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INEQUALITIES FOR THE ANGULAR DERIVATIVES OF CERTAIN CLASSES OF HOLOMORPHIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this paper, a boundary version of the Schwarz lemma is investigated. We take into consideration a function $f(z)=z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\cdots$ holomorphic in the unit disc and $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z}-\alpha\right|<\alpha$ for $|z|<1$, where $\frac{1}{2}<\alpha\leq\frac{1}{1+\lambda}$, $0\leq\lambda<1$. If we know the second and the third coefficient in the expansion of the function $f(z)=z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\cdots$, then we can obtain more general results on the angular derivatives of certain holomorphic function on the unit disc at boundary by taking into account c_{p+1}, c_{p+2} and zeros of $f(z)-z$. We obtain a sharp lower bound of $|f'(b)|$ at the point b, where $|b|=1$.

1. Introduction

Let f be a holomorphic function in the unit disc $D=\{z:|z|<1\}$, $f(0)=0$ and $|f(z)|<1$ for $|z|<1$. In accordance with the classical Schwarz lemma, for any point z in the disc D, we have $|f(z)|\leq|z|$ and $|f'(0)|\leq 1$. Equality in these inequalities (in the first one, for $z\neq 0$) occurs only if $f(z)=\gamma z$, $|\gamma|=1$ ([4], p. 329).

Let $f(z)=z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\cdots$, $p\in\mathbb{N}$ be a holomorphic function on D and let $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z}-\alpha\right|<\alpha$ for $|z|<1$, where $\frac{1}{2}<\alpha\leq\frac{1}{1+\lambda}$, $0\leq\lambda<1$.

Consider the functions

$$\varphi(z)=\frac{\vartheta(z)-\alpha}{\alpha},$$

where

$$\vartheta(z)=\frac{f(z)}{\lambda f(z)+(1-\lambda)z}$$

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and

$$\psi(z) = \frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)}.$$

$\varphi(z)$ and $\psi(z)$ are holomorphic functions in the unit disc D, $|\psi(z)| < 1$ for $|z| < 1$ and $\psi(0) = 0$. Therefore, from the Schwarz lemma, we obtain

$$|f(z)| \leq \frac{\alpha\,(1-\lambda)\,|z|\,(1+|z|^p)}{\alpha + (1-\alpha)\,|z|^p - \alpha\lambda\,(1+|z|^p)} \tag{1.1}$$

and

$$|c_{p+1}| \leq \frac{2\alpha - 1}{\alpha\,(1-\lambda)}. \tag{1.2}$$

Equality is achieved in (1.1) (for some nonzero $z \in D$) or in (1.2) if and only if $f(z)$ is the function of the form $f(z) = \frac{\alpha(1-\lambda)z\left(1+z^p e^{i\theta}\right)}{\alpha+(1-\alpha)z^p e^{i\theta} - \alpha\lambda(1+z^p e^{i\theta})}$, where θ is a real number.

H. Unkelbach ([9]) and Robert Osserman ([7]) have given the inequalities which are called the boundary Schwarz lemma. They have first showed that

$$|f'(b)| \geq \frac{2}{1+|f'(0)|} \tag{1.3}$$

and

$$|f'(b)| \geq 1 \tag{1.4}$$

under the assumption $f(0) = 0$ where f is a holomorphic function mapping the unit disc into itself and b is a boundary point to which f extends continuously and $|f(b)| = 1$. Moreover, equality in (1.3) holds if and only if f is of the form

$$f(z) = ze^{i\theta}\frac{z-a}{1-\overline{a}z},$$

where $\theta \in \mathbb{R}$ and $a \in D$ satisfies $\arg a = \arg b$. Also, the equality in (1.4) holds if and only if $f(z) = ze^{i\theta}$, $\theta \in \mathbb{R}$.

One does not need to assume that f extends continuously to b. For example, if f has a radial limit $f(b)$ at b, with $|f(b)| = 1$, and if f has a radial derivative at b, then that derivative also satisfies the inequalities (1.3) and (1.4). Accordingly, using the Möbius transformation, they have generalized the inequality on the case of $f(0) \neq 0$.

If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then by the Julia-Wolff lemma the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$ (see [8]).

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [4], [8]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1], [2], [3], [5], [6] and references therein).

Vladimir N. Dubinin ([2]) has continued this line of research and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, with a zero set $\{a_k\}$.

Some other types of the strengthening inequalities are obtained in (see [1], [6]). That is, in ([1]), we gave estimate below $|f'(b)|$ according to the first nonzero Taylor coefficient of f about two zeros, namely $z = 0$ and $z_0 \neq 0$. In ([6]), we obtained such type results for other than above mentioned class. An other interpretation of the results in ([6]) is given in ([5]).

2. Main results

In this section, we can obtain more general results on the angular derivatives of certain holomorphic function on the unit disc at boundary by taking into account c_{p+1}, c_{p+2} and zeros of $f(z) - z$ if we know the second and the third coefficient in the expansion of the function $f(z) = z + c_{p+1} z^{p+1} + c_{p+2} z^{p+2} + \cdots$. We obtain a sharp lower bound of $|f'(b)|$ at the point b, where $|b| = 1$.

Theorem 2.1. *Let* $f(z) = z + c_{p+1} z^{p+1} + c_{p+2} z^{p+2} + \cdots$, $c_{p+1} \neq 0$, $p > 1$ *be a holomorphic function in the unit disc* D *and let* $\left| \frac{f(z)}{\lambda f(z) + (1-\lambda) z} - \alpha \right| < \alpha$ *for* $|z| < 1$, *where* $\frac{1}{2} < \alpha \leq \frac{1}{1+\lambda}$, $0 \leq \lambda < 1$. *Further assume that, for some* $b \in \partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b) = 0$. *Then*

$$|f'(b)| \geq \frac{\alpha(1-\lambda)}{2\alpha - 1} \left[p + \frac{2\left(1 - \frac{\alpha(1-\lambda)}{2\alpha-1} |c_{p+1}|\right)^2}{1 - \left(\frac{\alpha(1-\lambda)}{2\alpha-1} |c_{p+1}|\right)^2 + \frac{\alpha(1-\lambda)}{2\alpha-1} |c_{p+2}|} \right]. \tag{1.5}$$

Moreover, the equality in (1.5) *occurs for the function*

$$f(z) = \frac{\alpha(1-\lambda) z (1 - z^p)}{\alpha - (1-\alpha) z^p - \alpha\lambda (1 - z^p)}.$$

Proof. Consider the functions

$$\psi(z) = \frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)} \varphi(z)}, \ \omega(z) = z^p.$$

$\psi(z)$ and $\omega(z)$ are holomorphic functions in D, and $|\psi(z)| < 1$, $|\omega(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in D$, we have the inequality

$$|\psi(z)| \leq |\omega(z)|.$$

Therefore, the absolute value of the holomorphic function

$$h(z) = \frac{\psi(z)}{\omega(z)}$$

in D is bounded by 1 in D.

In particular, we have

$$|h(0)| = \frac{\alpha(1-\lambda)}{2\alpha - 1} |c_{p+1}| \leq 1 \tag{1.6}$$

and

$$|h'(0)| = \frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+2}|.$$

Moreover, we can show that

$$\frac{b\psi'(b)}{\psi(b)} = |\psi'(b)| \geq |\omega'(b)| = \frac{b\omega'(b)}{\omega(b)}.$$

The function

$$T(z) = \frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}$$

is holomorphic in the unit disc D, $|T(z)| < 1$, $T(0) = 0$ and $|T(b)| = 1$ for $b \in \partial D$.

From (1.3), we obtain

$$\begin{aligned}
\frac{2}{1+|T'(0)|} \leq |T'(b)| &= \frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(b)\right|}|h'(b)| \\
&\leq \frac{1+|h(0)|}{1-|h(0)|}|h'(b)| \\
&= \frac{1+\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|}{1-\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|}\left\{|\psi'(b)| - |\omega'(b|)\right\} \\
&= \frac{1+\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|}{1-\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|}\left\{\frac{1-|\varphi(0)|^2}{\left|1-\overline{\varphi(0)}\varphi(b)\right|^2}|\varphi'(b)| - p\right\} \\
&= \frac{1+\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|}{1-\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|}\left\{\frac{1-\left(\frac{1-\alpha}{\alpha}\right)^2}{\left(1+\frac{1-\alpha}{\alpha}\right)^2}\frac{|\vartheta'(b)|}{\alpha} - p\right\}
\end{aligned}$$

and

$$\frac{2}{1+\frac{\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+2}|}{1-\left(\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right)^2}} \leq \frac{1+\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|}{1-\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|}\left\{\frac{2\alpha-1}{\alpha(1-\lambda)}|f'(b)| - p\right\}.$$

Therefore, we take the inequality (1.5).

To show that the inequality (1.5) is sharp, take the holomorphic function

$$f(z) = \frac{\alpha(1-\lambda)z(1-z^p)}{\alpha-(1-\alpha)z^p-\alpha\lambda(1-z^p)}.$$

Then

$$\begin{aligned}
f'(z) = \alpha(1-\lambda)&\frac{(1-(p+1)z^p)(\alpha(1-\alpha)z^p-\alpha\lambda(1-z^p))}{(\alpha-(1-\alpha)z^p-\alpha\lambda(1-z^p))^2} \\
&-\alpha(1-\lambda)\frac{\left(-p(1-\alpha)z^{p-1}+\alpha\lambda p z^{p-1}\right)\left(z-z^{p+1}\right)}{(\alpha-(1-\alpha)z^p-\alpha\lambda(1-z^p))^2}
\end{aligned}$$

and

$$f'(1) = -p\frac{\alpha(1-\lambda)}{2\alpha-1}.$$

Since $|c_{p+1}| = \frac{2\alpha-1}{\alpha(1-\lambda)}$, (1.5) is satisfied with equality. □

If $f(z) - z$ has no zeros different from $z = 0$ in Theorem 2.1, the inequality (1.5) can be further strengthened. This is given by the following theorem.

Theorem 2.2. *Let* $f(z) = z + c_{p+1}z^{p+1} + c_{p+2}z^{p+2} + \cdots$, $c_{p+1} \neq 0$, $p > 1$ *be a holomorphic function in the unit disc* D *and* $f(z) - z$ *has no zeros in* D *except* $z = 0$, *and let* $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z} - \alpha\right| < \alpha$ *for* $|z| < 1$, *where* $\frac{1}{2} < \alpha \leq \frac{1}{1+\lambda}$, $0 \leq \lambda < 1$. *Further assume that, for some* $b \in \partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b) = 0$. *Then*

$$|f'(b)| \geq \frac{\alpha(1-\lambda)}{2\alpha-1}\left[p - \frac{2|c_{p+1}|\left(\ln\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right)^2}{2|c_{p+1}|\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right) - |c_{p+2}|}\right] \tag{1.7}$$

and

$$|c_{p+2}| \leq 2\left|c_{p+1}\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right)\right|. \tag{1.8}$$

In addition, the equality in (1.7) *occurs for the function*

$$f(z) = \frac{\alpha(1-\lambda)z(1-z^p)}{\alpha - (1-\alpha)z^p - \alpha\lambda(1-z^p)}$$

and the equality in (1.8) *occurs for the function*

$$f(z) = z + z^{p+1}\frac{(2\alpha-1)e^Q}{\alpha + (1-\alpha)z^p e^Q - \alpha\lambda(1+z^p e^Q)},$$

where $0 < c_{p+1} < 1$, $\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}c_{p+1}\right) < 0$ *and* $Q = \frac{1+z}{1-z}\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}c_{p+1}\right)$.

Proof. We can assume that $c_{p+1} > 0$. Let $\psi(z)$, $h(z)$ and $\omega(z)$ be as in the proof of Theorem 2.1. Bearing in the mind inequality (1.6), we denote by $\ln h(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln h(0) = \ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}c_{p+1}\right) < 0.$$

The composite function

$$\Theta(z) = \frac{\ln h(z) - \ln h(0)}{\ln h(z) + \ln h(0)}$$

is holomorphic in the unit disc D, $|\Theta(z)| < 1$, $\Theta(0) = 0$ and $|\Theta(b)| = 1$ for $b \in \partial D$.

From (1.3), we obtain

$$\frac{2}{1+|\Theta'(0)|} \leq |\Theta'(b)| = \frac{|2\ln h(0)|}{|\ln h(b)+\ln h(0)|^2}\left|\frac{h'(b)}{h(b)}\right|,$$

$$\begin{aligned}
\frac{2}{1-\frac{|c_{p+2}|}{2|c_{p+1}|\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right)}} &\leq \frac{-2\ln h(0)}{\arg^2 h(b)+\ln^2 h(0}|h'(b)| \\
&= \frac{-2\ln h(0)}{\arg^2 h(b)+\ln^2 h(0)}\{|\psi'(b)|-|\omega'(b)|\} \\
&= \frac{-2\ln h(0)}{\arg^2 h(b)+\ln^2 h(0)}\left\{\frac{1-|\varphi(0)|^2}{\left|1-\overline{\varphi(0)}\varphi(b)\right|^2}|\varphi'(b)|-p\right\} \\
&= \frac{-2\ln h(0)}{\arg^2 h(b)+\ln^2 h(0)}\left\{\frac{1-\left(\frac{1-\alpha}{\alpha}\right)^2}{\left(1+\frac{1-\alpha}{\alpha}\right)^2}\frac{|\vartheta'(b)|}{\alpha}-p\right\} \\
&= \frac{-2\ln h(0)}{\arg^2 h(b)+\ln^2 h(0)}\left\{\frac{2\alpha-1}{\alpha(1-\lambda)}|f'(b)|-p\right\}
\end{aligned}$$

and replacing $\arg^2 h(b)$ by zero, we obtain (1.7) with an obvious equality case.

Similarly, $\Theta(z)$ satisfies the assumptions of the Schwarz lemma, we obtain

$$1 \geq |\Theta'(0)| = \frac{|2\ln h(0)|}{|\ln h(0)+\ln h(0)|^2}\left|\frac{h'(0)}{h(0)}\right| = \frac{-1}{2\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right)}\frac{|c_{p+2}|}{|c_{p+1}|}.$$

Therefore, we have the inequality (1.8).

Now, we shall show that the inequality (1.8) is sharp. Let

$$f(z) = z + z^{p+1}g(z),$$

where

$$g(z) = \frac{(2\alpha-1)\,e^{\frac{1+z}{1-z}\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}c_{p+1}\right)}}{\alpha+(1-\alpha)\,z^p e^{\frac{1+z}{1-z}\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}c_{p+1}\right)}-\alpha\lambda\left(1+z^p e^{\frac{1+z}{1-z}\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}c_{p+1}\right)}\right)}.$$

Then

$$g'(0) = c_{p+2}.$$

Under the simple calculations, we take

$$c_{p+2} = 2c_{p+1}\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}c_{p+1}\right).$$

Therefore, we obtain

$$|c_{p+2}| = 2\left|c_{p+1}\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right)\right|.$$

□

Relation (1.8) shows that inequality (1.7) is more stronger than inequality (1.5).

If $f(z)-z$ have zeros different from $z=0$, taking into account these zeros, the inequality (1.5) can be strengthened in another way. This is given by the following theorem.

Theorem 2.3. *Let* $f(z)=z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\cdots$, $c_{p+1}\neq 0$, $p>1$ *be a holomorphic function in the unit disc* D *and let* $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z}-\alpha\right|<\alpha$ *for* $|z|<1$, *where* $\frac{1}{2}<\alpha\leq\frac{1}{1+\lambda}$, $0\leq\lambda<1$. *Further assume that, for some* $b\in\partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b)=0$. *Let* $a_1,a_2,\ldots,a_n$ *be zeros of the function* $f(z)-z$ *in* D *that are different from zero. Then we have the inequality*

$$(1.9)\quad |f'(b)| \geq \frac{\alpha(1-\lambda)}{2\alpha-1}\left(p+\sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2}+\frac{2\left(1-\frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+1}|}{\prod_{k=1}^{n}|a_k|}\right)^2}{1-\left(\frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+1}|}{\prod_{k=1}^{n}|a_k|}\right)^2+\frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+2}|}{\prod_{k=1}^{n}|a_k|}}\right).$$

In addition, the equality in (1.9) *occurs for the function*

$$f(z)=\frac{\alpha(1-\lambda)z\left(1-z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)}{\alpha-(1-\alpha)z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}-\alpha\lambda\left(1-z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)},$$

where $a_1,a_2,\ldots,a_n$ *are positive real numbers.*

Proof. Let $\psi(z)$ be as in the proof of Theorem 2.1 and $a_1,a_2,\ldots,a_n$ be zeros of the function $f(z)-z$ in D that are different from zero.

$$B(z)=z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}$$

is a holomorphic function in D and $|B(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in D$, we have

$$|\psi(z)|\leq|B(z)|.$$

The function

$$s(z)=\frac{\psi(z)}{B(z)}$$

is a holomorphic function in D and $|s(z)| < 1$ for $|z| < 1$. In particular, we have

$$|s(0)| = \frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+1}|}{\prod\limits_{k=1}^{n}|a_k|} \leq 1$$

and

$$|s'(0)| = \frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+2}|}{\prod\limits_{k=1}^{n}|a_k|}.$$

Moreover, it can be seen that

$$\frac{b\psi'(b)}{\psi(b)} = |\psi'(b)| \geq |B'(b)| = \frac{bB'(b)}{B(b)}.$$

Besides, with the simple calculations, we take

$$|B'(b)| = \frac{bB'(b)}{B(b)} = p + \sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2}.$$

The auxiliary function

$$d(z) = \frac{s(z)-s(0)}{1-\overline{s(0)}s(z)}$$

is holomorphic in the unit disc D, $|d(z)| < 1$, $d(0) = 1$ and $|d(b)| = 1$ for $b \in \partial D$.

From (1.3), we obtain

$$\begin{aligned}
\frac{2}{1+|d'(0)|} \leq |d'(b)| &= \frac{1-|s(0)|^2}{\left|1-\overline{s(0)}s(b)\right|^2}|s'(b)| \\
&\leq \frac{1+|s(0)|}{1-|s(0)|}\{\{|\psi'(b)|-|B'(b)|\}\} \\
&= \frac{1+\frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+1}|}{\prod_{k=1}^{n}|a_k|}}{1-\frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+1}|}{\prod_{k=1}^{n}|a_k|}}\left\{\frac{1-|\varphi(0)|^2}{\left|1-\overline{\varphi(0)}\varphi(b)\right|^2}|\varphi'(b)|-|B'(b)|\right\}
\end{aligned}$$

and

$$\frac{2}{1+\frac{\frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+2}|}{\prod_{k=1}^{n}|a_k|}}{1-\left(\frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+1}|}{\prod_{k=1}^{n}|a_k|}\right)^2}} \leq \frac{1+\frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+1}|}{\prod_{k=1}^{n}|a_k|}}{1-\frac{\alpha(1-\lambda)}{2\alpha-1}\frac{|c_{p+1}|}{\prod_{k=1}^{n}|a_k|}}\left\{\frac{2\alpha-1}{\alpha(1-\lambda)}|f'(b)|-|B'(b)|\right\}.$$

Therefore, we take the inequality (1.9) with an obvious equality case. □

We note that the inequality (1.3) has been used in the proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3. Therefore, there are both c_{p+1} and c_{p+2} in the right side of the inequalities. But, if we use (1.4) instead of (1.3), we obtain weaker but more simpler inequality (not including c_{p+2}). It is formulated in the following theorem.

Theorem 2.4. *Under the hypotheses of Theorem 2.2, we have the inequality*

$$|f'(b)| \geq \frac{\alpha(1-\lambda)}{2\alpha-1}\left[p - \frac{1}{2}\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right)\right]. \tag{1.10}$$

The equality in (1.10) *holds if and only if*

$$f(z) = \frac{\alpha(1-\lambda)z\left(1+z^p e^Q\right)}{\alpha + (1-\alpha)z^p e^Q - \alpha\lambda\left(1+z^p e^Q\right)},$$

where $0 < c_{p+1} < 1$, $\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}c_{p+1}\right) < 0$, $Q = \ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}c_{p+1}\right)\frac{1+ze^{i\theta}}{1-ze^{i\theta}}$ *and* θ *is a real number.*

Proof. From the proof of Theorem 2.2, using the inequality (1.4) for the function $\Theta(z)$, we obtain

$$\begin{aligned} 1 \leq |\Theta'(b)| &= \frac{|2\ln h(0)|}{|\ln h(b) + \ln h(0)|^2}\left|\frac{h'(b)}{h(b)}\right| \\ &= \frac{-2\ln h(0)}{\arg^2 h(b) + \ln^2 h(0)}\left\{|\psi'(b)| - |\omega'(b)|\right\} \end{aligned}$$

and replacing $\arg^2 h(b)$ by zero

$$\begin{aligned} 1 &\leq \frac{-2}{\ln h(0}\left\{\frac{1-|\varphi(0)|^2}{\left|1-\overline{\varphi(0)}\varphi(b)\right|^2}|\varphi'(b)| - p\right\} \\ &= \frac{-2}{\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right)}\left\{\frac{2\alpha-1}{\alpha(1-\lambda)}|f'(b)| - p\right\}. \end{aligned} \tag{1.11}$$

Therefore, we have the inequality (1.10).

If $|f'(b)| = \frac{\alpha(1-\lambda)}{2\alpha-1}\left[p - \frac{1}{2}\ln\left(\frac{\alpha(1-\lambda)}{2\alpha-1}|c_{p+1}|\right)\right]$ from (1.11) and $|\Theta'(b)| = 1$, we obtain

$$f(z) = \frac{\alpha(1-\lambda)z\left(1+z^p e^Q\right)}{\alpha + (1-\alpha)z^p e^Q - \alpha\lambda\left(1+z^p e^Q\right)}.$$

□

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