho_i}(\alpha)$ is the ρ_i-length of α. The classical spectral rigidity question asks whether MLS(ρ_1) = MLS(ρ_2) implies $\rho_1 = \rho_2$. Here we consider a natural refinement: whether MLS(ρ_1) $\leq$ MLS(ρ_2) implies $\rho_1 = \rho_2$. We show:

Theorem 3.1. *Let t and λ be a Thurston metric and a hyperbolic metric on S, respectively. If* MLS(t) $\leq$ MLS(λ), *then $t = \lambda$.*

Theorem 3.1 implies that the underlying complex structure (hyperbolic metric) of a complex projective structure is strongly determined, via a uniform inequality, by the marked length spectrum of the Thurston metric corresponding to the complex projective structure.

Remark 1.1. As a consequence of Theorem 3.1, we have the following spectral rigidity result proved by Kim ([9], Theorem 2): If MLS(t) = MLS(λ), then $t = \lambda$.

Let $gr = \pi \circ Gr : T(S) \times MF(S) \to T(S)$ be the conformal grafting mapping, where $\pi : P(S) \to T(S)$ is the forgetful mapping which only remembers the complex structures. Before stating the next theorem, we recall the following two inequalities concerning length comparisons.

First, let Z be a complex projective structure and t_Z the corresponding Thurston metric. By the uniformization theorem, the underlying complex structure $\pi(Z)$ corresponds to a hyperbolic metric λ_Z. Then it is easy to know (see Lemma 2.1) that $\lambda_Z \leq t_Z$ and hence $l_{\lambda_Z}(\alpha) \leq l_{t_Z}(\alpha)$ for every $\alpha \in MF(S)$.

Second, let $\beta \in MF(S)$ and $X \in T(S)$. By abuse of the notations, let $Gr_\beta(X)$ be the Thurston metric corresponding to the complex projective structure obtained by grafting X along β, and let $gr_\beta(X) = \pi(Gr_\beta(X))$ be the underlying hyperbolic metric. Then McMullen [15] showed:

Theorem A ([15], Theorem 3.1)**.** *For any $\alpha, \beta \in MF(S)$ and $X \in T(S)$, we have*

$$l_{gr_\beta(X)}(\alpha) \leq l_X(\alpha) + i(\alpha, \beta),$$

where $i(\alpha, \beta)$ is the intersection number. The inequality is strict if both α and β are non-zero.

For $\epsilon > 0$, let $T_\epsilon(S)$ be the ϵ-thick part of $T(S)$ consisting of those points whose injectivity radius is at least $\epsilon/2$. In Theorem 4.1 in Section 4, we show an inequality concerning points in $T_\epsilon(S)$ which gives length comparisons between the Thurston metric and its underlying hyperbolic metric. This is in some sense converse to Lemma 2.1.

Theorem 4.1. *For any $\epsilon > 0$, there exists a uniform constant C_ϵ depending only on ϵ and S such that for any $X \in T(S)$ and $\beta \in MF(S)$ with $gr_\beta(X) \in T_\epsilon(S)$, we have*

$$l^2_{gr_\beta(X)}(\alpha) \geq C_\epsilon \frac{l^2_{Gr_\beta(X)}(\alpha)}{4\pi(g-1) + l_X(\beta)}$$

for any $\alpha \in MF(S)$.

With exactly the same hypothesis as in Theorem 4.1, we have some corollaries which are in some sense converse to Theorem A and Lemma 2.2.

Corollary 4.1.

$$l^2_{gr_\beta(X)}(\alpha) \geq C_\epsilon \frac{l^2_X(\alpha)}{4\pi(g-1)+l_X(\beta)}$$

for any $\alpha \in MF(S)$.

Corollary 4.2.

$$(l_X(\alpha)+i(\alpha,\beta))^2 \geq C_\epsilon \frac{l^2_{Gr_\beta(X)}(\alpha)}{4\pi(g-1)+l_X(\beta)}$$

for any $\alpha \in MF(S)$.

Corollary 4.3.

$$l^2_X(\alpha) \geq C_\epsilon \frac{l^2_{Gr_\beta(X)}(\alpha)}{4\pi(g-1)+l_X(\beta)}$$

for any $\alpha \in MF(S)$ *with* $i(\alpha,\beta)=0$.

Remark 1.2. Given $\epsilon > 0$, for $\beta \in MF(S)$ and $c > 0$ with $c \geq \epsilon$, let $B(\beta, c) = \{X \in T(S) : l_X(\beta) \leq c\}$. For those X in $B(\beta,c) \cap gr_\beta^{-1}(T_\epsilon(S))$, the denominator in the right-hand side of each of the inequalities in Theorem 4.1 and Corollaries 4.1, 4.2 and 4.3 can be replaced by a uniform constant depending only on β, c and g. Note that (cf. [5, 8]) $B(\beta, c)$ is compact for any filling measured foliation β and any $c > 0$.

In Section 5 we describe certain distance distortions under conformal graftings, with respect to the Teichmüller metric d_T, the length spectrum metric d_L and Thurston's asymmetric metrics d_{P_i}, $i = 1, 2$. Let $\mathcal{S}(S)$ be the set of homotopy classes of simple closed curves on S. For $X \in T(S)$ and $\alpha \in \mathcal{S}(S)$, let $D(gr_\alpha(X), \phi_\alpha)$ be the Teichmüller disk determined by $gr_\alpha(X)$ and ϕ_α, where ϕ_α is the Jenkins-Strebel differential of α on $gr_\alpha(X)$. Then:

Theorem 5.1. *For any* $\epsilon > 0$, *there exists* $\mathcal{A}_\epsilon$ *depending only on* ϵ *such that for any fixed* $X \in T(S)$ *and* $\alpha \in \mathcal{S}(S)$,

$$d_T(gr_\alpha(X), gr_\alpha(R)) \leq d_{P_i}(X, R) + \mathcal{A}_\epsilon, \;\; i = 1, 2$$

holds for any $R \in T_\epsilon(S) \cap gr_\alpha^{-1}(D(gr_\alpha(X), \phi_\alpha))$.

Similarly, Corollaries 5.1–5.5 in Section 5 describe some other distance distortions with respect to the Teichmüller metric, the length spectrum metric and Thurston's asymmetric metrics.

2. Preliminaries

In this section, we present some necessary backgrounds on related topics. For references, see [3, 6, 15, 21].

Complex projective structure. Let S be a closed surface with genus $g \geq 2$. A complex projective structure Z on S is [6, 15] a maximal atlas of charts mapping open sets in S into the complex projective space $\mathbb{C}P^1$, such that the transition maps are restrictions of projective maps, i.e., the restriction of elements of $\mathrm{PSL}(2,\mathbb{C})$. We also call the pair (S, Z), or simply Z, a complex projective surface. Since projective maps are conformal maps, a complex projective structure induces a underlying complex structure which (by the uniformization theorem) corresponds to a hyperbolic metric. Thus on a complex projective surface Z, we have a underlying hyperbolic metric λ_Z.

Grafting, Thurston metric. Let $P(S)$ be the space of complex projective structures on S. Thurston discovered a connection between the complex projective structures and hyperbolic geometry by showing that $P(S)$ is parametrized via the grafting homeomorphism (cf. [6]) $Gr : T(S)\times MF(S) \to P(S)$. Roughly speaking, given $X \in T(S)$ and a weighted simple closed curve $s\gamma \in MF(S)$, a complex projective structure $Gr_{s\gamma}(X)$ is obtained by cutting X open along γ and inserting a flat cylinder A of height s and circumference $l_X(\gamma)$. The grafting along a general $\mu \in MF(S)$ is obtained from the density of weighted simple closed curves in $MF(S)$ by continuity.

There is also a conformal grafting $gr : T(S) \times MF(S) \to T(S)$ such that $gr = \pi \circ Gr$, where $\pi : P(S) \to T(S)$ is the forgetful mapping remembering only the underlying complex structures.

Let Z be a complex projective surface. For each point $x \in Z$ and each tangent vector $v \in T_xZ$, define the length of v by

$$t_Z(v) = \inf_{f:\mathbb{D}\to Z} \lambda_{\mathbb{D}}(f^*v),$$

where the infimum is taken over all projective immersion $f : \mathbb{D} \to Z$ with $x \in f(\Delta)$, and $\lambda_{\mathbb{D}}$ is the hyperbolic metric on the unit disk $\mathbb{D}$. Then the metric t_Z is the Thurston metric corresponding to Z [21].

The hyperbolic (Kobayashi) metric on a Riemann surface is defined by taking the infimum over all holomorphic immersions from the unit disk, which is a category larger than that of projective immersions. Thus one gets:

Lemma 2.1. *Let $Z = Gr_\mu(X)$ for $X \in T(S)$ and $\mu \in MF(S)$. Then $\lambda_Z \leq t_Z$, hence $l_{\lambda_Z}(\alpha) \leq l_{t_Z}(\alpha)$ for every $\alpha \in MF(S)$.*

According to the grafting operation, the Thurston metric corresponding to $Gr_\mu(X)$ agrees with the hyperbolic metric on X and with the flat metric on the inserted cylinder A. Therefore, it is easy to get the following:

Lemma 2.2. *Let $X \in T(S)$ and $\mu \in MF(S)$. Then*

$$l_{Gr_\mu(X)}(\alpha) \geq l_X(\alpha)$$

for every $\alpha \in MF(S)$.

Marked length spectrum. Let $\mathcal{C}(S)$ be the set of homotopy classes of essential closed curves on S. Throughout the paper, if not otherwise stated, there is no difference between a closed curve and its homotopy class. For a (isotopy class of a) metric ρ on S, the marked length spectrum $\mathrm{MLS}(\rho)$ of ρ is defined to be the function

$$\begin{aligned}\mathcal{C}(S) &\to \mathbb{R}\\ \alpha &\mapsto l_\rho(\alpha),\end{aligned}$$

where $l_\rho(\alpha)$ is the ρ-length of α.

Spectral rigidity. Let $\mathfrak{m}(S)$ be a collection of (isotopy classes of) metrics on S. Then a general problem is the following spectral rigidity problem: whether the map

$$\begin{aligned}\mathfrak{m}(S) &\to \mathbb{R}^{\mathcal{C}(S)}\\ \rho &\mapsto \mathrm{MLS}(\rho)\end{aligned}$$

is injective. If such a map is injective, then $\mathcal{C}(S)$ is said to be spectrally rigid over $\mathfrak{m}(S)$. There are many important results in this direction, see, e.g., [2, 3, 17], etc. For example, it has been known since Fricke that $\mathcal{C}(S)$ is spectrally rigid over $T(S)$, the space of isotopy classes of hyperbolic metrics.

In this paper we will consider the following natural refinement of the spectral rigidity problem. For $\rho_1,\ \rho_2 \in \mathfrak{m}(S)$, write $\mathrm{MLS}(\rho_1) \leq \mathrm{MLS}(\rho_2)$ if $l_{\rho_1}(\alpha) \leq l_{\rho_2}(\alpha)$ for every $\alpha \in \mathcal{C}(S)$. Then is it true that $\mathrm{MLS}(\rho_1) \leq \mathrm{MLS}(\rho_2)$ implies $\rho_1 = \rho_2$?

For the hyperbolic metrics, we have:

Lemma 2.3 ([15], [22, Theorem 3.1])**.** *Let* $X_1,\ X_2 \in T(S)$. *If* $\mathrm{MLS}(X_1) \leq \mathrm{MLS}(X_2)$, *then* $X_1 = X_2$.

3. A rigidity result

Theorem 3.1. *Let t and λ be a Thurston metric and a hyperbolic metric on S, respectively. If* $\mathrm{MLS}(t) \leq \mathrm{MLS}(\lambda)$, *then* $t = \lambda$.

Proof. Suppose $\mathrm{MLS}(t) \leq \mathrm{MLS}(\lambda)$. Then by Lemma 2.1 we have

$$\mathrm{MLS}(\lambda_t) \leq \mathrm{MLS}(t) \leq \mathrm{MLS}(\lambda),$$

where λ_t is the underlying hyperbolic metric of the complex projective surface corresponding to the Thurston metric t. From Lemma 2.3, one concludes that $\lambda_t = \lambda$. Consequently, $\mathrm{MLS}(\lambda_t) = \mathrm{MLS}(\lambda) = \mathrm{MLS}(t)$. But this is only possible if, in the grafting operation generating t, the grafting is performed along the zero foliation. This implies $\lambda_t = t$. Therefore, $t = \lambda$. $\square$

Remark 3.1. By Lemma 2.1, we always have $\mathrm{MLS}(\lambda_t) \leq \mathrm{MLS}(t)$ for any Thurston metric t. Thus generally speaking, for a hyperbolic metric λ and a Thurston metric t, the implication from $\mathrm{MLS}(\lambda) \leq \mathrm{MLS}(t)$ to $t = \lambda$ might not hold.

4. Length comparisons

In this section, we describe some length comparisons with respect to the hyperbolic metric, the Thurston metric and its underlying hyperbolic metric.

For a Riemann surface X, let $Q(X)$ be the space of holomorphic quadratic differentials on X. The following two norms may be endowed on $Q(X)$, each making $Q(X)$ a Banach space. The first one is the L_1-norm

$$||q||_1 = \int_X |q(z)|dxdy,\ z = x + iy.$$

The second is Bers' sup-norm

$$||q||_\infty = \sup_X \frac{|q(z)|}{\lambda(z)},$$

where $\lambda(z)|dz|^2$ is the hyperbolic metric on X.

By the uniformization theorem, a closed Riemann surface X with genus $g \geq 2$ can be represented as $\mathbb{H}/\Gamma_X$, where Γ_X is a Fuchsian group acting on the upper half plane $\mathbb{H}$. For each hyperbolic element $g \in \Gamma_X$, we have [5] $|\mathrm{Tr}(g)| = 2\cosh(l_X(g)/2)$, where $\mathrm{Tr}(g)$ is the trace of g and $l_X(g)$ is the hyperbolic length of the closed geodesic on X corresponding to g. Consequently the ϵ-thick part $T_\epsilon(S)$ consists of those X such that $|\mathrm{Tr}(g)| \geq c_\epsilon$ for every non-trivial $g \in \Gamma_X$, where c_ϵ is a constant depending only on ϵ.

Now, a quick check of the proof of Lemma 1 and Theorem 1 in [10] leads to the following uniform comparison between the above norms on thick parts of the Teichmüller space.

Lemma 4.1 ([10])**.** *For any $\epsilon > 0$, there exists a uniform constant M_ϵ depending only on ϵ and S, such that for any $X \in T_\epsilon(S)$ and any $q \in Q(X)$,*

$$||q||_\infty \leq M_\epsilon ||q||_1.$$

Note that the Fuchsian groups considered in [10] maybe finitely or infinitely generated, while for our purpose we only need the finitely generated case.

The main result in this section is the following Theorem 4.1, which gives length comparisons between the Thurston metric and its underlying hyperbolic metric.

Theorem 4.1. *For any $\epsilon > 0$, there exists a uniform constant C_ϵ depending only on ϵ and S such that for any $X \in T(S)$ and $\beta \in MF(S)$ with $gr_\beta(X) \in T_\epsilon(S)$, we have*

$$l^2_{gr_\beta(X)}(\alpha) \geq C_\epsilon \frac{l^2_{Gr_\beta(X)}(\alpha)}{4\pi(g-1) + l_X(\beta)}$$

for any $\alpha \in MF(S)$.

Proof. It is sufficient to prove the theorem for the case when α is a simple closed curve, while the situation for measured foliations can be accomplished by the density of weighted simple closed curves in $MF(S)$ from continuity.

In the following two paragraphs, we make general discussions with no assumption made on $gr_\beta(X)$. By definition [1], the extremal length of α with respect to $gr_\beta(X)$ is given by

$$ext_{gr_\beta(X)}(\alpha) = \sup_\rho \frac{l_\rho^2(\alpha)}{A_\rho}, \tag{1}$$

where the supremum is taken over all the conformal metrics ρ on $gr_\beta(X)$, and A_ρ is the ρ-area of $gr_\beta(X)$. Note that the Thurston metric corresponding to $Gr_\beta(X)$ is compatible with the underlying complex structure $gr_\beta(X)$, and that its area is $4\pi(g-1) + l_X(\beta)$. Thus from (1) we obtain

$$ext_{gr_\beta(X)}(\alpha) \geq \frac{l^2_{Gr_\beta(X)}(\alpha)}{4\pi(g-1)+l_X(\beta)}. \tag{2}$$

On the other hand, from the extremality of the metric induced by the Jenkins-Strebel differential $\phi_\alpha \in Q(gr_\beta(X))$ of α with unit L_1-norm [20], we get

$$ext_{gr_\beta(X)}(\alpha) = l^2_{\phi_\alpha}(\alpha). \tag{3}$$

For any simple closed curve $\tilde\alpha$ in the homotopy class of α,

$$\begin{aligned}\int_{\tilde\alpha} \sqrt{|\phi_\alpha(z)|}|dz| &= \int_{\tilde\alpha} \frac{\sqrt{|\phi_\alpha(z)|}}{\sqrt{gr_\beta(X)(z)}} \sqrt{gr_\beta(X)(z)}|dz| \\ &\leq \sqrt{||\phi_\alpha||_\infty} \int_{\tilde\alpha} \sqrt{gr_\beta(X)(z)}|dz|,\end{aligned}$$

where in the above expression we have still used the notation $gr_\beta(X)$ for the hyperbolic metric corresponding to $gr_\beta(X)$. This implies

$$l_{\phi_\alpha}(\alpha) \leq \sqrt{||\phi_\alpha||_\infty}\, l_{gr_\beta(X)}(\alpha). \tag{4}$$

Now we specialize to the situation that $gr_\beta(X) \in T_\epsilon(S)$. By Lemma 4.1, (4) implies

$$l^2_{\phi_\alpha}(\alpha) \leq M_\epsilon l^2_{gr_\beta(X)}(\alpha). \tag{5}$$

Then (3) and (5) lead to

$$ext_{gr_\beta(X)}(\alpha) \leq M_\epsilon l^2_{gr_\beta(X)}(\alpha). \tag{6}$$

Therefore, by (2) and (6) we conclude

$$M_\epsilon l^2_{gr_\beta(X)}(\alpha) \geq \frac{l^2_{Gr_\beta(X)}(\alpha)}{4\pi(g-1)+l_X(\beta)}.$$

□

Under the same hypothesis as in Theorem 4.1, we have some corollaries.

Combined with Lemma 2.2, Theorem 4.1 implies the following corollary which is in some sense converse to Theorem A.

Corollary 4.1.
$$l^2_{gr_\beta(X)}(\alpha) \geq C_\epsilon \frac{l^2_X(\alpha)}{4\pi(g-1)+l_X(\beta)}$$
for any $\alpha \in MF(S)$.

Together with Theorem A, Theorem 4.1 also implies the following:

Corollary 4.2.
$$(l_X(\alpha)+i(\alpha,\beta))^2 \geq C_\epsilon \frac{l^2_{Gr_\beta(X)}(\alpha)}{4\pi(g-1)+l_X(\beta)}$$
for any $\alpha \in MF(S)$.

In particular, Corollary 4.2 has a direct consequence which is in some sense converse to Lemma 2.2.

Corollary 4.3.
$$l^2_X(\alpha) \geq C_\epsilon \frac{l^2_{Gr_\beta(X)}(\alpha)}{4\pi(g-1)+l_X(\beta)}$$
for any $\alpha \in MF(S)$ *with* $i(\alpha,\beta)=0$.

5. Distance distortions under conformal graftings

In this section we describe certain distance distortions under conformal graftings, with respect to the Teichmüller metric, the length spectrum metric and Thurston's asymmetric metrics.

Let us first recall the definitions of related metrics. The Teichmüller metric is defined as [1]
$$d_T(X_1,X_2) = \log \inf_f \{K(f)\},$$
where the infimum is taken over all the quasiconformal mappings $f: X_1 \to X_2$ in the homotopy class of $id: S \to S$ and $K(f)$ is the maximal dilatation of f. The length spectrum metric is defined as [18, 19]
$$d_L(X_1,X_2) = \log \sup_{\alpha\in\mathcal{C}(S)} \{\frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)}, \frac{l_{X_1}(\alpha)}{l_{X_2}(\alpha)}\}.$$
Thurston's asymmetric metrics are defined as [22]
$$d_{P_1}(X_1,X_2) = \log \sup_{\alpha\in\mathcal{C}(S)} \{\frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)}\}$$
and
$$d_{P_2}(X_1,X_2) = \log \sup_{\alpha\in\mathcal{C}(S)} \{\frac{l_{X_1}(\alpha)}{l_{X_2}(\alpha)}\}.$$

Let $\mathcal{S}(S)$ be the set of homotopy classes of simple closed curves on S. By Thurston's result [22], the supremum in the definitions of d_L and d_{P_i}, $i = 1, 2$ can be equivalently taken over $\mathcal{S}(S)$. It is known that these metrics are topologically equivalent [19, 11] while not bi-Lipschitz equivalent [12, 13].

A pair $X \in T(S)$ and $q \in Q(X)$ determines (cf. ([16], p. 149–150)) a Teichmüller disk which is given by $\{[z\overline{q}/|q|] : z \in \mathbb{D}\} \subset T(S)$, where $[\cdot]$ represents a Teichmüller equivalence class of Beltrami differential. For $X \in T(S)$ and $\alpha \in \mathcal{S}(S)$, let $D(gr_\alpha(X), \phi_\alpha)$ be the Teichmüller disk determined by $gr_\alpha(X)$ and $\phi_\alpha \in Q(gr_\alpha(X))$, where ϕ_α is the Jenkins-Strebel differential [20] of α on $gr_\alpha(X)$. Note that points in $D(gr_\alpha(X), \phi_\alpha)$ are determined by those Teichmüller mappings whose initial differential is ϕ_α.

The main result in this section is:

Theorem 5.1. *For any $\epsilon > 0$, there exists $\mathcal{A}_\epsilon$ depending only on ϵ such that for any fixed $X \in T(S)$ and $\alpha \in \mathcal{S}(S)$,*

$$d_T(gr_\alpha(X), gr_\alpha(R)) \leq d_{P_i}(X, R) + \mathcal{A}_\epsilon, \;\; i = 1, 2$$

holds for any $R \in T_\epsilon(S) \cap gr_\alpha^{-1}(D(gr_\alpha(X), \phi_\alpha))$.

The proof of this theorem relies on the following quantitative comparison between the extremal length and the hyperbolic length.

Lemma 5.1. *For any $\epsilon > 0$, there exists a uniform constant $\mathcal{B}_\epsilon$ depending only on ϵ such that*

$$\mathcal{B}_\epsilon l_X(\gamma) \leq ext_{gr_\gamma(X)}(\gamma)$$

holds for any $X \in T_\epsilon(S)$ and any simple closed curve γ.

For any $X \in T(S)$,

$$ext_{gr_\gamma(X)}(\gamma) \leq l_X(\gamma)$$

holds for any simple closed curve γ.

Proof. By (2) and Lemma 2.2,

$$ext_{gr_\gamma(X)}(\gamma) \geq \frac{l_X^2(\gamma)}{4\pi(g-1) + l_X(\gamma)}$$

holds for any simple closed curve γ. Then the first part of the lemma follows from this inequality.

Let A be an embedded cylinder in X. Then (cf. [7]) A is, with respect to the conformal structure induced from that of X, conformally equivalent to exactly one flat cylinder up to change of scale. The modulus of A is defined to be the modulus of this flat cylinder given by s/l, where s is the height and l is the circumference of the flat cylinder. By the geometric definition of the extremal length [7],

$$ext_X(\alpha) = 1/\text{mod}(\alpha),$$

where $\mathrm{mod}(\alpha)$ is the supremum of the moduli of all cylinders embedded in X with core curve isotopic to α. Therefore, according to the grafting operation,

$$ext_{gr_{s\alpha}(X)}(\alpha) \leq \frac{l_X(\alpha)}{s}$$

holds for any weighted simple closed curve $s\alpha$. Then we obtain the second part of the lemma by taking $s = 1$. □

Now the proof of Theorem 5.1 follows easily.

Proof. (of Theorem 5.1) By a result of Kerckhoff ([7], Theorem 4), the Teichmüller metric can be described in terms of the extremal lengths:

$$d_T(gr_\alpha(X), gr_\alpha(R)) = \log \sup_{\beta \in MF(S)} \{\frac{ext_{gr_\alpha(X)}(\beta)}{ext_{gr_\alpha(R)}(\beta)}\}.$$

From Teichmüller's theorem, the supremum in this expression is realized by the horizontal foliation of the initial differential ϕ_α of the Teichmüller mapping from $gr_\alpha(X)$ to $gr_\alpha(R)$. In our settings, this means

$$d_T(gr_\alpha(X), gr_\alpha(R)) = \log \frac{ext_{gr_\alpha(X)}(\alpha)}{ext_{gr_\alpha(R)}(\alpha)} \tag{7}$$

for all $R \in gr_\alpha^{-1}(D(gr_\alpha(X), \phi_\alpha))$.

As a consequence of Lemma 5.1 and (7), we obtain

$$d_T(gr_\alpha(X), gr_\alpha(R)) \leq d_{P_2}(X, R) + \mathcal{A}_\epsilon. \tag{8}$$

The case for d_{P_1} follows by switching the roles of X and R in (8). □

By a well-known distortion result [18, 23],

$$l_Y(f(\gamma)) \leq K(f) l_X(\gamma)$$

holds for any simple closed curve γ, where $f : X \to Y$ is a quasiconformal mapping with maximal dilatation $K(f)$. Then, we have from the definitions that

$$d_{P_i} \leq d_L \leq d_T,\ i = 1, 2.$$

Therefore, with the same hypothesis as in Theorem 5.1, we have the following consequences on distance distortions under conformal graftings.

Corollary 5.1.

$$d_T(gr_\alpha(X), gr_\alpha(R)) \leq d_L(X, R) + \mathcal{A}_\epsilon.$$

Corollary 5.2.

$$d_T(gr_\alpha(X), gr_\alpha(R)) \leq d_T(X, R) + \mathcal{A}_\epsilon.$$

Corollary 5.3.

$$d_L(gr_\alpha(X), gr_\alpha(R)) \leq d_{P_i}(X, R) + \mathcal{A}_\epsilon,\ i = 1, 2.$$

Corollary 5.4.

$$d_L(gr_\alpha(X), gr_\alpha(R)) \leq d_L(X,R) + \mathcal{A}_\epsilon.$$

Corollary 5.5.

$$d_{P_i}(gr_\alpha(X), gr_\alpha(R)) \leq d_{P_i}(X,R) + \mathcal{A}_\epsilon,\ \ i = 1,2.$$

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