Bull. Korean Math. Soc. **53** (2016), No. 2, pp. 373–386
http://dx.doi.org/10.4134/BKMS.2016.53.2.373

ON DIFFERENCE QUOTIENTS OF CHEBYSHEV POLYNOMIALS

Seon-Hong Kim and Jung Hee Lee

Abstract. In this paper, we investigate analytic and algebraic properties, and derive some identities satisfied by difference quotients of Chebyshev polynomials of the first kind.

1. Introduction

Let $f(x)$ and $g(y)$ be polynomials in the single independent variables x and y with coefficients in the field $\mathbb{C}$ of complex numbers. Cassels et al. [2, 3] studied factorizations of $f(x) - g(y)$ as the polynomial in the pair of variables x and y. They also considered a trivial case when f and g are the same polynomial since $f(x)-f(y)$ is divisible by $x-y$. In this case, obtaining the factors of the polynomial

$$\frac{f(x)-f(y)}{x-y} \tag{1}$$

is in general rather complicated. The polynomial of the form (1) also arises in Bezout matrices that have appeared in the literature for a long time. Given

$$u(x) = \sum_{j=0}^{n} u_j x^j \quad (u_n \neq 0), \qquad v(x) = \sum_{j=0}^{n} v_j x^j,$$

let

$$\frac{u(x)v(y)-u(y)v(x)}{x-y} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \gamma_{ij} x^i y^j. \tag{2}$$

Then $B(u,v) = (\gamma_{ij})_{i,j=0}^{n-1}$ is called the Bezout matrix of $u(x)$ and $v(x)$. As a special case of (2) when $v(x) = 1$, we have

$$\frac{u(x)-u(y)}{x-y}$$

Received July 3, 2014; Revised October 31, 2014.
2010 *Mathematics Subject Classification.* Primary 33C47; Secondary 11B83.
Key words and phrases. difference quotients, Chebyshev polynomials.
This research was supported by the Sookmyung Women's University Research Grants 2015.

that is of the same form with (1). Bezout matrix have many applications in inertia and stability problems of polynomials, control theory and system theory and so on, see [1, 5]. The Bezoutian matrix for Chebyshev polynomials of the second kind has been studied in [9]. They have used Chebyshev polynomials of the second kind to obtain a Barnett-type factorization formula and a triangular-type formula for a generalized Bezout matrix.

Chebyshev polynomials are of great importance in many areas of mathematics, particularly approximation theory. Many papers and books ([6], [8]) have been written about these polynomials. Let $T_n(z)$ and $U_n(z)$ be the Chebyshev polynomials of first kind and of the second kind, respectively. These polynomials satisfy the recurrence relations

$$\begin{aligned} &T_0(z)=1, \quad T_1(z)=z, \quad T_{n+1}(z)=2zT_n(z)-T_{n-1}(z) \quad (n\geq 1)\\ &U_0(z)=1, \quad U_1(z)=2z, \quad U_{n+1}(z)=2zU_n(z)-U_{n-1}(z) \quad (n\geq 1). \end{aligned}$$

Due to frequent occurrences of Chebyshev polynomials in mathematics and other sciences, it is natural to investigate analytic and algebraic properties, and to derive some identities satisfied by polynomials of the form (1) when $f(x)$ is a Chebyshev polynomial. In this paper we particularly study polynomials of z with integer coefficients

$$QT_n(a,z) := \frac{T_n(z)-T_n(a)}{z-a} \qquad (z\neq a)$$

for a constant a, and

$$QT_n(a,a) := \lim_{z\to a} QT_n(a,z).$$

Since $T_n'(a)=nU_{n-1}(a)$, $QT_n(a,z)$ approaches to $nU_{n-1}(a)$ as z approaches to a. Also when n is odd, the polynomial $QT_n(a,z)$ can be considered as a generalization of $T_n(z)$ in that

$$QT_n(0,z)=\frac{T_n(z)}{z}\in\mathbb{Z}[z]$$

and $T_n(z)$ is a polynomial of z with integer coefficients multiplied by z.

2. New results

In this section we list our new results about $QT_n(a,z)$, and their proofs will be given in Section 3. Throughout the paper we use n to represent a positive integer. We begin with introducing the recurrence relations of $QT_n(a,z)$

$$\begin{aligned} QT_{n+4}(a,z) &= 2(a+z)QT_{n+3}(a,z)-2(1+2az)QT_{n+2}(a,z)\\ &\quad +2(a+z)QT_{n+1}(a,z)-QT_n(a,z) \end{aligned}$$

whose proof is straightforward. The proof only depends on the recurrence relations of $T_n(z)$.

A first natural step for studying $QT_n(a,z)$ seems to be to express $QT_n(a,z)$ in terms of powers of z. Let

$$QT_n(a,z) = \sum_{k=0}^{n-1} \alpha(k,a) z^k,$$

where $\alpha(n-1,a) = 2^{n-1}$. The following three propositions are devoted to study the coefficients $\alpha(k,a)$.

Proposition 1. *For an even integer n,*

$$\alpha(k-1,a) = \begin{cases} a\alpha(k,a), & \text{if } k \text{ is odd,} \\ a\alpha(k,a) + (-1)^{\frac{n-k}{2}} 2^{k-1} \binom{\frac{n+k}{2}}{\frac{n-k}{2}} \frac{n}{\frac{n+k}{2}}, & \text{if } k \text{ is even,} \end{cases}$$

and for an odd integer n,

$$\alpha(k-1,a) = \begin{cases} a\alpha(k,a), & \text{if } k \text{ is even,} \\ a\alpha(k,a) + (-1)^{\frac{n-k}{2}} 2^{k-1} \binom{\frac{n+k}{2}}{\frac{n-k}{2}} \frac{n}{\frac{n+k}{2}}, & \text{if } k \text{ is odd,} \end{cases}$$

where $\alpha(n,a) = 0$.

In the example below, we list some coefficients $\alpha(k,a)$ of $QT_n(a,z)$ when n is even, where for an odd integer k, $\alpha(k-1,a) = a\alpha(k,a)$ is easily shown. So we only need to compute $\alpha(k-1,a)$, where k is even.

Example 2. For an even integer n, the first few terms of $\alpha(k-1,a)$, where k is even, starting from $\alpha(n-1,a)$ are

$$\begin{aligned}
\alpha(n-1,a) &= 2^{n-1}, \\
\alpha(n-3,a) &= 2^{n-1}a^2 - 2^{n-3}\binom{n-1}{1}\frac{n}{n-1} = 2^{n-3}(4a^2-n), \\
\alpha(n-5,a) &= a^2\alpha(n-3,a) + 2^{n-5}\binom{n-2}{2}\frac{n}{n-2}, \\
\alpha(n-7,a) &= a^4\alpha(n-3,a) + 2^{n-7}\left[4a^2\binom{n-2}{2}\frac{n}{n-2} - \binom{n-3}{3}\frac{n}{n-3}\right], \\
\alpha(n-9,a) &= a^2\alpha(n-7,a) + 2^{n-9}\binom{n-4}{4}\frac{n}{n-4}, \\
\alpha(n-11,a) &= a^4\alpha(n-7,a) + 2^{n-11}\left[4a^2\binom{n-4}{4}\frac{n}{n-4} - \binom{n-5}{5}\frac{n}{n-5}\right].
\end{aligned}$$

Proposition 3. *If $a > 0$ and $n < 4a^2$, then $\alpha(k,a) > 0$ for all k, $0 \le k \le n-1$.*

Proposition 4. *We have*

$$\alpha(0,a)=\begin{cases}(a-1)\sum_{k=1}^{n-1}\alpha(k,a), & \text{if } n \text{ is even and } 4\mid n,\\ 2+(a-1)\sum_{k=1}^{n-1}\alpha(k,a), & \text{if } n \text{ is even and } 4\nmid n,\\ 1+(a-1)\sum_{k=1}^{n-1}\alpha(k,a), & \text{if } n \text{ is odd.}\end{cases}$$

It is an immediate consequence of Propositions 3 and 4 above that using Rouché's theorem, for $a>1$, $QT_n(a,z)$ has no roots in $|z|<1$ when $4a^2-n>0$. In fact we shall show in Theorem 7 below that the roots of $QT_n(a,z)$ lie on an ellipse closely inside $|z|=a$. This ellipse will be independent of the degree of $QT_n(a,z)$.

An arbitrary polynomial of degree n can be written in terms of the Chebyshev polynomials of the first kind. In the next theorem, we will see that the polynomial $QT_n(a,z)$ are nicely written in terms of the Chebyshev polynomials of the first kind. Its coefficients are obtained from the Chebyshev polynomials of the second kind.

Theorem 5. *We have*

$$QT_n(a,z)=\sum_{k=0}^{n-1}{}' \, 2U_{n-k-1}(a)T_k(z),$$

where $\sum'$ *means a sum with the first term halved. In particular,*

$$QT_n(a,a)=nU_{n-1}(a)$$

and

$$U_n(z)=\frac{2}{n}\sum_{k=1}^{n}U_{n-k}(z)T_k(z).$$

We now turn to the zero location problems of $QT_n(a,z)$. The study of discriminants has a long history, and the concept of discriminant connects with the ratios of roots of polynomials. The discriminant is also a special case of the resultant between a polynomial and its derivative. More specifically, the resultant of polynomials p and q is defined by

$$\mathrm{Res}\,(p,q)=a^{\deg q}b^{\deg p}\prod_{\substack{p(x)=0\\ q(y)=0}}(x-y),$$

where a and b are leading coefficients of p and q, respectively, and the discriminant of p is

$$\Delta_z(p)=(-1)^{\binom{\deg p}{2}}\frac{1}{a}\mathrm{Res}\,(p,p').$$

For more details, see [4]. The discriminants of the Chebyshev polynomials $T_n(z)$ is given by simple and elegant formula (see [8])

$$2^{(n-1)^2}n^n.$$

For Theorem 7 and its own purposes we present an explicit formula for the discriminant of $QT_n(a,z)$.

Theorem 6. *The discriminant of $QT_n(a,z)$ is*

$$\begin{cases} 2^{(n-1)^2} n^{n-2}(1-a^2)^{\frac{n-1}{2}} U_{n-1}^{n-3}(a), & \text{if } n \text{ is odd,} \\ 2^{(n-1)^2} n^{n-2}(1-a^2)^{\frac{n}{2}-1} U_{n-1}^{n-4}(a)(1+T_n(a)), & \text{if } n \text{ is even.} \end{cases}$$

For $-1 \le a \le 1$, all the roots of $QT_n(a,z)$ are real and lie in $[-1,1]$ because $-1 \le T_n(a) \le 1$. However we will see in the next theorem that for $a > 1$, the roots of $QT_n(a,z)$ are always lie on an ellipse.

Theorem 7. *If $a > 1$, then the roots of $QT_n(a,z)$ lie on the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-1} = 1 \qquad (z = x+iy)$$

and are simple. This ellipse is independent of n. In fact, the roots are

$$\zeta_j = a\cos\frac{2j\pi}{n} + i\sqrt{a^2-1}\sin\frac{2j\pi}{n}, \quad 1 \le j \le n-1.$$

By Theorem 7,

$$QT_n(a,z) = 2^{n-1}\prod_{k=1}^{n-1}(z - e(a, \sqrt{a^2-1}, k, n)),$$

where

$$e(a,b,k,n) = a\cos\frac{2k\pi}{n} + ib\sin\frac{2k\pi}{n}.$$

We now follow the theory of Cyclotomic polynomials to obtain another representation of $QT_n(a,z)$. For real numbers a and b, define the polynomial

$$\phi_n(z) = \prod_{\substack{1\le k\le n \\ \gcd(k,n)=1}} (z - e(a,b,k,n)).$$

Theorem 8. *We have*

$$\prod_{d|n}\phi_d(z) = \prod_{k=1}^{n}(z - e(a,b,k,n)),$$

and with $a > 1$ and $b = \sqrt{a^2-1}$,

$$QT_n(a,z) = 2^{n-1}\prod_{\substack{d|n \\ d\ne 1}}\phi_d(z).$$

The composition identity $T_{mn}(z) = T_m(T_n(z))$ is well known. A version for quotients of Chebyshev polynomials can be stated as follows.

Theorem 9. *We have*

$$QT_{mn}(a,z) = QT_n(T_m(a), T_m(z))\, QT_m(a,z)$$

and so if n divides m, then $QT_n(a,z)$ is a divisor of $QT_m(a,z)$.

For more divisibility properties of $QT_n(a,z)$, one may consider irreducibility of $QT_n(a,z)$. Cassels [2] posed a problem polynomial for which polynomials f is the polynomial

$$\frac{f(x) - f(y)}{x - y} \tag{3}$$

reducible. In [3], Engler and Khanduja contributed this problem by showing that for $f(x) \in \mathbb{Q}[x]$, (3) is irreducible over $\mathbb{Q}$ when $\deg f(x)$ is prime. However if y is a fixed constant, the problem is in different situation. In our problem, using Theorem 9, we can easily show that for an integer a, if $QT_n(a,z)$ is irreducible over $\mathbb{Z}$, then n is prime. But the converse is not generally true. In fact, for an odd prime n and $a = 1$, we can prove the following.

Theorem 10. *Let p be an odd prime. Then*

$$T_p(z) - 1 = (z-1)f(z)^2, \quad T_p(z) + 1 = (z+1)g(z)^2,$$

where

$$U_{p-1}(z) = f(z)g(z)$$

and both $f(z)$ and $g(z)$ are irreducible polynomials over $\mathbb{Z}$. So $QT_p(1,z)$ is the square of an irreducible polynomial over $\mathbb{Z}$.

By above theorem, for an odd prime p, $QT_p(1,z)$ is not irreducible over $\mathbb{Z}$. However due to numerical evidences, one may conjecture that for $a \in \mathbb{Z}$ and $a \neq \pm 1$, $QT_p(a,z)$ is an irreducible polynomial over $\mathbb{Z}$. If this is true, it will be a generalization of a well known fact that for an odd integer n, n is a prime if and only $\frac{T_n(z)}{z}$ is an irreducible polynomial over $\mathbb{Z}$ (See Theorem 3 of [7]) because $T_p(0) = 0$. This remains an open problem.

3. Proofs

We first prove Proposition 1.

Proof of Proposition 1. Let for $n \geq 1$,

$$QT_n(a,z) = \frac{T_n(z) - T_n(a)}{z - a} = \sum_{k=0}^{n-1} \alpha(k,a) z^k.$$

Then

$$\begin{aligned} &T_n(z) \\ &= T_n(a) + (z-a)\sum_{k=0}^{n-1} \alpha(k,a) z^k \end{aligned} \tag{4}$$

$$= T_n(a) - a\alpha(0,a) + \sum_{k=1}^{n-1} \left[\alpha(k-1,a) - a\alpha(k,a)\right] z^k + \alpha(n-1,a)z^n.$$

The left side of (4) is given by the well known formula

$$T_n(z) = \frac{1}{2}\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2z)^{n-2k},$$

and the proposition follows from comparing coefficients of z^k of each side. □

Proof of Proposition 3. We only prove the case when n is even and $n < 4a^2$. The case for an odd integer n can be shown in the same way. By Example 2,

$$\alpha(n-1,a) = 2^{n-1} > 0, \quad \alpha(n-3,a) = 2^{n-3}(4a^2 - n) > 0,$$

and so it suffices to consider the signs of

$$\alpha(n-7,a),\ \alpha(n-11,a),\ \alpha(n-15,a),\ \ldots,$$

i.e., $\alpha(n-(4k-1),a)$ for $2 \le k \le n/4$ because if these are positive, so are all others. For $2 \le k \le n/4$, one can compute

$$\begin{aligned}
&\alpha(n-(4k-1),a)\\
&= a^4\alpha(n-(4k-5),a)\\
&\quad + 2^{n-(4k-1)}\left[4a^2\binom{n-(2k-2)}{2k-2}\frac{n}{n-(2k-2)} - \binom{n-(2k-1)}{2k-1}\frac{n}{n-(2k-1)}\right].
\end{aligned}$$

If we show that the second term of the right side of the above equation is positive, then the proposition is proven because $a^4\alpha(n-(4k-5),a) > 0$ when $k = 2$. On the one hand,

$$\begin{aligned}
&4a^2\binom{n-(2k-2)}{2k-2}\frac{n}{n-(2k-2)} - \binom{n-(2k-1)}{2k-1}\frac{n}{n-(2k-1)}\\
&> n\binom{n-(2k-2)}{2k-2}\frac{n}{n-(2k-2)} - \binom{n-(2k-1)}{2k-1}\frac{n}{n-(2k-1)}\\
&= n\frac{(n-(2k-2))(n-(2k-1))(n-2k)\cdots(n-(4k-5))}{(2k-2)!}\frac{n}{n-(2k-2)}\\
&\quad - \frac{(n-(2k-1))(n-2k)(n-(2k+1))\cdots(n-(4k-3))}{(2k-1)!}\frac{n}{n-(2k-1)}\\
&= \frac{n}{(2k-1)!}(n-2k)\cdots(n-(4k-5))\\
&\quad [(2k-1)n(n-(2k-1)) - (n-(4k-4))(n-(4k-3))] > 0.
\end{aligned}$$

□

Proof of Proposition 4. Since

$$\sum_{k=1}^{n-1}\alpha(k,a) = \frac{T_n(1) - T_n(a)}{1-a} - \alpha(0,a),$$

we have

$$(a-1)\sum_{k=1}^{n-1}\alpha(k,a) = T_n(a) - T_n(1) - a\alpha(0,a) + \alpha(0,a).$$

But by (4),

$$T_n(a) - a\alpha(0,a) = T_n(0) = \begin{cases} (-1)^{n/2}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$(a-1)\sum_{k=1}^{n-1}\alpha(k,a) = T_n(0) - 1 + \alpha(0,a),$$

which proves the results. □

Proof of Theorem 5. We use induction on n. The case $n=1$ is obvious. Suppose it holds up to $n-1$. Then

$$\begin{aligned}
\frac{T_n(z)-T_n(a)}{z-a} &= \frac{2zT_{n-1}(z)-T_{n-2}(z)-2aT_{n-1}(a)+T_{n-2}(a)}{z-a} \\
&= 2T_{n-1}(z) + 2a\frac{T_{n-1}(z)-T_{n-1}(a)}{z-a} - \frac{T_{n-2}(z)-T_{n-2}(a)}{z-a} \\
&= 2T_{n-1}(z) + 2a{\sum_{k=0}^{n-2}}' 2U_{n-k-2}(a)T_k(z) - {\sum_{k=0}^{n-3}}' 2U_{n-k-3}(a)T_k(z) \\
&= 2T_{n-1}(z) + {\sum_{k=0}^{n-2}}' 2T_k(z)[U_{n-k-1}(a)+U_{n-k-3}(a)] \\
&\quad - {\sum_{k=0}^{n-3}}' 2T_k(z)U_{n-k-3}(a) \\
&= {\sum_{k=0}^{n-1}}' 2T_k(z)U_{n-k-1}(a).
\end{aligned}$$

Using the identity

$$U_k(z)T_j(z) = \frac{1}{2}\left(U_{k+j}(z)+U_{k-j}(z)\right)$$

for all k, j, where $U_{-n}(z) = -U_{n-2}(z)$ and $U_{-1}(a) = 0$, we have

$$\begin{aligned}
QT_n(a,a) &= {\sum_{k=0}^{n-1}}' 2U_{n-k-1}(a)T_k(a) \\
&= \frac{1}{2}\left(U_{n-1}(a)+U_{n-1}(a)\right) + \sum_{k=1}^{n-1}\left(U_{n-1}(a)+U_{n-2k-1}(a)\right)
\end{aligned}$$

$$= nU_{n-1}(a) + \sum_{k=1}^{\left[\frac{n-1}{2}\right]} U_{n-2k-1}(a) + \sum_{k=\left[\frac{n+1}{2}\right]}^{n-1} U_{n-2k-1}(a)$$
$$= nU_{n-1}(a) + \sum_{k=1}^{\left[\frac{n-1}{2}\right]} U_{n-2k-1}(a) - \sum_{k=\left[\frac{n+1}{2}\right]}^{n-1} U_{-1+2k-n}(a).$$

Observe that

$$\sum_{k=1}^{\left[\frac{n-1}{2}\right]} U_{n-2k-1}(a) - \sum_{k=\left[\frac{n+1}{2}\right]}^{n-1} U_{-1+2k-n}(a) = 0,$$

and so $QT_n(a,a) = nU_{n-1}(a)$. Hence

$$nU_{n-1}(a) = {\sum_{k=0}^{n-1}}' 2U_{n-k-1}(a)T_k(a)$$
$$= U_{n-1}(a) + \sum_{k=1}^{n-1} 2U_{n-k-1}(a)T_k(a).$$

This equation holds for any a, and so we can replace a by the variable z. $\square$

Proof of Theorem 6. We note that

$$QT_n'(a,z) = \frac{1}{(z-a)^2}\left[T_n(a) - T_n(z) + nU_{n-1}(z)(z-a)\right]$$
$$= \frac{1}{z-a}\left[-QT_n(a,z) + nU_{n-1}(z)\right].$$

Let $\xi_1, \ldots, \xi_{n-1}$ be the roots of $QT_n(a,z)$. Then for any ξ_j,

$$QT_n'(a,\xi_j) = \frac{nU_{n-1}(\xi_j)}{\xi_j - a},$$

and the discriminant of $QT_n(a,z)$ is

$$\Delta_z(QT_n(a,z))$$
$$= (-1)^{\binom{n-1}{2}} \frac{1}{2^{n-1}} \mathrm{Res}(QT_n(a,z), QT_n'(a,z))$$
$$= (-1)^{\binom{n-1}{2}} \frac{1}{2^{n-1}} 2^{(n-1)(n-2)} \prod_{j=1}^{n-1} QT_n'(a,\xi_j)$$
$$= (-1)^{\binom{n-1}{2}} 2^{(n-1)(n-3)} \prod_{j=1}^{n-1} \frac{nU_{n-1}(\xi_j)}{\xi_j - a}$$
$$= (-1)^{\binom{n-1}{2}} 2^{(n-1)(n-3)} n^{n-1} (-1)^{n-1} 2^{n-1} \frac{1}{QT_n(a,a)} \prod_{j=1}^{n-1} U_{n-1}(\xi_j).$$

By Theorem 5,

$$QT_n(a,a) = nU_{n-1}(a)$$

and hence

$$\Delta_z(QT_n(a,z)) = (-1)^{\frac{1}{2}n(n-1)}2^{(n-1)(n-2)}n^{n-2}\frac{1}{U_{n-1}(a)}\prod_{j=1}^{n-1}U_{n-1}(\xi_j). \tag{5}$$

We now compute $\prod_{j=1}^{n-1}U_{n-1}(\xi_j)$. Let $\alpha_1,\ldots,\alpha_{n-1}$ be the roots of $U_{n-1}(z)$, where $\alpha_j = \cos(j\pi/n)$. Then $T_n(\alpha_j) = \cos(j\pi) = (-1)^j$ and

$$\begin{aligned}
\prod_{j=1}^{n-1}U_{n-1}(\xi_j) &= \frac{1}{2^{(n-1)(n-1)}}\mathrm{Res}(QT_n(a,z),U_{n-1}(z)) \\
&= (-1)^{(n-1)^2}\frac{1}{2^{(n-1)^2}}\mathrm{Res}(U_{n-1}(z),QT_n(a,z)) \\
&= (-1)^{(n-1)^2}\frac{1}{2^{(n-1)^2}}2^{(n-1)(n-1)}\prod_{j=1}^{n-1}QT_n(a,\alpha_j) \\
&= (-1)^{(n-1)^2}\frac{\prod_{j=1}^{n-1}((-1)^j - T_n(a))}{\prod_{j=1}^{n-1}(\alpha_j - a)} \\
&= (-1)^{(n-1)^2}2^{n-1}(-1)^{n-1}\frac{\prod_{j=1}^{n-1}((-1)^j - T_n(a))}{U_{n-1}(a)}.
\end{aligned}$$

Substituting this into (5) yields

$$\Delta_z(QT_n(a,z)) = (-1)^{3n(n-1)/2}2^{(n-1)^2}n^{n-2}\frac{\prod_{j=1}^{n-1}((-1)^j - T_n(a))}{U_{n-1}^2(a)}.$$

Using an identity $1 - T_n^2(z) = (1-z^2)U_{n-1}^2(z)$, we may compute that for an odd integer n,

$$\begin{aligned}
\frac{\prod_{j=1}^{n-1}((-1)^j - T_n(a))}{U_{n-1}^2(a)} &= (-1)^{\frac{n-1}{2}}\frac{\left(1-T_n^2(a)\right)^{\frac{n-1}{2}}}{U_{n-1}^2(a)} \\
&= (-1)^{\frac{n-1}{2}}\frac{((1-a^2)U_{n-1}^2(a))^{\frac{n-1}{2}}}{U_{n-1}^2(a)} \\
&= (-1)^{\frac{n-1}{2}}(1-a^2)^{\frac{n-1}{2}}U_{n-1}^{n-3}(a),
\end{aligned}$$

and for an even integer n,

$$\begin{aligned}
\frac{\prod_{j=1}^{n-1}((-1)^j - T_n(a))}{U_{n-1}^2(a)} &= (-1)^{\frac{n}{2}}\frac{\left(1-T_n^2(a)\right)^{\frac{n}{2}-1}(1+T_n(a))}{U_{n-1}^2(a)} \\
&= (-1)^{\frac{n}{2}}\frac{((1-a^2)U_{n-1}^2(a))^{\frac{n-2}{2}}(1+T_n(a))}{U_{n-1}^2(a)} \\
&= (-1)^{\frac{n}{2}}(1-a^2)^{\frac{n}{2}-1}U_{n-1}^{n-4}(a)(1+T_n(a)).
\end{aligned}$$

These complete the proof. $\square$

For the proof of Theorem 7, we need the following lemma that is easily checked.

Lemma 11. *For a complex number w and $A > 1$, if*

$$w^n + w^{-n} = A^n + A^{-n},$$

then

$$w^n = A^n \quad \textit{or} \quad A^{-n}.$$

Proof of Theorem 7. For $z = 1/2(w + w^{-1})$, we have

$$T_n(z) = \frac{1}{2}(w^n + w^{-n}),$$
$$T_n(a) = \frac{1}{2}\left((a - \sqrt{a^2-1})^n + (a + \sqrt{a^2-1})^n\right).$$

The equation $T_n(z) = T_n(a)$ gives

$$w^n = (a - \sqrt{a^2-1})^n \quad \text{or} \quad (a + \sqrt{a^2-1})^n$$

by Lemma 11. So the roots of $QT_n(a, z)$ satisfy

$$z = \frac{1}{2}(w + w^{-1}),$$

where

$$|w| = a \pm \sqrt{a^2-1}.$$

We may assume that $|w| = a + \sqrt{a^2-1} =: r_a$, i.e.,

$$w = r_a \cos\frac{2j\pi}{n} + ir_a \sin\frac{2j\pi}{n},$$
$$w^{-1} = {r_a}^{-1} \cos\frac{2j\pi}{n} - i{r_a}^{-1} \sin\frac{2j\pi}{n}, \qquad 0 \le j \le n-1,$$

and so

$$z = \frac{1}{2}\left((r_a + {r_a}^{-1})\cos\frac{2j\pi}{n} + i(r_a - {r_a}^{-1})\sin\frac{2j\pi}{n}\right)$$
$$= a\cos\frac{2j\pi}{n} + i\sqrt{a^2-1}\sin\frac{2j\pi}{n}, \qquad 0 \le j \le n-1.$$

For $j = 0$, $z = a$ and so the roots of $QT_n(a, z)$ are

$$a\cos\frac{2j\pi}{n} + i\sqrt{a^2-1}\sin\frac{2j\pi}{n}, \qquad 1 \le j \le n-1,$$

and they lie on the standard ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-1} = 1.$$

By Theorem 6, the roots of $QT_n(a, z)$ are simple. $\square$

Next we prove Theorem 8. To do this, we need the lemma below.

Lemma 12. *Let $n = n_1 d$. Then*

$$\phi_d(z) = \prod_{\substack{1 \le k \le n \\ \gcd(k,n)=n_1}} (z - e(a,b,k,n)) .$$

Proof. By definition,

$$\phi_d(z) = \prod_{\substack{1 \le r \le d \\ \gcd(r,d)=1}} (z - e(a,b,r,d)) .$$

Now

$$\frac{r}{d} = \frac{n_1 r}{n}$$

and

$$\gcd(r,d) = 1 \quad \text{if and only if} \quad \gcd(n_1 r, n) = n_1;$$

also $r \le d$ if and only if $n_1 r \le n$. Writing $k = n_1 r$, we see that $\phi_d(z)$ equates to the stated product. □

Proof of Theorem 8. Every k such that $1 \le k \le n$ has $\gcd(k,n) = n_1$ for some $n_1 \mid n$. When d runs through the divisors of n, so does $n_1 = n/d$. Hence

$$\prod_{d|n} \phi_d(x) = \prod_{k=1}^{n} (x - e(a,b,k,n)).$$

Since

$$QT_n(a,z) = 2^{n-1} \prod_{k=1}^{n-1} (z - e(a, \sqrt{a^2-1}, k, n))$$

by Theorem 7, we have

$$QT_n(a,z) = 2^{n-1} \prod_{\substack{d|n \\ d \ne 1}} \phi_d(z)$$

with $a > 1$ and $b = \sqrt{a^2-1}$. □

Proof of Theorem 9. The proof follows from

$$\begin{aligned} QT_{mn}(a,z) &= \frac{T_{mn}(z) - T_{mn}(a)}{z-a} = \frac{T_m(T_n(z)) - T_m(T_n(a))}{z-a} \\ &= \frac{T_m(T_n(z)) - T_m(T_n(a))}{T_m(z) - T_m(a)} QT_m(a,z) \\ &= \frac{T_n(T_m(z)) - T_n(T_m(a))}{T_m(z) - T_m(a)} QT_m(a,z) \\ &= QT_n(T_m(a), T_m(z))\, QT_m(a,z). \end{aligned}$$

□

For the proof of Theorem 10, we need a result of [7] below. Consider a fixed integer $n \geq 2$. Let $h \mid 2n+2$ and

$$S_h = \{k : \gcd(k, 2n+2) = h,\ 1 \leq k \leq n\}.$$

Then

$$l_h := |S_h| = \frac{\phi\left(\frac{2n+2}{h}\right)}{2}.$$

Now let

$$E_h(z) = 2^{l_h} \prod_{\substack{1 \leq k \leq n \\ \gcd(k,2n+2)=h}} (z - \eta_k),$$

where η_k are the zeros of $U_n(x)$.

Theorem 13 ([7]). *For any integer $n \geq 2$,*

$$U_n(z) = \prod_{\substack{1 \leq h \leq n \\ h \mid 2n+2}} E_h(z),$$

where the $E_h(z)$ are irreducible over $\mathbb{Z}$.

We finally prove Theorem 10.

Proof of Theorem 10. Using an identity

$$T_n(z)^2 - 1 = (z^2 - 1)U_{n-1}(z)^2,$$

for an odd prime p we have

$$\begin{aligned}(T_p(z)+1)(T_p(z)-1) &= (z+1)(z-1) \prod_{\substack{1 \leq h \leq p-1 \\ h \mid 2p}} E_h(x)^2 \\ &= (z+1)(z-1)E_1(z)^2 E_2(z)^2,\end{aligned}$$

where

$$E_1(z) = 2^{l_1} \prod_{\substack{1 \leq k \leq p-1 \\ \gcd(k,2p)=1}} (z - \eta_k) = 2^{l_1} \prod_{\substack{1 \leq k \leq p-1 \\ k \text{ odd}}} (z - \eta_k)$$

and

$$E_2(z) = 2^{l_2} \prod_{\substack{1 \leq k \leq p-1 \\ \gcd(k,2p)=2}} (z - \eta_k) = 2^{l_2} \prod_{\substack{1 \leq k \leq p-1 \\ k \text{ even}}} (z - \eta_k)$$

are irreducible polynomials over $\mathbb{Z}$. The facts $T_p(1) = 1$, $T_p(-1) = -1$, $\deg T_p(z) = p$ complete the proof. $\square$

Acknowledgments. The authors wish to thank to Professor Kenneth B. Stolarsky who let the author know some questions in this paper.

References

[1] S. Barnett, *A note on the Bezoutian matrix*, SIAM J. Appl. Math. **22** (1972), 84–86
[2] J. W. S. Cassels, *Factorization of polynomials in several variables*, in Proc. 15th Scandinavian Congress Oslo, 1968, 1–17, Springer Lecture Notes in Mathematics, **118**, 1970.
[3] A. J. Engler and S. K. Khanduja, *On irreducible factors of the polynomial* $f(x) - g(x)$, Internat. J. Math. **21** (2010), no. 4, 407–418.
[4] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.
[5] P. Lancaster and M. Tismenetsky, *The Theory of Matrices with Applications*, 2nd ed., Academic Press, New York, 1985.
[6] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapman and Hall/CRC, Boca Raton, 2003.
[7] M. O. Rayes, V. Trevisan, and P. S. Wang, *Factorization of Chebyshev Polynomials*, Tech. Rep. ICM-199802-0001, Kent State University, 1998.
[8] T. J. Rivlin, *Chebyshev Polynomials*, From approximation theory to algebra and number theory. Pure and Applied Mathematics (New York). John Wiley and Sons, 1990.
[9] Z. H. Yang and B. F. Cui, *On the Bezoutian matrix for Chebyshev polynomials*, Appl. Math. Comput. **219** (2012), no. 3, 1183–1192.

Seon-Hong Kim
Department of Mathematics
Sookmyung Women's University
Seoul 140-742, Korea
E-mail address: shkim17@sookmyung.ac.kr

Jung Hee Lee
Department of Mathematics
Sookmyung Women's University
Seoul 140-742, Korea
E-mail address: dosk12@daum.net