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A NOTE ON THE VALUE DISTRIBUTION OF $f^2(f')^n$ FOR $n \geq 2$

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Abstract. Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$, and a be a nonzero constant. We give a quantitative estimate of the characteristic function $T(r,f)$ in terms of $N(r, 1/(f^2(f')^n - a))$, which states as following inequality, for positive integers $n \geq 2$,

$$T(r,f) \leq \left(3 + \frac{6}{n-1}\right) N\left(r, \frac{1}{af^2(f')^n - 1}\right) + S(r,f).$$

1. Introduction and results

Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$. In this article, the standard symbols $T(r,f)$, $N(r,f)$, $\bar{N}(r,f)$, $m(r,f)$, $S(r,f)$ are due to Nevanlinna [4]. Particularly, $S(r,f)$ is used to denote an error term $v(r)$ satisfying $v(r) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of finite linear measure. A small function (with respect to f) means a function $\varphi(z)$ meromorphic in $\mathbb{C}$ satisfying $T(r,\varphi) = S(r,f)$. A meromorphic function f is rational if and only if $T(r,f) = O(\log r)$ [3].

The First Fundamental Theorem [3]**.** *Let f be a non-constant meromorphic function. Then*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + \varepsilon(r,a)$$

for each complex number $a \neq \infty$, where $\varepsilon(r,a) = O(1)$ as $r \to \infty$.

The Second Fundamental Theorem [3]**.** *Suppose that $f(z)$ is a meromorphic function that is not identically constant, and let $a_1, \ldots, a_q$ be distinct complex numbers, one of which may be equal to ∞. Then*

$$\sum_{v=1}^{q} m(r, a_v) \leq 2T(r,f) - N_1(r) + S(r,f).$$

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This theorem also implies the following inequality:

$$(q-2)T(r,f) \leq \sum_{v=1}^{q} \bar{N}(r,a_v) + S(r,f). \tag{1}$$

The quantity

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,1/(f-a))}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,1/(f-a))}{T(r,f)}$$

is called the deficiency of f at the point a. Note that $0 \leq \delta(a,f) \leq 1$.

In the last decades, a flock of articles on the topic of value distribution of $f^l\left(f^{(k)}\right)^n$ were sprang up, where l, n, k are positive integers. All of these fruits make this topic surprisingly developed, some results are even very concise and simple, for example, Lahiri and Dewan [7] proved for $l \geq 3$,

$$T(r,f) \leq \frac{1}{l-2} N\left(r, \frac{1}{f^l\left(f^{(k)}\right)^n - a}\right) + S(r,f).$$

Tse and Yang [9], Li and Yang [8], Alotaibi [2] and Wang [10] gave different estimates when $l = 1$, but with the restriction to n or some other additional conditions. Zhang [13] obtained that the inequality $T(r,f) < 6N(r,1/(f^2f' - 1)) + S(r,f)$ holds. Later on, Huang and Gu [5] extended function f^2f' to higher derivatives as follows.

Theorem A ([5, 13])**.** *Let f be a meromorphic and transcendental function in the plane and let k be a positive integer. Then*

$$T(r,f) \leq 6N\left(r, \frac{1}{f^2f^{(k)} - 1}\right) + S(r,f).$$

There are also papers relating another general form of $f^l\left(f'\right)^n$. For $l \geq 3$, a result was given by Lahiri and Dewan [7, Theorem 3.2]. For $l = 1$, Alotaibi [1, 2] discussed this case in separate discussion on $n = 1$ and $n > 1$. For the case $l = 2$, $n = 1$, the estimate is as Theorem A [5, 13] showed. However we noticed that the discuss on the forms of $f^2\left(f'\right)^n$ are less involved all the years. But there are still some related results with some additional conditions. For instance, Yang and Hu [11, Theorem 2] proved that, for positive integers k, l, n and a nonzero finite complex number a, $f^2\left(f'\right)^n - a$ has infinitely many zeros if $\delta(0,f) > 3/(3(l+n)+1)$, where $\delta(0,f)$ is the deficiency of f at the value 0. In fact, the case $n = 1$ is usually discussed as a special case to deal with, since the method to gain an estimate is always different from that for the case $n > 1$. This phenomenon can be discovered from Alotaibi's two papers [1, 2] or Zhang and Huang's papers [5, 13].

We set our aim at removing the additional conditions to get an estimate of $T(r,f)$ with respect to the function $f^2\left(f'\right)^n$ for $n > 2$. We tried on it by improving the method due to Li and Yang [8], in their paper, they gave an estimate of $T(r,f)$ with respect to $f\left(f^{(k)}\right)^n$ for $n > 2$.

Theorem 1.1. *Let f be a transcendental meromorphic function in $\mathbb{C}$, $n(\geq 2)$ be a positive integer and $a(\not\equiv 0)$ be a small function with respect to f. Then*

$$T(r,f) \leq \left(3+\frac{6}{n-1}\right) N\left(r, \frac{1}{af^2\left(f'\right)^n-1}\right)+S(r,f). \tag{2}$$

Theorem 1.2. *Under the conditions of Theorem 1.1, we have*

$$\delta(a, f^2(f')^n) \leq 1-\frac{n-1}{6(n+1)^2}.$$

Since $a(\not\equiv 0)$ is a small function with respect to f, the notation $\delta(a, f^2(f')^n)$ can be understood as $\delta(0, f^2(f')^n - a)$.

2. Lemmas

We need two lemmas to proceed our proofs. The first lemma is a simple result from another paper partly contributed by the author [6].

Lemma 2.1 ([6])**.** *Let f be a transcendental meromorphic function in the plane. Then the differential monomial*

$$\nu = f^l\left(f^{(k)}\right)^n$$

is transcendental, where l, n and k are positive integers.

Lemma 2.2 ([12])**.** *Let f be a meromorphic function satisfying $f^{(k)} \not\equiv 0$. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right)+k\bar{N}(r,f)+S(r,f).$$

3. Proof of Theorem 1.1

Set $\psi = af^2\left(f'\right)^n - 1$. Then

$$\psi' = a\left(f'\right)^n\left(\frac{a'}{a}f^2+2ff'+nf^2\frac{f''}{f'}\right) = a(f')^n F,$$

where $F = -hf^2 + 2f'f$, $h = -\frac{a'}{a} - n\frac{f''}{f'}$.

Except for zeros and poles of $a = a(z)$, any zero of f' can only probably be a simple pole of F, any pole of f is also a pole of F. Then

$$N\left(r, \frac{1}{\psi'}\right) \geq N\left(r, \frac{1}{a\left(f'\right)^n}\right)+N\left(r, \frac{1}{F}\right)-\bar{N}\left(r, \frac{1}{f'}\right),$$

which implies

$$\begin{aligned} N\left(r, \frac{1}{F}\right) &\leq N\left(r, \frac{1}{\psi'}\right)-N\left(r, \frac{1}{a\left(f'\right)^n}\right)+\bar{N}\left(r, \frac{1}{f'}\right) \\ &\leq N\left(r, \frac{1}{\psi'}\right)-nN\left(r, \frac{1}{f'}\right)+\bar{N}\left(r, \frac{1}{f'}\right)+S(r,f). \end{aligned} \tag{3}$$

Since f is nonconstant, then f' can be written by

$$f' = \frac{1}{2}\left(\frac{F}{f} + hf\right).$$

Set $G = F/f = -hf + 2f'$. Assume $a \neq 0, \infty$ at the poles of f, then after some simple calculations, we know that poles of f can not be zeros of G since poles of f are also poles of G, which means poles of f can not be zeros of G. Therefore

$$N\left(r, \frac{1}{G}\right) \leq N\left(r, \frac{1}{F}\right) + S(r, f). \tag{4}$$

Differentiating f' to be

$$f'' = \frac{1}{2}\left((h' + \frac{1}{2}h^2)f + G' + \frac{1}{2}hG\right).$$

We write $f'' = \frac{1}{2}(Df + E)$, where $E = G' + \frac{1}{2}hG$, $D = h' + \frac{1}{2}h^2$, and D is a differential polynomial in h with multiplicities of poles at most 2. Then

$$\left(\frac{f''}{f'}h - D\right)f = E - \frac{f''}{f'}G.$$

Set

$$E - \frac{f''}{f'}G = GG^*,$$

where

$$\begin{aligned} G^* &= \frac{E}{G} - \frac{f''}{f'} \\ &= \frac{G'}{G} + \frac{1}{2}h - \frac{f''}{f'} \\ &= \frac{G'}{G} + \left(\frac{1}{2} + \frac{1}{n}\right)h + \frac{a'}{na}. \end{aligned} \tag{5}$$

Let $N_{1)}(r, 1/f)$ denote the counting function with respect to the simple zeros of f, then any simple zero of f is not a pole of $\left(\frac{f''}{f'}h - D\right)$. Hence

$$\begin{aligned} N_{1)}\left(r, \frac{1}{f}\right) &\leq \bar{N}\left(r, \frac{1}{GG^*}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{G^*}\right) + S(r, f). \end{aligned} \tag{6}$$

Since $h = -a'/a - nf''/f'$, by the definition of G^* in (5), we know that the poles of G^* come from the zeros of G, or the poles of G or h with the multiplicities at most 1. Nevertheless, $G = -hf + 2f'$, poles of G come from poles of f and h. Poles of h come from zeros or poles of f'. Because $m(r, G^*) = S(r, G) = S(r, f)$, then

$$T(r, G^*) \leq N\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{f'}\right) + \bar{N}(r, f) + S(r, f). \tag{7}$$

It follows from inequalities (6) and (7) that

$$\bar{N}_{1)}\left(r, \frac{1}{f}\right) \leq 2N\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{f'}\right) + \bar{N}(r, f) + S(r, f). \tag{8}$$

Since

$$\bar{N}\left(r, \frac{1}{\psi + 1}\right) = \bar{N}\left(r, \frac{1}{af^2(f')^n}\right) \leq N_{1)}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f'}\right) + S(r, f). \tag{9}$$

Note that ψ is transcendental by Lemma 2.1, then it follows from Lemma 2.2 that

$$N\left(r, \frac{1}{\psi'}\right) \leq N\left(r, \frac{1}{\psi}\right) + \bar{N}(r, f) + S(r, f). \tag{10}$$

By inequalities (1) and (8), (9), (3), (4), (10), we have for $n \geq 2$,

$$\begin{aligned} T(r, \psi) &\leq \bar{N}\left(r, \frac{1}{\psi}\right) + \bar{N}\left(r, \frac{1}{\psi + 1}\right) + \bar{N}(r, \psi) + S(r, \psi) \\ &\leq \bar{N}\left(r, \frac{1}{\psi}\right) + 2N\left(r, \frac{1}{G}\right) + 2\bar{N}\left(r, \frac{1}{f'}\right) + 2\bar{N}(r, f) + S(r, f) \\ &\leq 2N\left(r, \frac{1}{\psi'}\right) + \bar{N}\left(r, \frac{1}{\psi}\right) + 2(2-n)N\left(r, \frac{1}{f'}\right) + 2\bar{N}(r, f) + S(r, f) \\ &\leq 3N\left(r, \frac{1}{\psi}\right) + 4\bar{N}(r, f) + S(r, f). \end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned} T(r, f^2(f')^n) \geq N(r, f^2(f')^n) &= 2N(r, f) + nN(r, f') \\ &= (2+n)N(r, f) + n\bar{N}(r, f) \\ &\geq 2(n+1)\bar{N}(r, f). \end{aligned}$$

Then it follows from the inequality (11) that

$$T(r, f^2(f')^n) \leq \left(3 + \frac{6}{n-1}\right) N\left(r, \frac{1}{\psi}\right) + S(r, f). \tag{12}$$

Since $m\left(r, f^{(k)}/f\right) = S(r, f)$, then

$$\begin{aligned} m(r, (f')^n) &= \frac{n}{n+2} m(r, (f')^{n+2}) \\ &\leq \frac{n}{n+2} m\left(r, \left(\frac{f'}{f}\right)^2\right) + \frac{n}{n+2} m(r, f^2(f')^n) + O(1) \\ &\leq \frac{n}{n+2} m(r, f^2(f')^n) + S(r, f). \end{aligned} \tag{13}$$

In addition,

$$
\begin{aligned}
N(r,(f')^n) &= N(r,f^2(f')^n) - 2N(r,f) \\
&\le N(r,f^2(f')^n) - 2\bar{N}(r,f). \qquad (14)
\end{aligned}
$$

By inequalities (13) and (14)

$$
\begin{aligned}
T(r,f) = \frac{1}{2}T(r,f^2) &\le T(r,(f')^n) + T(r,f^2(f')^n) \\
&\le T(r,f^2(f')^n).
\end{aligned}
$$

Hence, it follows from inequality (12) that

$$T(r,f) \le \left(3+\frac{6}{n-1}\right) N\left(r,\frac{1}{\psi}\right) + S(r,f).$$

4. Proof of Theorem 1.2

Set $\varphi = \frac{1}{a}f^2(f')^n - 1$. Since

$$T(r,f) \le \left(3+\frac{6}{n-1}\right) N\left(r,\frac{1}{a\varphi}\right) + S(r,f).$$

$$
\begin{aligned}
T(r,a\varphi) &\le (2n+2)T(r,f) + S(r,f) \\
&\le \frac{6(n+1)^2}{n-1} N\left(r,\frac{1}{a\varphi}\right) + S(r,f). \qquad (15)
\end{aligned}
$$

We have

$$N\left(r,\frac{1}{a\varphi}\right) \ge \frac{n-1}{6(n+1)^2}T(r,a\varphi) - S(r,f).$$

Since

$$f^2 = (a\varphi + a)(f')^{-n},$$

by the First Fundamental Theorem,

$$
\begin{aligned}
T(r,f) = \frac{1}{2}T(r,f^2) &\le T\left(r,(f')^n\right) + T(r,a\varphi) + S(r,f) \\
&\le O\left(T\left(r,a\varphi\right)\right).
\end{aligned}
$$

Therefore,

$$\liminf_{r\to\infty} \frac{S(r,f)}{T(r,a\varphi)} = \liminf_{r\to\infty} \frac{S(r,f)}{T(r,f)}\frac{T(r,f)}{T(r,a\varphi)} = 0.$$

By the definition of deficiency, we deduce that

$$
\begin{aligned}
\delta\left(a,f^2(f')^n\right) = \delta(0,a\varphi) &= 1 - \limsup_{r\to\infty} \frac{N\left(r,\frac{1}{a\varphi}\right)}{T(r,a\varphi)} \\
&\le 1 - \limsup_{r\to\infty} \frac{\frac{n-1}{6(n+1)^2}T(r,a\varphi) - S(r,f)}{T(r,a\varphi)}
\end{aligned}
$$

$$\leq 1 - \frac{n-1}{6(n+1)^2} + \liminf_{r\to\infty} \frac{S(r,f)}{T(r,a\varphi)}$$
$$= 1 - \frac{n-1}{6(n+1)^2}.$$

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