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ENTROPY-BASED GOODNESS OF FIT TEST FOR A COMPOSITE HYPOTHESIS

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Abstract. In this paper, we consider the entropy-based goodness of fit test (Vasicek's test) for a composite hypothesis. The test measures the discrepancy between the nonparametric entropy estimate and the parametric entropy estimate obtained from an assumed parametric family of distributions. It is shown that the proposed test is asymptotically normal under regularity conditions, but is affected by parameter estimates. As a remedy, a bootstrap version of Vasicek's test is proposed. Simulation results are provided for illustration.

1. Introduction

For decades, the goodness of fit (gof) test for statistical models has been a core issue in statistical analysis. The gof test has a long history and various methodologies have been developed by many researchers. See, for instance, D'Agostino and Stephens [3]. The entropy based gof test, the entropy based gof test has been very popular among practitioners in diverse fields. In particular, the entropy test of Vasicek [11] has been studied extensively in the literature. His approach involves a nonparametric estimate (m-spacing estimate) of Shannon's entropy. Thus far, a number of articles exist on the distributional properties of Vasicek's test: see, for instance, Kashimov [5], van Es [10], Beirant et al. [2], Song [8], and the references therein. Among them, Song [8] rigorously verifies that Vasicek's estimator is consistent and asymptotically normal under certain regularity conditions. This result is easily applicable to simple vs. simple gof tests. However, attention has not yet been paid to composite hypothesis tests. In the literature, it is well known that gof tests are often affected by parameter estimation, and their limiting distributions rely on the choice of parameter estimators. This phenomenon is prominent in the empirical process of the gof tests, as seen in Durbin [4], and often leads practitioners to a burdensome situation. This difficulty may be overcome by using the transformation method proposed by Khmaladze [6], Bai [1] and Lee [7], which, however, is not

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easy to implement owing to a time consuming computational process. In this study, we focus on the entropy test to measure the discrepancy between the nonparametric entropy estimate (Vasicek's estimate) and the parametric entropy estimate obtained from the assumed parametric family of distributions. Although simple and natural, to our knowledge, no literature has explicitly considered this test. It may be because the proposed test is severely affected by parameter estimation, and thus, is not as useful in actual implementation. Conventionally, gof methods depending on asymptotic theories do not perform well for small samples, and particularly, in the implementation of Vasicek's test, the choice of spacing parameter m escalates this difficulty. As a remedy, it is natural to adopt the parametric bootstrap approach and construct a bootstrap version of the tests. Thus, we propose a bootstrap version of Vasicek's test for a composite hypothesis and investigate its finite sample behavior through a simulation study. The organization of this paper is as follows. In Section 2, we introduce Vasicek's test and show that a certain asymptotic expansion form holds for Vasicek's test with plugged-in estimators and leads to a result having asymptotic normality. Further, we introduce a bootstrap version of Vasicek's test and demonstrate that its usage is justifiable. In Section 3, we conduct a simulation study to evaluate the proposed bootstrap test and compare its performance with other existing tests. Concluding remarks are provided in Section 4.

2. Main result

Given an i.i.d. random sample $X_1, \ldots, X_n$ with common distribution F, Vasicek (1992) proposed as an estimate of $H(F) = -\int \log f(x) f(x) dx$ the following:

$$V_{mn} = \frac{1}{n} \sum_{i=1}^{n} \log \frac{n}{2m} (X_{(i+m)} - X_{(i-m)}), \tag{2.1}$$

where $f = F^{'}$, $X_{(i)}$ denotes the ordered r.v.s., and $X_{(i)} = X_{(1)}$ for $i < 1$ and $X_{(i)} = X_{(n)}$ for $i > n$. Later, Song (2000) showed that if the following conditions are fulfilled

(R1) $E(\log f(X_1))^2 < \infty$;

(R2) $\sup_{\phi(F) < x < \psi(F)} F(x)(1 - F(x)) \frac{|f^{'}(x)|}{f^2(x)} < \infty$;

(R3) $m = m_n$ satisfies $\log n/m = o(1)$ and $m(\log n)^{2/3}/n^{1/3} = o(1)$ as $n \to \infty$,

where $\phi(F) = \sup\{x : F(x) = 0\}$ and $\psi(F) = \inf\{x : F(x) = 1\}$, then

$$n^{1/2}(V_{mn} - H(F) + \log 2m + \gamma - R_{2m-1}) \xrightarrow{d} \mathcal{N}(0, \sigma^2(F)), \tag{2.2}$$

where $R_n = \sum_{i=1}^{n} 1/i$, $\gamma = \lim_{n\to\infty}(R_n - \log n)$, and $\sigma^2(F) = Var(\log f(X_1))$.

The result in (2.2) is applicable to a goodness of fit test for a composite hypothesis such as

$$\mathcal{H}_0 : X_i \sim F = F_{\theta_0} \quad \text{vs.} \quad \mathcal{H}_1 : X_i \sim F \notin \{F_\theta\},$$

where $\{F_\theta\}$ is a parameter family indexed with $\theta \in \Theta$, a subset of R^d, $d \geq 1$, and θ_0 is an interior point of Θ. The result in (2.2) indicates that under (R1) and (R2), with F replaced by F_{θ_0}, and (R3),

$$n^{1/2}(V_{mn} - H(F_{\theta_0}) + \log 2m + \gamma - R_{2m-1}) \xrightarrow{d} \mathcal{N}(0, \sigma^2(F_{\theta_0})), \tag{2.3}$$

where $f_\theta = F_\theta^{'}$ is continuous in θ, $H(F_\theta) = -\int \log(f_\theta(x)) f_\theta(x) dx$ and $\sigma^2(F_\theta) = Var_\theta(\log f_\theta(X_1))$ for all θ. Here, Var_θ and E_θ denote the variance and expectation under F_θ, respectively.

The argument in (2.3) suggests a test based on the difference between H_n and $H(\hat{\theta}_n)$, where $\hat{\theta}_n$ is a consistent estimator of θ_0, since $H(\theta_0)$ is unknown. As usual, we assume $n^{1/2}(\hat{\theta}_n - \theta_0) = O_P(1)$ under $\mathcal{H}_0$. In view of the proof of Theorem 1 of Song [8], it can be seen that

$$V_{mn} = H_n - \log 2m - \gamma + R_{2m-1} + o_P(n^{-1/2}), \tag{2.4}$$

where $H_n = -\frac{1}{n}\sum_{i=1}^n \log f_{\theta_0}(X_i)$. Hence,

$$V_{mn} - H(F_{\hat{\theta}_n}) + \log 2m + \gamma - R_{2m-1} = H_n - H(F_{\hat{\theta}_n}) + o_P(n^{-1/2}). \tag{2.5}$$

Then, we have the result addressed below.

Theorem 1. *Suppose that*

(R1)$'$ $E_{\theta_0}(\log f(X_1))^2 < \infty$.

(R2)$'$ $\sup_{\phi(F_{\theta_0}) < x < \psi(F_{\theta_0})} F_{\theta_0}(x)(1 - F_{\theta_0}(x)) \frac{|f_{\theta_0}^{'}(x)|}{f_{\theta_0}^2(x)} < \infty$.

Further, assume (R3),

(R4) $E_{\theta_0} \frac{\partial}{\partial\theta} \log f_{\theta_0}(X_1) = 0$ *and* $u(\theta) = \frac{\partial}{\partial\theta} H(\theta) = -\int \log f_\theta(x) \frac{\partial}{\partial\theta} f_\theta(x) dx$ *is continuous in* θ.

(R5) $\hat{\theta}_n - \theta_0 = n^{-1}\sum_{i=1}^n l_{\theta_0}(X_i) + o_P(n^{-1/2})$, *where* l_θ *is a* $d \times 1$ *vector function with* $E_{\theta_0} l_{\theta_0}(X_1) = 0$ *and* $Var_{\theta_0}||l_{\theta_0}(X_1)|| < \infty$, *where* $||\cdot||$ *is a Euclidean norm.*

Then, under $\mathcal{H}_0$,

$$T_n := n^{1/2}(V_{mn} - H(F_{\hat{\theta}_n}) + \log 2m + \gamma - R_{2m-1}) \xrightarrow{d} N(0, \tau^2)$$

with $\tau^2 = Var_{\theta_0}(\log f_{\theta_0}(X_1) + H(F_{\theta_0}) + l_{\theta_0}(X_1)^T u(\theta_0))$.

Proof. By the mean value theorem, $H(F_{\hat{\theta}_n}) - H(F_{\theta_0}) = (\hat{\theta}_n - \theta_0)^T u(\theta_0) + \rho_n$, where $||\rho_n|| \leq \eta_n \sup_{||\theta - \theta_0|| \leq \eta_n} ||u(\theta) - u(\theta_0)|| = o_P(n^{-1/2})$ with $\eta_n = ||\hat{\theta}_n - \theta_0||$ owing to (R4) and (R5). Combining this and (2.5), the theorem is validated. □

Theorem 1 indicates that the parameter estimation affects the null limiting distribution, and further, there is a serious difficulty in estimating τ^2 when the explicit form of l_θ is unknown.

Remark. One may consider another test based on $\hat{H}_n = -\frac{1}{n}\sum_{i=1}^n \log f_{\hat{\theta}_n}(X_i)$. Put $\Delta_n = n^{1/2}(\hat{H}_n - H_n)$ and suppose that the following holds:

(R6) $\frac{\partial f_\theta}{\partial \theta}$ is continuous in θ and for some $\epsilon > 0$,

$$\sum_{i=1}^d \sup_{|\theta_i - \theta_{0i}| \le \epsilon} \left| \frac{\frac{\partial f_\theta(x)}{\partial \theta_i}}{f_\theta(x)} \right| \le g(x), \quad E_{\theta_0} g(X_1) < \infty,$$

where θ_i denotes the i-th entry of θ. By the mean value theorem, we can express

$$\Delta_n = \sqrt{n}(\theta_0 - \hat{\theta}_n)^T \frac{1}{n}\sum_{i=1}^n \frac{\frac{\partial f_{\theta_n'}}{\partial \theta}}{f_{\theta_n'}},$$

where $\theta_n^{'}$ is between θ_0 and $\hat{\theta}_n$. Then, using (R6), we can easily see that for all $i = 1, \ldots, d$,

$$\frac{1}{n}\sum_{i=1}^n \frac{\frac{\partial f_{\theta_n'}}{\partial \theta_i}}{f_{\theta_n'}} - \frac{1}{n}\sum_{i=1}^n \frac{\frac{\partial f_{\theta_0}}{\partial \theta_i}}{f_{\theta_0}} = o_P(1).$$

Together with (R4), this entails (2.3), and thus, $\Delta_n = o_P(1)$. Then, in view of (2.4), (2.5), and Theorem 1, we have that under (R1)$^{'}$, (R2)$^{'}$, and (R3)–(R6),

$$T_n^{'} := n^{1/2}(\hat{H}_n - H(F_{\hat{\theta}_n})) \xrightarrow{d} N(0, \tau^2).$$

Meanwhile, Song's approach can be also extended to a rowwise independent double array of random variables, say, $X_{n1}, \ldots, X_{nn}$. Suppose that $X_{ni}, i = 1, \ldots, n$ follows from F_{θ_n} where $\{\theta_n\}$ is a sequence in Θ that converges to an interior point $\theta_0 \in \Theta$ as n tends to ∞. In this case, we can consider the estimator

$$DV_{mn} = \frac{1}{n}\sum_{i=1}^n \log \frac{n}{2m}(X_{n,(i+m)} - X_{n,(i-m)}),$$

where $X_{n,(i)}$ are analogously defined as $X_{(i)}$.

In what follows, we assume

(R1)$^{''}$ For some $\epsilon > 0$,

$$\int \sup_{||\theta - \theta_0|| \le \epsilon} (1 + (\log f_\theta(x))^2) f_\theta(x) dx < \infty,$$

(R2)$^{''}$ $\phi(F_\theta) = \phi$ and $\psi(F_\theta) = \psi$ for all θ. Further, for some $\epsilon > 0$,

$$\sup_{\phi < x < \psi} \sup_{||\theta - \theta_0|| \le \epsilon} F_\theta(x)(1 - F_\theta(x)) \frac{|f_\theta^{'}(x)|}{f_\theta^2(x)} < \infty.$$

Then, if we put

$$DH_n = -\frac{1}{n}\sum_{i=1}^{n} \log f_{\theta_n}(X_{ni}),$$

following essentially the same lines as in the proof of Theorem 1 of Song [8], one can check that provided (R3) holds,

$$DV_{mn} + \log 2m + \gamma - R_{2m-1} = DH_n + o_P(n^{-1/2}).$$

Then, if the following condition is satisfied:

(R4)$'$ $E_\theta \frac{\partial}{\partial\theta} \log f_\theta(X_1) = 0$ for all θ and $u(\theta)$ in (R4) is continuous in θ,

and if the estimator $\hat{\theta}_{nn}$ of θ_n based on $X_{ni}, i = 1, \ldots, n$ satisfies:

(R5)$'$ $\hat{\theta}_{nn} - \theta_n = n^{-1}\sum_{i=1}^{n} l_{\theta_n}(X_{ni}) + o_P(n^{-1/2})$, where l_θ is continuous in θ, $E_\theta l_\theta(X_1) = 0$ for all θ, and $\int \sup_{||\theta-\theta_0||\leq\epsilon} ||l_\theta(x)||^2 f_\theta(x)dx < \infty$ for some $\epsilon > 0$,

using the dominated convergence theorem and Lindeberg's central limit theorem, we can have

$$\sqrt{n}(DH_n - H(F_{\hat{\theta}_{nn}})) \xrightarrow{d} \mathcal{N}(0, \tau^2),$$

and subsequently, we have the following.

Theorem 2. *Under* (R1)$''$, (R2)$''$, (R3), (R4)$'$ *and* (R5)$'$,

$$\sqrt{n}(DV_{mn} - H(F_{\hat{\theta}_{nn}}) + \log 2m + \gamma - R_{2m-1}) \xrightarrow{d} \mathcal{N}(0, \tau^2). \tag{2.6}$$

The argument in (2.6) suggests that a bootstrap test can be designed for the composite hypothesis test in Theorem 1. Here, we use the parametric bootstrap method as in Stute et al. [9]. Given sample $X_1, \ldots, X_n$, we estimate θ_0 by $\hat{\theta}_n$ and generate the bootstrap sample from $F_{\hat{\theta}_n}$, say, $X_1^*, \ldots, X_n^*$, and put

$$V_{mn}^* = \frac{1}{n}\sum_{i=1}^{n} \log \frac{n}{2m}(X_{(i+m)}^* - X_{(i-m)}^*).$$

Then, if $\hat{\theta}_n \to \theta_0$ a.s., $\hat{\theta}_n^*$ is the estimator based on the bootstrap sample, and (R5)$'$ holds for any sequence $\{\theta_n\}$ that converges to θ_0: in other words, l_θ in (R5)$'$ satisfies

(R5)$''$ $\hat{\theta}_n^* - \hat{\theta}_n = n^{-1}\sum_{i=1}^{n} l_{\hat{\theta}_n}(X_i^*) + o_P(n^{-1/2})$,

we can conclude that under (R1)$''$, (R2)$''$, (R3) and (R4)$'$,

$$T_n^* := \sqrt{n}(V_{mn}^* - H(F_{\hat{\theta}_n^*}) + \log 2m + \gamma - R_{2m-1}) \xrightarrow{d} \mathcal{N}(0, \tau^2) \quad \text{a.s.}. \tag{2.7}$$

By obtaining $|T_n^*|$ for the bootstrapped sample B times, say, $|T_n^{*b}|$, $b = 1, \ldots, B$, we can calculate sample quantiles, say $c = c(n, \alpha)$, given any significance level α. Then, we reject $\mathcal{H}_0$ if $|T_n| \geq c$. This bootstrap method provides a more stable test, unaffected by the choice of spacing parameter m, especially in handling small samples, as seen in the simulation study below, where we focus on the finite sample behavior of T_n^* and investigate its empirical sizes and powers.

3. Simulation

In this simulation study, we evaluate the bootstrap Vasicek's test T_n^* (T) and compare its performance with the Kolmogorov-Smirnov (KS), Cramer-von Mises (CV), and Anderson-Darling (AD) tests. To be fair, we also employ the bootstrap versions of KS, CV, and AD tests.

For this, we consider

Group1: Laplace$(0,1)$, Normal$(0,1)$, and Student's $t(3)$ distributions and

Group2: Gamma, Inverse-Gaussian (IG), and Weibull distributions with skewness equal to 1.414. The shape parameter of the Gamma, IG, and Weibull are 2, 4.5, and 1.259, respectively, and the scale parameter of the distribution is equal to 1 in all cases.

The figures in Tables 1-6 (Tables 1-3 for Group 1 and Tables 4-6 for Group 2) exhibit the proportion of the number of rejections of the null hypothesis out of 500 repetitions with $B = 500$. Here, we use $(n = 20, m = 4, 5, 6, 7)$, $(n = 50, m = 6, 7, 8, 9)$, $(n = 100, m = 8, 9, 10, 11)$, nominal level 0.05, and repetition number 1,000. In all the cases, the sizes turn out to be close to the nominal level regardless of the choice of n, m and the power tends to increase as the sample size increases. In particular, it is shown that none of the tests outperform the others perfectly: our test significantly outperforms other tests in the cases of Student's t vs. Normal and Weibull vs. Inverse-Gaussian. As seen in the tables, the choice of m can affect the performance of the test in power. Thus, it may be an important issue to choose an optimal m that produces the best powers, but it is difficult to set up a rule theoretically to obtain such m. Our past experience suggests that one may choose $m = c_1 + c_2 n^{1/3}$ for some suitable $c_1, c_2 > 0$, but this cannot be directly applied to all situations. In practice, for a given gof test, one could obtain an optimal m empirically through a simulation. Overall, our findings show that the bootstrap Vasicek's test performs adequately and is compatible with other existing tests.

4. Concluding remarks

To perform a gof test for a composite hypothesis, we suggested to use of a bootstrap Vasicek's test. A simulation study indicates that the bootstrap test performs adequately in terms of size and power. The comparison study with other tests such as the KS, CV, and AD tests indicates that none of these tests outperform the others completely. Vasicek's test appears to outperform the others in some situations and is proven to be a useful tool to perform a gof test. Manifestly, it would be of great interest to extend our method to dependent data sets, especially the residuals from time series models such as autoregressive and GARCH models. Thus, we leave this as a task of our future study.

Table 1. Laplace null model: sizes and powers

	distribution	n	m	T	KS	CV	AD
Size	Laplace(0,1)	20	4	0.064	0.052	0.052	0.048
		20	5	0.050	0.052	0.056	0.054
		20	6	0.058	0.050	0.052	0.050
		20	7	0.064	0.056	0.058	0.054
		50	6	0.048	0.048	0.072	0.062
		50	7	0.046	0.048	0.046	0.044
		50	8	0.036	0.056	0.068	0.064
		50	9	0.058	0.036	0.038	0.042
		100	8	0.046	0.058	0.054	0.046
		100	9	0.056	0.056	0.038	0.046
		100	10	0.050	0.060	0.040	0.048
		100	11	0.042	0.052	0.050	0.052
Power	Normal(0,1)	20	4	0.195	0.076	0.084	0.078
		20	5	0.222	0.096	0.076	0.068
		20	6	0.252	0.096	0.094	0.082
		20	7	0.240	0.088	0.094	0.082
		50	6	0.442	0.180	0.170	0.164
		50	7	0.482	0.168	0.152	0.138
		50	8	0.492	0.180	0.158	0.142
		50	9	0.568	0.212	0.192	0.152
		100	8	0.690	0.364	0.404	0.342
		100	9	0.692	0.386	0.420	0.370
		100	10	0.696	0.420	0.420	0.378
		100	11	0.718	0.406	0.396	0.348
Power	$t(3)$	20	4	0.070	0.058	0.058	0.072
		20	5	0.080	0.064	0.072	0.092
		20	6	0.048	0.060	0.058	0.066
		20	7	0.072	0.078	0.062	0.066
		50	6	0.078	0.068	0.064	0.076
		50	7	0.060	0.074	0.058	0.068
		50	8	0.082	0.098	0.100	0.011
		50	9	0.056	0.086	0.076	0.009
		100	8	0.046	0.104	0.092	0.110
		100	9	0.064	0.098	0.088	0.100
		100	10	0.054	0.078	0.070	0.074
		100	11	0.088	0.078	0.074	0.088

Table 2. Normal null model: sizes and powers

	distribution	n	m	T	KS	CV	AD
Size	Normal(0,1)	20	4	0.044	0.062	0.060	0.058
		20	5	0.054	0.064	0.050	0.044
		20	6	0.068	0.048	0.060	0.060
		20	7	0.050	0.070	0.052	0.056
		50	6	0.050	0.068	0.058	0.064
		50	7	0.052	0.050	0.046	0.044
		50	8	0.058	0.056	0.044	0.044
		50	9	0.056	0.042	0.054	0.062
		100	8	0.032	0.066	0.058	0.050
		100	9	0.052	0.058	0.054	0.056
		100	10	0.066	0.036	0.046	0.042
		100	11	0.034	0.046	0.050	0.040
Power	$t(3)$	20	4	0.136	0.226	0.290	0.320
		20	5	0.104	0.252	0.302	0.324
		20	6	0.116	0.304	0.326	0.366
		20	7	0.092	0.246	0.302	0.332
		50	6	0.276	0.470	0.550	0.570
		50	7	0.248	0.524	0.612	0.626
		50	8	0.158	0.444	0.514	0.548
		50	9	0.120	0.474	0.586	0.622
		100	8	0.528	0.704	0.786	0.824
		100	9	0.514	0.748	0.844	0.864
		100	10	0.404	0.722	0.808	0.844
		100	11	0.348	0.712	0.832	0.850
Power	Laplace(0,1)	20	4	0.062	0.232	0.268	0.274
		20	5	0.048	0.230	0.248	0.274
		20	6	0.048	0.244	0.296	0.292
		20	7	0.034	0.208	0.230	0.236
		50	6	0.210	0.458	0.580	0.576
		50	7	0.128	0.456	0.558	0.556
		50	8	0.110	0.434	0.536	0.548
		50	9	0.068	0.400	0.522	0.528
		100	8	0.428	0.690	0.812	0.806
		100	9	0.380	0.716	0.824	0.840
		100	10	0.318	0.742	0.836	0.836
		100	11	0.258	0.704	0.814	0.816

Table 3. Student's t null model: sizes and powers

	distribution	n	m	T	KS	CV	AD
Size	$t(3)$	20	4	0.066	0.060	0.050	0.056
		20	5	0.060	0.072	0.054	0.050
		20	6	0.056	0.058	0.052	0.056
		20	7	0.054	0.048	0.048	0.053
		50	6	0.048	0.048	0.040	0.040
		50	7	0.042	0.028	0.042	0.046
		50	8	0.048	0.036	0.042	0.042
		50	9	0.044	0.052	0.046	0.054
		100	8	0.028	0.060	0.056	0.050
		100	9	0.052	0.044	0.042	0.040
		100	10	0.058	0.058	0.052	0.050
		100	11	0.074	0.044	0.050	0.054
Power	Laplace(0,1)	20	4	0.068	0.044	0.038	0.036
		20	5	0.086	0.044	0.046	0.042
		20	6	0.066	0.052	0.054	0.056
		20	7	0.072	0.058	0.060	0.056
		50	6	0.080	0.058	0.048	0.046
		50	7	0.072	0.052	0.056	0.058
		50	8	0.082	0.058	0.044	0.028
		50	9	0.074	0.042	0.052	0.042
		100	8	0.104	0.056	0.048	0.044
		100	9	0.116	0.078	0.060	0.052
		100	10	0.070	0.064	0.048	0.052
		100	11	0.068	0.058	0.038	0.040
Power	Normal(0,1)	20	4	0.312	0.060	0.054	0.048
		20	5	0.358	0.060	0.052	0.040
		20	6	0.356	0.052	0.042	0.030
		20	7	0.328	0.044	0.030	0.026
		50	6	0.804	0.052	0.048	0.050
		50	7	0.854	0.044	0.048	0.046
		50	8	0.816	0.048	0.050	0.050
		50	9	0.810	0.030	0.042	0.013
		100	8	0.990	0.044	0.060	0.130
		100	9	0.996	0.062	0.096	0.158
		100	10	0.992	0.064	0.084	0.144
		100	11	0.984	0.068	0.088	0.136

Table 4. Gamma null model: sizes and powers

	distribution	n	m	T	KS	CV	AD
Size	Gamma k=2	20	4	0.060	0.062	0.058	0.050
		20	5	0.048	0.054	0.052	0.046
		20	6	0.044	0.056	0.048	0.048
		20	7	0.046	0.072	0.078	0.074
		50	6	0.040	0.046	0.050	0.056
		50	7	0.034	0.034	0.038	0.038
		50	8	0.044	0.048	0.046	0.050
		50	9	0.046	0.036	0.038	0.034
		100	8	0.056	0.050	0.05	0.056
		100	9	0.058	0.058	0.052	0.054
		100	10	0.048	0.048	0.056	0.054
		100	11	0.056	0.056	0.054	0.060
Power	Inverse Gaussian λ=4.5	20	4	0.068	0.078	0.080	0.084
		20	5	0.072	0.078	0.096	0.098
		20	6	0.044	0.110	0.100	0.100
		20	7	0.034	0.082	0.080	0.082
		50	6	0.110	0.124	0.132	0.152
		50	7	0.084	0.114	0.152	0.168
		50	8	0.098	0.128	0.152	0.174
		50	9	0.084	0.148	0.158	0.166
		100	8	0.164	0.198	0.240	0.278
		100	9	0.150	0.224	0.268	0.298
		100	10	0.126	0.248	0.272	0.308
		100	11	0.128	0.208	0.248	0.264
Power	Weibull k=1.259	20	4	0.078	0.054	0.064	0.064
		20	5	0.066	0.070	0.072	0.070
		20	6	0.034	0.066	0.050	0.048
		20	7	0.054	0.062	0.082	0.084
		50	6	0.056	0.078	0.076	0.072
		50	7	0.072	0.064	0.062	0.066
		50	8	0.060	0.062	0.056	0.062
		50	9	0.070	0.058	0.066	0.068
		100	8	0.074	0.080	0.090	0.088
		100	9	0.066	0.070	0.072	0.070
		100	10	0.088	0.096	0.094	0.084
		100	11	0.086	0.074	0.092	0.088

Table 5. Inverse Gaussian null model: sizes and powers

	distribution	n	m	T	KS	CV	AD
Size	Inverse Gaussian λ=4.5	20	4	0.086	0.060	0.074	0.076
		20	5	0.040	0.052	0.048	0.054
		20	6	0.042	0.050	0.050	0.044
		20	7	0.062	0.040	0.044	0.040
		50	6	0.062	0.064	0.056	0.048
		50	7	0.050	0.066	0.074	0.068
		50	8	0.056	0.048	0.052	0.052
		50	9	0.056	0.038	0.044	0.042
		100	8	0.054	0.052	0.060	0.054
		100	9	0.052	0.062	0.048	0.050
		100	10	0.048	0.052	0.048	0.058
		100	11	0.058	0.052	0.052	0.046
Power	Gamma k=2	20	4	0.162	0.266	0.324	0.322
		20	5	0.150	0.270	0.314	0.324
		20	6	0.106	0.306	0.324	0.328
		20	7	0.076	0.280	0.336	0.348
		50	6	0.416	0.562	0.644	0.654
		50	7	0.346	0.562	0.632	0.638
		50	8	0.306	0.518	0.600	0.618
		50	9	0.296	0.586	0.660	0.678
		100	8	0.680	0.814	0.868	0.874
		100	9	0.628	0.824	0.884	0.898
		100	10	0.608	0.840	0.880	0.888
		100	11	0.574	0.832	0.864	0.870
Power	Weibull k=1.259	20	4	0.342	0.476	0.516	0.524
		20	5	0.286	0.468	0.514	0.532
		20	6	0.256	0.504	0.540	0.560
		20	7	0.220	0.522	0.550	0.552
		50	6	0.730	0.834	0.880	0.890
		50	7	0.700	0.826	0.872	0.876
		50	8	0.632	0.818	0.854	0.870
		50	9	0.616	0.816	0.872	0.870
		100	8	0.978	0.994	0.998	0.998
		100	9	0.944	0.984	0.994	0.994
		100	10	0.938	0.984	0.998	0.998
		100	11	0.924	0.968	0.994	0.994

Table 6. Weibull null model: sizes and powers

	distribution	n	m	T	KS	CV	AD
Size	Weibull k=1.259	20	4	0.058	0.044	0.032	0.042
		20	5	0.034	0.060	0.052	0.046
		20	6	0.046	0.050	0.062	0.066
		20	7	0.036	0.054	0.050	0.056
		50	6	0.040	0.044	0.040	0.058
		50	7	0.058	0.048	0.052	0.062
		50	8	0.044	0.054	0.044	0.056
		50	9	0.042	0.050	0.052	0.052
		100	8	0.060	0.050	0.046	0.050
		100	9	0.054	0.074	0.058	0.062
		100	10	0.050	0.078	0.074	0.080
		100	11	0.072	0.048	0.044	0.052
Power	Inverse Gaussian λ=4.5	20	4	0.228	0.158	0.190	0.190
		20	5	0.248	0.168	0.210	0.214
		20	6	0.248	0.144	0.192	0.198
		20	7	0.222	0.174	0.212	0.208
		50	6	0.668	0.378	0.524	0.582
		50	7	0.660	0.412	0.518	0.564
		50	8	0.638	0.330	0.486	0.560
		50	9	0.602	0.372	0.470	0.534
		100	8	0.942	0.688	0.840	0.890
		100	9	0.938	0.664	0.834	0.904
		100	10	0.920	0.692	0.828	0.898
		100	11	0.940	0.704	0.852	0.914
Power	Gamma k=2	20	4	0.068	0.070	0.068	0.066
		20	5	0.060	0.064	0.058	0.070
		20	6	0.056	0.058	0.076	0.084
		20	7	0.044	0.080	0.074	0.072
		50	6	0.092	0.074	0.092	0.084
		50	7	0.086	0.082	0.086	0.078
		50	8	0.090	0.082	0.086	0.094
		50	9	0.084	0.082	0.088	0.092
		100	8	0.140	0.094	0.124	0.132
		100	9	0.118	0.108	0.136	0.130
		100	10	0.114	0.084	0.108	0.128
		100	11	0.150	0.110	0.146	0.156

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