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ON A COMPOSITE FUNCTIONAL EQUATION RELATED TO THE GOLAB-SCHINZEL EQUATION

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Abstract. Let X be a vector space over a field K of real or complex numbers and $k \in \mathbb{N}$. We prove the superstability of the following generalized Golab–Schinzel type equation

$$f(x_1 + \sum_{i=2}^{p} x_i f(x_1)^k f(x_2)^k \cdots f(x_{i-1})^k) = \prod_{i=1}^{p} f(x_i),\ x_1, x_2, \ldots, x_p \in X,$$

where $f : X \to K$ is an unknown function which is hemicontinuous at the origin.

1. Introduction and preliminaries

Let X be a vector space over a field K of real or complex numbers. The Golab–Schinzel equation

$$f(x + f(x)y) = f(x)f(y) \quad \text{for} \quad x, y \in X, \tag{1.1}$$

and its generalization

$$f(x + f(x)^k y) = \lambda f(x)f(y) \quad \text{for} \quad x, y \in X, \tag{1.2}$$

where $k \in \mathbb{N},\ \lambda \in K \setminus \{0\}$ are fixed and $f : X \to K$ is an unknown function, are intensively studied in the last half-century. The solutions of (1.1) and (1.2) have been investigated under various regularity assumptions, e.g., in [1], [4, 5, 6, 7, 8, 9, 10] and [14, 23]. For more details concerning (1.1) and (1.2), its applications and further generalizations we refer the reader to a survey paper [10] (see also [6, 7, 8, 9, 10, 11, 12, 13, 14] and [25, 26, 27]).

The stability problem for (1.1) and (1.2) has been considered in [17, 18, 19, 20, 21]. It has been proved in [19] that for every $k \in \mathbb{N}$, Eq. (1.2) is superstable in the class of functions $f : X \to K$ continuous at 0 on rays, i.e., every such function satisfying the inequality

$$\left| f(x + f(x)^k y) - \lambda f(x)f(y) \right| \leq \varepsilon \quad \text{for} \quad x, y \in X,$$

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where ε is a fixed positive real number, either is bounded or satisfies (1.2). The first results of that kind have been studied in [3] for the exponential equation, in [2] for the cosine equation on an abelian group and in [30, 31, 32, 33, 34] for trigonmetric functional equations on any group. For further information regarding superstability of functional equations we refer to [24]. Recently in [15], it has been proved that the functional equation

$$f(x+f(x)^k y+f(x)^k f(y)^k z)=f(x)f(y)f(z), \quad x,y,z\in X,$$

is superstable. Let $p\in\mathbb{N}$ such that $p\geq 2$ and given a function $f:X\to K$, we will denote its difference by an operator $Df:X^p\to K$ as

$$Df(x_1,x_2,\ldots,x_p)=\left|f(x_1+\sum_{i=2}^{p}x_i f(x_1)^k f(x_2)^k\cdots f(x_{i-1})^k)-\prod_{i=1}^{p}f(x_i)\right|$$

for all $x_1,x_2,\ldots,x_p\in X$.

In the present paper, we deal with the superstability problem for the generalized Golab–Schinzel type equation

$$f(x_1+\sum_{i=2}^{p}x_i f(x_1)^k f(x_2)^k\cdots f(x_{i-1})^k)=\prod_{i=1}^{p}f(x_i) \text{ for } x_1,x_2,\ldots,x_p\in X, \tag{1.3}$$

where f is defined on a linear space X over the field K of real or complex numbers and takes its values in K, namely, we investigate the condition

$$Df(x_1,x_n,\ldots,x_p)\leq\varphi(x_2,x_3,\ldots,x_p), \quad x_1,x_2,\ldots,x_p\in X.$$

As consequences, we give some applications.

In what follows $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of all positive integers and real numbers, respectively. X is a vector space over a field K of real or complex numbers and p and k are nonnegative integer constants such that $p\geq 2$.

2. Auxiliary results

To formulate the main result of the paper, we need the following definition (cf. [19]).

Definition 1. A function $f:X\to K$ is hemicontinuous at the origin provided, for every $x\in X$, the function $f_x:K\to K$, given by

$$f_x(t)=f(tx)$$

for $t\in K$ is continuous at 0.

The functional equation (1.3) is connected with the equation (1.2) as follows:

Lemma 1. *A function $f:X\to K$ satisfies the functional equation*

$$f(x_1+\sum_{i=2}^{p}x_i f(x_1)^k f(x_2)^k\cdots f(x_{i-1})^k)=\prod_{i=1}^{p}f(x_i) \tag{2.1}$$

for all $x_1, x_2, \dots, x_p \in X$ if and only if f satisfies the functional equation

$$f(x + f(x)^k y) = f(0)^{p-2} f(x) f(y) \tag{2.2}$$

for all $x, y \in X$.

Proof. Setting in (2.1) $x_3 = x_4 = \cdots = x_p = 0$, we clearly see that, (2.1) implies (2.2). Thus let us assume that f satisfies (2.2). Putting $x = y = 0$ in the identity (2.2) we obtain that

$$f(0)^p = f(0).$$

Then we get $f(0)^{p-1} = 1$ if $f \neq 0$, and we have

$$\begin{aligned} & f(x_1 + \sum_{i=2}^{p} x_i f(x_1)^k f(x_2)^k \cdots f(x_{i-1})^k) \\ &= f(x_1 + f(x_1)^k \left\{ x_2 + \sum_{i=3}^{p} x_i f(x_2)^k \cdots f(x_{i-1})^k \right\} \\ &= f(0)^{p-2} f(x_1) f(x_2 + \sum_{i=3}^{p} x_i f(x_2)^k \cdots f(x_{i-1})^k) \end{aligned}$$

for all $x_1, x_2, \dots, x_p \in X$. By induction, we get

$$\begin{aligned} & f(x_1 + \sum_{i=2}^{p+1} x_i f(x_1)^k f(x_2)^k \cdots f(x_{i-1})^k) \\ &= f(0)^{p-2} f(x_1)^k f(0)^{p-2} f(x_2)^k f(x_3 + \sum_{i=4}^{p} x_i f(x_2) \cdots f(x_{i-1})^k). \\ &= (f(0)^{p-2})^{p-1} \prod_{i=1}^{p} f(x_i) \\ &= \prod_{i=1}^{p} f(x_i) \end{aligned}$$

for all $x_1, x_2, \dots, x_p \in X$. □

Remark 1. A function $f : X \to K$ with $f(0) \geq 0$ satisfies the functional equation (2.1) if and only if f satisfies the functional equation:

$$f(x + f(x)^k y) = f(x) f(y) \quad \text{for} \quad x, y \in X.$$

Lemma 2. *Let $f : X \to K$ be a bounded function satisfying*

$$\left| f(x_1 + \sum_{i=2}^{p} x_i f(x_1)^k f(x_2)^k \cdots f(x_{i-1})^k) - \prod_{i=1}^{p} f(x_i) \right| \leq \varepsilon \tag{2.3}$$

for all $x_1, x_2, \dots, x_p \in X$. Then

$$|f(x)| \leq \frac{1 + \sqrt{1 + 4\varepsilon}}{2} \quad \textit{for all } x \in X.$$

Proof. Assume that f is a bounded function satisfying the inequality (2.3) and let $M = \sup|f|$. Then we get, for all $x_1, x_2, \dots, x_p \in X$, that

$$\left|\prod_{i=1}^{p} f(x_i)\right| \leq \varepsilon + M,$$

from which we obtain that $M^p - M - \varepsilon \leq 0$. If $M \geq 1$, then

$$M^2 - M - \varepsilon \leq M^p - M - \varepsilon \leq 0.$$

This inequality shows that

$$M \leq \frac{1+\sqrt{1+4\varepsilon}}{2},$$

and so

$$|f(x)| \leq \mathrm{Max}(1, \frac{1+\sqrt{1+4\varepsilon}}{2}) = \frac{1+\sqrt{1+4\varepsilon}}{2}$$

for all $x \in X$. □

3. Technical lemmas

We assume throughout the rest of this section that f is an unbounded function satisfying (2.3). Let $a_1, a_2, \dots, a_{p-2}$ be fixed elements of X such that $f(a_i) \neq 0$ for all $i \in \{1, 2, \dots, p-2\}$ and $(y_n : n \in \mathbb{N})$ be a sequence of elements of $X \backslash f^{-1}(0)$ such that

$$\lim_{n \longrightarrow +\infty} |f(y_n)| = +\infty.$$

We will now introduce some notations that we will use throughout the rest of the paper.

Definition 2. For all $x_1, x_2, \dots, x_p$ in X and $n \in \mathbb{N}$ put

$$B(x_1, x_2, \dots, x_p) = x_1 + \sum_{i=2}^{p} x_i f(x_1)^k f(x_2)^k \cdots f(x_{i-1})^k,$$

and define the sequences of functions d_n, α_n and c_n by

$$\begin{aligned} d_n(x_1, x_2, \dots, x_p) &= B(y_n, a_1, a_2, \dots, a_{p-2}, B(x_1, x_2, \dots, x_p)), \\ \alpha_n(x_1) &= B(y_n, a_1, a_2, \dots, a_{p-2}, x_1), \\ c_n(x_1, x_2, \dots, x_p) &= B(\alpha_n(x_1), x_2, x_3, \dots, x_p). \end{aligned}$$

Lemma 3. *Let f be an unbounded function satisfying (2.3). Then, for all $x_1, x_2, \dots, x_p \in X$, we have*

$$\lim_{n\to+\infty} \frac{f(d_n(x_1,x_2,\dots,x_p))}{f(y_n)} = f(a_1)f(a_2)\cdots f(a_{p-2})f(B(x_1, x_2, \dots, x_p)), \tag{3.1}$$

$$\lim_{n\to+\infty} \frac{f(\alpha_n(x_1))}{f(y_n)} = f(a_1)f(a_2)\cdots f(a_{p-2})f(x_1). \tag{3.2}$$

For all $x_1 \in X \backslash f^{-1}(0)$ *and* $x_2, \ldots, x_p \in X$*, we have*

$$\lim_{n\to+\infty} \frac{f(c_n(x_1,x_2,\ldots,x_p))}{f(\alpha_n(x_1))} = f(x_2)f(x_3)\cdots f(x_p), \tag{3.3}$$

$$\lim_{n\to+\infty} \frac{f(c_n(x_1,x_2,\ldots,x_p))}{f(y_n)} = f(a_1)f(a_2)\cdots f(a_{p-2})f(x_1)f(x_2)f(x_3)\cdots f(x_p), \tag{3.4}$$

and we have $c_n(x_1, x_2, \ldots, x_p) \in X \backslash f^{-1}(0)$ *for all* $x_1, x_2, \ldots, x_p \in X \backslash f^{-1}(0)$.

Proof. For every $x_1, x_2, \ldots, x_p$ in X and $n \in \mathbb{N}$, using (2.3) to get

$$|f(d_n(x_1, x_2, \ldots, x_p)) - f(y_n)f(a_1)f(a_2)\cdots f(a_{p-2})f(B(x_1, x_2, \ldots, x_p))| \leq \epsilon.$$

Dividing the above inequality by $f(y_n)$, then we get (3.1). Putting $x_2 = x_3 = \cdots = x_p = 0$ in the identity (3.1) to obtain (3.2). Hence

$$\lim_{n\longrightarrow+\infty} |f(\alpha_n(x_1))| = +\infty$$

for all $x_1 \in X \backslash f^{-1}(0)$. By virtue of (2.3), we have

$$\left| f(c_n(x_1, x_2, \ldots, x_p)) - f(\alpha_n(x_1) \prod_{i=2}^{p} f(x_i) \right| \leq \epsilon.$$

Dividing by $f(\alpha_n)$ and passing to the limit as $n \to +\infty$ with the use of $\lim_{n\to+\infty} |f(\alpha_n(x_1))| = +\infty$, we obtain (3.3). Thus, taking into account (3.2) and (3.3), we get (3.4) from which we conclude that, for all $x_1, x_2, \ldots, x_p \in X \backslash f^{-1}(0)$, we have

$$c_n(x_1, x_2, \ldots, x_p) \in X \backslash f^{-1}(0).$$

□

Definition 3. We define

$$l_n(x_1, x_2, \ldots, x_p) = \frac{d_n(x_1, x_2, \ldots, x_p) - c_n(x_1, x_2, \ldots, x_p)}{f(c_n(x_1, x_2, \ldots, x_p))^k} \tag{3.5}$$

for all $x_1, x_2, \ldots, x_p \in X \backslash f^{-1}(0)$.

Lemma 4. *We have*

i)

$$\begin{aligned} d_n(x_1, x_2, \ldots, x_p) &= c_n(x_1, x_2, \ldots, x_p) \\ &\quad + l_n(x_1, x_2, \ldots, x_p) f(c_n(x_1, x_2, \ldots, x_p))^k, \end{aligned} \tag{3.6}$$

ii)

$$\lim_{n\to+\infty} l_n(x_1, x_2, \ldots, x_p) = 0 \tag{3.7}$$

for all $x_1, x_2, \ldots, x_p \in X \backslash f^{-1}(0)$.

Proof. i) Equality (3.6) is an immediate consequence of (3.5).

ii) For every $x_1, x_2, \dots, x_p \in X \backslash f^{-1}(0)$, we have

$$\begin{aligned} &l_n(x_1, x_2, \dots, x_p) \\ &= \frac{f(y_n)^k}{f(c_n(x_1, \dots, x_p))^k} \left((f(a_1) f(a_2) \cdots f(a_{p-2}) f(x_1))^k - \frac{f(\alpha_n(x_1))^k}{f(y_n)^k} \right) \\ &\quad \times B(x_2, x_3, \dots, x_p). \end{aligned}$$

By virtue of (3.2) and (3.4), we obtain

$$\lim_{n \to +\infty} l_n(x_1, x_2, \dots, x_p) = 0.$$

Thus, as f is hemicontinuous at the origin, we get

$$\lim_{n \to +\infty} z_{2,n}(x_1, x_2, \dots, x_p) = 0,$$

and inductively, since $f(0) \neq 0$, we obtain that

$$\lim_{n \to +\infty} z_{i,n}(x_1, x_2, \dots, x_p) = 0 \quad \text{for all } 2 \leq i \leq p, \tag{3.8}$$

as desired. □

Definition 4. We define the sequences of functions $z_{1,n}, z_{2,n}, \dots, z_{p,n}$ by

$$\begin{cases} z_{1,n}(x_1, x_2, \dots, x_p) = c_n(x_1, x_2, \dots, x_p), \\ z_{2,n}(x_1, x_2, \dots, x_p) = \frac{1}{p-1} l_n(x_1, x_2, \dots, x_p), \\ z_{i,n}(x_1, x_2, \dots, x_p) = \frac{1}{p-1} \frac{l_n(x_1, \dots, x_p)}{(f(z_{2,n}(x_1, \dots, x_p)) f(z_{3,n}(x_1, \dots, x_p)) \cdots f(z_{i-1,n}(x_1, \dots, x_p)))^k}, \\ \qquad i \geq 3, \end{cases}$$

for all $x_1, x_2, \dots, x_p \in X \setminus f^{-1}(0)$.

Remark 2. The sequences of functions $z_{i,n}$, $i \in \{1, 2, \dots, p\}$ are well defined. Moreover, as f is hemicontinuous at the origin, we get

$$\lim_{n \to +\infty} f(z_{i,n}(x_1, x_2, \dots, x_p)) = f(0) \quad \text{for all } 2 \leq i \leq p.$$

Then, for all $i \in \{1, 2, \dots, p\}$, from a certain *positive integer* N_i, we shall have $f(z_{i,n}(x_1, x_2, \dots, x_p)) \neq 0$. For $n \in \mathbb{N}$ such that $n \geq N := \max_{i \in \{1,2,\dots,p\}}(N_i)$ we get that $f(z_{i,n}(x_1, \dots, x_p)) \neq 0$ for all $i \in \{1, 2, \dots, p\}$.

Lemma 5. *For every* $x_1, x_2, \dots, x_p \in X \backslash f^{-1}(0)$ *we have*

i)

$$\lim_{n \longrightarrow +\infty} z_{i,n}(x_1, x_2, \dots, x_p) = 0 \quad \textit{for all } 1 \leq i \leq p,$$

ii)

$$\begin{aligned} &B(z_{1,n}(x_1, x_2, \dots, x_p), z_{2,n}(x_1, x_2, \dots, x_p), \dots, z_{p,n}(x_1, x_2, \dots, x_p)) \\ &= c_n(x_1, x_2, \dots, x_p) + l_n(x_1, x_2, \dots, x_p) f(c_n(x_1, x_2, \dots, x_p))^k. \end{aligned}$$

Proof. i) Using (3.7), we have $\lim_{n\to+\infty} z_{1,n}(x_1, x_2, \ldots, x_p) = 0$. Also, f is hemicontinuous at the origin, then we get

$$\lim_{n\to+\infty} z_{2,n}(x_1, x_2, \ldots, x_p) = 0,$$

since $f(0) \neq 0$, we obtain by induction, $\lim_{n\to+\infty} z_{i,n}(x_1, x_2, \ldots, x_p) = 0$ for all $2 \leq i \leq p$ and $x_1, x_2, \ldots, x_p \in X \backslash f^{-1}(0)$.

ii) We have

$$\begin{aligned}
& z_{1,n}(x_1, x_2, \ldots, x_p) \\
& \quad + \sum_{i=2}^{p} z_{i,n}(x_1, \ldots, x_p)(f(z_{1,n}(x_1, \ldots, x_p))f(z_{2,n}(x_1, \ldots, x_p)) \\
& \qquad \cdots f(z_{i-1,n}(x_1, \ldots, x_p)))^k \\
&= c_n(x_1, x_2, \ldots, x_p) + \sum_{i=2}^{p} \frac{1}{p-1} l_n(x_1, x_2, \ldots, x_p) f(z_{1,n}(x_1, x_2, \ldots, x_p))^k \\
&= c_n(x_1, x_2, \ldots, x_p) + l_n(x_1, x_2, \ldots, x_p) f(z_{1,n}(x_1, x_2, \ldots, x_p))^k \\
&= c_n(x_1, x_2, \ldots, x_p) + l_n(x_1, x_2, \ldots, x_p) f(c_n(x_1, x_2, \ldots, x_p))^k \\
&= d_n(x_1, x_2, \ldots, x_p)
\end{aligned}$$

for all $x_1, x_2, \ldots, x_p \in X \backslash f^{-1}(0)$, as desired. □

4. Superstability of (1.3)

In this section we investigate the superstability of the functional equation (1.3).

Theorem 1. *Assume that $f : X \to K$ and $\varphi : X^{p-1} \to \mathbb{R}^+$ are two hemicontinuous functions at the origin satisfying*

$$Df(x_1, x_2, \ldots, x_p) \leq \varphi(x_2, x_3, \ldots, x_p), \quad x_1, x_2, \ldots, x_p \in X. \tag{4.1}$$

Then either f is bounded or it satisfies the functional equation (1.3).

Proof. Suppose that f is unbounded. Letting $x_2 = x_3 = \cdots = x_p = 0$ in (4.1), we obtain

$$f(x_1)(f(0)^{p-1} - 1) \leq \varphi(0, 0, \ldots, 0),$$

and so $f(0)^{p-1} = 1$ since f is unbounded. Let $a_1, a_2, \ldots, a_{p-2}$ be fixed such that $f(a_i) \neq 0$ for all $i \in \{1, 2, \ldots, p-2\}$ and $(y_n : n \in \mathbb{N})$ be a sequence of elements of $X \backslash f^{-1}(0)$ such that

$$\lim_{n\to+\infty} |f(y_n)| = +\infty.$$

Replacing $x_1, x_2, \ldots, x_p$ in (4.1) by $z_{1,n}(x_1, x_2, \ldots, x_p)$, $z_{2,n}(x_1, x_2, \ldots, x_p), \ldots,$ $z_{p,n}(x_1, x_2, \ldots, x_p)$, respectively, using (3.6) and Lemma 5(ii), we get

$$\left| f(d_n(x_1, x_2, \ldots, x_p)) - \prod_{i=1}^{p} f(z_i(x_1, x_2, \ldots, x_p)) \right|$$

$$\leq \varphi(z_{2,n}(x_1, x_2, \ldots, x_p), \ldots, z_{p,n}(x_1, x_2, \ldots, x_p)).$$

Moreover, in view of Lemma 5(i), we have

$$\lim_{n\to+\infty} z_{i,n}(x_1, x_2, \ldots, x_p) = 0 \quad \text{for all } 2 \leq i \leq p, \tag{4.2}$$

for all $x_1, x_2, \ldots, x_p \in X\backslash f^{-1}(0)$. This yields, since φ is hemicontinuous at the origin, that

$$\lim_{n\to+\infty} \varphi(z_{2,n}(x_1, x_2, \ldots, x_p), \ldots, z_{p,n}(x_1, x_2, \ldots, x_p)) = \varphi(0, 0, \ldots, 0).$$

So, we get

$$\lim_{n\to+\infty} \frac{f(d_n(x_1, x_2, \ldots, x_p)) - \prod_{i=1}^{p} f(z_{i,n}(x_1, x_2, \ldots, x_p))}{f(y_n)}$$

$$= \lim_{n\to+\infty} \frac{f(d_n(x_1, x_2, \ldots, x_p)) - f(c_n(x_1, x_2, \ldots, x_p)) \prod_{i=2}^{p} f(z_i(x_1, x_2, \ldots, x_p))}{f(y_n)}$$

$$= 0.$$

Therefore, taking into account (3.1) and (3.4), we have

(4.3)

$$\begin{aligned} &f(a_1)f(a_2)\cdots f(a_{p-2})f(B(x_1, x_2, \ldots, x_p)) \\ &= f(a_1)f(a_2)\cdots f(a_{p-2})f(x_1)f(x_2)f(x_3)\cdots f(x_p)\prod_{i=2}^{p}\lim_{n\to+\infty} f(z_i(x_1, x_2, \ldots, x_p)) \\ &= f(a_1)f(a_2)\cdots f(a_{p-2})f(x_1)f(x_2)f(x_3)\cdots f(x_p)f^{n-1}(0) \end{aligned}$$

for all $x_1, x_2, \ldots, x_p \in X\backslash f^{-1}(0)$. Using (4.3) and the fact that $f^{n-1}(0) = 1$, we get

$$f(B(x_1, x_2, \ldots, x_p)) = f(x_1)f(x_2)f(x_3)\cdots f(x_p)$$

for all $x_1, x_2, \ldots, x_p \in X\backslash f^{-1}(0)$. From which we deduce that

$$f(\alpha_n(x_1)) = f(y_n)f(a_1)f(a_2)\cdots f(a_{p-2})f(x_1)$$

for all $x_1 \in X\backslash f^{-1}(0)$ and $n \in \mathbb{N}$. Thus, we obtain

$$\begin{aligned} &c_n(x_1, x_2, \ldots, x_p) \\ &= B(\alpha_n(x_1), x_2, x_3, \ldots, x_p) \\ &= \alpha_n(x_1) + \sum_{i=2}^{p} x_i (f(\alpha_n(x_1))f(x_2)\cdots f(x_{i-1}))^k \\ &= \alpha_n(x_1) + \sum_{i=2}^{p} x_i (f(y_n)f(a_1)f(a_2)\cdots f(a_{p-2})f(x_1)f(x_2)\cdots f(x_{i-1}))^k \\ &= B(y_n, a_1, \ldots, a_{p-2}, x_1) \end{aligned}$$

$$+ (f(y_n)f(a_1)f(a_2)\cdots f(a_{p-2}))^k \sum_{i=2}^{p} x_i(f(x_1)\cdots f(x_{i-1}))^k$$

$$= y_n + \sum_{i=1}^{p-2} a_i(f(y_n)f(a_1)\cdots f(a_{i-1}))^k + x_1(f(y_n)f(a_1)\cdots f(a_{p-2}))^k$$

$$+ (f(y_n)f(a_1)f(a_2)\cdots f(a_{p-2}))^k \sum_{i=2}^{p} x_i(f(x_1)f(x_2)\cdots f(x_{i-1}))^k$$

$$= y_n + \sum_{i=1}^{p-2} a_i(f(y_n)f(a_1)\cdots f(a_{i-1}))^k$$

$$+ B(x_1, x_2, \ldots, x_p)(f(y_n)f(a_1)\cdots f(a_{p-2}))^k$$

$$= B(y_n, a_1, a_2, \ldots, a_{p-2}, B(x_1, x_2, \ldots, x_p))$$

for all $x_1 \in X\backslash f^{-1}(0)$ and all $x_2, \ldots, x_p \in X$. It follows that

$$c_n(x_1, x_2, \ldots, x_p) = d_n(x_1, x_2, \ldots, x_p)$$

for all $x_1 \in X\backslash f^{-1}(0)$ and $x_2, \ldots, x_p \in X$. Taking into account (3.1) and (3.4), we get

$$f(B(x_1, x_2, \ldots, x_p)) = f(x_1)f(x_2)f(x_3)\cdots f(x_p)$$

for all $x_1 \in X\backslash f^{-1}(0)$ and $x_2, \ldots, x_p \in X$ since in the case when $f(x_1) = 0$, (1.3) trivially holds. Therefore the proof of the theorem is complete. □

Corollary 1. *Let $\varepsilon > 0$ be given. Assume that a function $f : X \to K$ is a hemicontinuous function at the origin satisfying*

$$Df(x_1, x_n, \ldots, x_p) \le \varepsilon$$

for all $x_1, x_2, \ldots, x_p \in X$. Then either

$$|f(x)| \le \frac{1 + \sqrt{1 + 4\varepsilon}}{2} \quad \text{for all } x \in X$$

or f satisfies the functional equation (1.3).

Proof. We put $\varphi = \varepsilon$ in Theorem 1, and then the result follows by Lemma 2. □

As a consequences of Theorem 1, we have the following results.

Corollary 2. *Assume that $f : X \to K$ and $\varphi : X^{p-1} \to \mathbb{R}^+$ are two hemicontinuous functions at the origin satisfying*

$$Df(x_1, x_2, \ldots, x_p) \le \varphi(x_2) \text{ or respectively } \varphi(x_3) \text{ or } \cdots \text{ or } \varphi(x_p)$$

for all $x_1, x_2, \ldots, x_p \in X$. Then either f is bounded or it satisfies the functional equation (1.3).

Corollary 3. *Let $f : X \to K$ and $\varphi : X \to \mathbb{R}^+$ be two hemicontinuous functions at the origin satisfying*

$$|f(x + f(x)y) - f(x)f(y)| \leq \varphi(y), \; x, y \in X. \tag{4.4}$$

Then either f is bounded or

$$f(x + f(x)y) = f(x)f(y) \tag{4.5}$$

for all $x, y \in X$.

Corollary 4 ([15])**.** *Assume that $f : X \to K$ and $\varphi : X \to \mathbb{R}^+$ are two hemicontinuous functions at the origin satisfying*

$$\left|f(x + f(x)^k y) - f(x)f(y)\right| \leq \varphi(y), \; x, y \in X.$$

Then either f is bounded or $f(x + f(x)^k y) = f(x)f(y)$ for all $x, y \in X$.

Corollary 5 ([15])**.** *Let $f : X \to K$ and $\varphi : X \times X \to \mathbb{R}^+$ be two hemicontinuous functions at the origin satisfying*

$$\left|f(x + f(x)^k y + f(x)^k f(y)^k z) - f(x)f(y)f(z)\right| \leq \varphi(y, z), \; x, y, z \in X.$$

Then either f is bounded or $f(x + f(x)^k y + f(x)^k f(y)^k z) = f(x)f(y)f(z)$ for all $x, y, z \in X$.

From Theorem 1, we can obtain the following three corollaries with particular cases of φ as natural results.

Corollary 6. *Let $\alpha_1, \alpha_2, \ldots, \alpha_{p-1}$, θ be nonnegative real numbers. Assume that $f : X \to K$ is a hemicontinuous function at the origin satisfying*

$$Df(x_1, x_2, \ldots, x_p) \leq \theta(\|x_2\|^{\alpha_1} + \|x_3\|^{\alpha_2} + \cdots + \|x_p\|^{\alpha_{p-1}}), \; x_1, x_2, \ldots, x_p \in X.$$

Then either f is bounded or it satisfies the functional equation (1.3)*.*

Corollary 7. *Let α, θ be nonnegative real numbers. Let $f : X \to K$ be a hemicontinuous function at the origin satisfying*

$$\left|f(x + f(x)^k y) - f(x)f(y)\right| \leq \theta \; \|y\|^{\alpha}, \; x, y \in X.$$

Then either f is bounded or $f(x + f(x)^k y) = f(x)f(y)$ for all $x, y \in X$.

Corollary 8. *Let α, β, θ be nonnegative real numbers. Let $f : X \to K$ be a hemicontinuous function at the origin satisfying*

$$\left|f(x + f(x)^k y + f(x)^k f(y)^k z) - f(x)f(y)f(z)\right| \leq \theta(\|y\|^{\alpha} + \|z\|^{\beta}), \; x, y, z \in X.$$

Then either f is bounded or $f(x + f(x)^k y + f(x)^k f(y)^k z) = f(x)f(y)f(z)$ for all $x, y, z \in X$.

Remark 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = 3x + 4$ for all $x \in \mathbb{R}$. Then $|f(x + f(x)y) - f(x)f(y)| = |9x + 12|$, but f is unbounded and f does not satisfy the equation (4.5). This shows that the condition (4.4) is essential in Corollary 3. Therefore the condition (4.1) is essential in Theorem 1.

References

[1] J. Aczel and S. Golab, *Remarks on one-parameter subsemigroups of the affine group and their homo- and isomorphisms*, Aequationes Math. **4** (1970), 1–10.

[2] J. A. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc. **80** (1980), no. 3, 411–416.

[3] J. A. Baker, J. Lawrence, and F. Zorzitto, *The stability of the equation* $f(x+y) = f(x)f(y)$, Proc. Amer. Math. Soc. **74** (1979), no. 2, 242–246.

[4] K. Baron, *On the continuous solutions of the Golab–Schinzel equation*, Aequationes Math. **38** (1989), no. 2-3, 155–162.

[5] N. Brillouët-Belluot, *On some functional equations of Golab–Schinzel type*, Aequationes Math. **42** (1991), no. 2-3, 239–270.

[6] N. Brillouët-Belluot, J. Brzdek, and K. Cieplinski, *On some recent developments in Ulam's type stability*, Abstr. Appl. Anal. **2012** (2012), Art. ID 716936, 41 pp.

[7] N. Brillouët-Belluot and J. Dhombres, *Équations fonctionnelles et recherche de sous-groupes*, Aequationes Math. **31** (1986), no. 2-3, 253–293.

[8] J. Brzdek, *Subgroups of the group* $\mathbb{Z}_n$ *and a generalization of the Golab–Schinzel functional equation*, Aequationes Math. **43** (1992), no. 1, 59–71.

[9] ———, *Some remarks on solutions of the functional equation* $f(x + f(x)^n y) = tf(x)f(y)$, Publ. Math. Debrecen **43** (1993), no. 1-2, 147–160.

[10] ———, *Golab–Schinzel equation and its generalizations*, Aequationes Math. **70** (2005), no. 1-2, 14–24.

[11] ———, *Stability of the generalized Golab–Schinzel equation*, Acta Math. Hungar. **113** (2006), 115–126.

[12] ———, *On the quotient stability of a family of functional equations*, Nonlinear Anal. **71** (2009), no. 10, 4396–4404.

[13] ———, *On stability of a family of functional equations*, Acta Math. Hungar. **128** (2010), no. 1-2, 139–149.

[14] J. Brzdek and K. Cieplinski, *Hyperstability and superstability*, Abstr. Appl. Anal. **2013** (2013), Article ID 401756, 13 pages.

[15] A. Chahbi, *On the superstability of the generalized Golab–Schinzel equation*, Internat. J. Math. Anal. **6** (2012), no. 54, 2677–2682.

[16] A. Charifi, B. Bouikhalene, S. Kabbaj, and J. M. Rassias, *On the stability of Pexiderized Golab-Schinzel equation*, Comput. Math. Appl. **59** (2010), no. 9, 3193–3202.

[17] J. Chudziak, *Approximate solutions of the Golab–Schinzel equation*, J. Approx. Theory **136** (2005), no. 1, 21–25.

[18] ———, *Stability of the generalized Golab–Schinzel equation*, Acta Math. Hungar. **113** (2006), no. 1-2, 133–144.

[19] ———, *Approximate solutions of the generalized Golab–Schinzel equation*, J. Inequal. Appl. **2006** (2006), Article ID 89402, 8 pp.

[20] ———, *Stability problem for the Golab–Schinzel type functional equations*, J. Math. Anal. Appl. **339** (2008), no. 1, 454–460.

[21] J. Chudziak and J. Tabor, *On the stability of the Golab–Schinzel functional equation*, J. Math. Anal. Appl. **302** (2005), no. 1, 196–200.

[22] R. Ger and P. Šemrl, *The stability of the exponential equation*, Proc. Amer. Math. Soc. **124** (1996), no. 3, 779–787.

[23] S. Golab and A. Schinzel, *Sur l'equation fonctionnelle* $f(x + yf(x)) = f(x)f(y)$, Publ. Math. Debrecen **6** (1959), 113–125.

[24] D. H. Hyers, G. I. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and their Applications **34**, Birkhäuser, Boston, Inc., Boston, MA, 1998.

[25] E. Jablonska, *On solutions of some generalizations of the Golab–Schinzel equation*, In: Functional Equations in Mathematical Analysis, pp. 509–521. Springer Optimization and its Applications vol. 52. Springer 2012.

[26] ______, *On continuous solutions of an equation of the Golab–Schinzel type*, Bull. Aust. Math. Soc. **87** (2013), no. 1, 10–17.

[27] ______, *On locally bounded above solutions of an equation of the Golab–Schinzel type*, Aequationes Math. **87** (2014), no. 1-2, 125–133.

[28] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[29] Th. M. Rassias, *On the stability of linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[30] A. Roukbi, D. Zeglami, and S. Kabbaj, *Hyers-Ulam stability of Wilson's functional equation*, Math. Sci. Adv. Appl. **22** (2013), 19–26.

[31] D. Zeglami, *The superstability of a variant of Wilson's functional equations on an arbitrary group*, Afr. Mat. **26** (2015), 609–617.

[32] D. Zeglami and S. Kabbaj, *On the supesrtability of trigonometric type functional equations*, British J. Math. & Comput. Sci. **4** (2014), no. 8, 1146–1155.

[33] D. Zeglami, S. Kabbaj, A. Charifi, and A. Roukbi, *μ-Trigonometric functional equations and Hyers-Ulam stability problem in hypergroups*, Functional Equations in Mathematical Analysis, pp. 337–358, Springer, New York, 2012.

[34] D. Zeglami, A. Roukbi, and S. Kabbaj, *Hyers-Ulam stability of generalized Wilson's and d'Alembert's functional equations*, Afr. Mat. **26** (2015), 215–223.

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