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Stationary bootstrap test for jumps in high-frequency financial asset data

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Abstract

We consider a jump diffusion process for high-frequency financial asset data. We apply the stationary bootstrapping to construct a bootstrap test for jumps. First-order asymptotic validity is established for the stationary bootstrapping of the jump ratio test under the null hypothesis of no jump. Consistency of the stationary bootstrap test is proved under the alternative of jumps. A Monte-Carlo experiment shows the advantage of a stationary bootstrapping test over the test based on the normal asymptotic theory. The proposed bootstrap test is applied to construct continuous-jump decomposition of the daily realized variance of the KOSPI for the year 2008 of the world-wide financial crisis.

Keywords: stationary bootstrap, jump diffusion process, ratio test, realized variation, realized bipower variation

1. Introduction

Jump is an important feature of financial asset processes as it causes sudden changes for asset processes. Financial markets generate significant discontinuities in financial variables; therefore, jump diffusion processes are considered more appropriate empirical models for financial data than Ito processes whose volatility parts consist only of generalized Wiener processes. Proper characterization of the jump feature (if any) of a financial asset is important for its volatility analysis which is crucial for asset pricing, risk management and portfolio allocation.

The recent availability of high frequency financial asset data has resulted in considerable studies on tests and estimations regarding jumps using high frequency data. Tests for jumps were proposed by Aït-Sahalia (2002), Carr and Wu (2003), Huang and Tauchen (2005), Barndorff- Nielsen and Shephard (2006), and Lee and Mykland (2008). Barndorff-Nielsen and Shephard (2004, 2006) addressed realized bipower variation to estimate the contribution of jumps to the variation of assets and to form robust tests, and derived asymptotic distribution theory for nonparametric tests. More recently, Aït-Sahalia and Jacod (2009) proposed a test to determine whether jumps are present in asset returns or other discrete samples processes. Aït-Sahalia *et al.* (2012) dealt with testing for jumps in noisy high frequency data. Podolskij and Vetter (2009), Jacod and Reiss (2014), Jacod and Todorov (2014), and Jing *et al.* (2014) discussed efficient estimations of integrated volatility in the presence of jumps.

Bootstrap method is attractive to approximate the sampling distributions of test statistics or estimators. For statistical inference for financial assets based on high frequency data, some papers

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reported better performances of statistical methods by bootstrap applications than by central limit theorem applications. Gonçalves and Meddahi (2009) and Dovonon *et al.* (2013) considered the i.i.d. bootstrap and the wild bootstrap for realized volatility and for realized regression coefficients, covariances, and correlations of multivariate high frequency returns, respectively. Hwang and Shin (2013a, 2013b) applied stationary bootstrapping to the realized volatility with and without market microstructural noise. As for the bootstrap jump tests, using the asymptotic theory of Barndorff-Nielsen and Shephard (2006), Hwang and Shin (2014) considered the i.i.d. bootstrap to construct a bootstrap test for jumps and Dovonon *et al.* (2014) applied the local Gaussian bootstrap to construct a bootstrap jump test.

We note that the observed log return series are not independent because of conditional heteroscedasticity. The dependent structure of a sample is better carried into a bootstrap sample by a block bootstrap method than by an i.i.d. bootstrap method. We apply stationary bootstrapping for jump diffusion processes (the most widely used block bootstrap method) to a test for jump via bootstrap realized quadratic and bipower variations that produce a stationary bootstrap version of the jump test by Barndorff-Nielsen and Shephard (2006) and the i.i.d bootstrap test of Hwang and Shin (2014). The first-order asymptotic validity of the distribution of the stationary bootstrap test is established under the null of no jump and consistency of the stationary bootstrap test is proved under the alternative of jumps, which is the main result of this paper.

The remaining of the paper is organized as follows. In Section 2, we describe the setup and the existing theory used in this work. In Section 3, the stationary bootstrap procedure is applied to construct a bootstrap jump test as well as to establish an asymptotic validity. In Section 4, a Monte-Carlo experiment compares the bootstrap test and the normal test based on central limit theorem. In Section 5, the proposed bootstrap test is applied to the daily realized variance of the KOSPI for the year 2008. In Section 6, a concluding remark is presented. In Section 7, technical results and proofs are given.

2. Preliminary setup

The preliminary setup reviews the jump test by Barndorff-Nielsen and Shephard (2006). A large part of this section is reproduced from Hwang and Shin (2014). We consider the log-price process $\{Y_t : t \geq 0\}$ of an asset, which is assumed to follow a jump diffusion process in a Brownian semimartingale plus jump:

$$Y_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \sum_{j=1}^{N_t} c_j, \tag{2.1}$$

where a_t is the drift term, σ_t is a volatility process, W_t is the standard Brownian motion, N_t is a simple counting process, and c_j, $j = 1, 2, \ldots$, are real-valued random variables. In this work we adopt the setup and notations of Barndorff-Nielsen and Shephard (2006) and Hwang and Shin (2014).

We are interested in testing the presence of jump based on a high frequency data. Without loss of generality, we assume the time interval for the sample to be one unit interval $[0, 1]$. The last term in (2.1) is due to jump. The absence of the jump component corresponds to $c_j = 0$, for all j in the interval $[0, 1]$.

The quadratic variation (QV) of Y_t over $[0, 1]$ is

$$\mathrm{QV} := \mathrm{p}\lim_{n\to\infty} \sum_{j=1}^{n-1} (Y_{t_{j+1}} - Y_{t_j})^2$$

for any sequence of partitions $0 = t_0 < t_1 < \cdots < t_n = 1$ with $\sup_j |t_{j+1} - t_j| \to 0$ as $n \to \infty$. The QV is decomposed into the continuous part $\int_0^1 \sigma_s^2 ds$ and purely discontinuous part $\sum_{0\le u\le 1} \Delta Y_u^2 = \sum_{j=1}^{N_1} c_j^2$ where $\Delta Y_u = Y_u - Y_{u-}$ are the jumps in Y. Then we have

$$\mathrm{QV} = \int_0^1 \sigma_s^2 ds + \sum_{j=1}^{N_1} c_j^2.$$

The two terms in the QV, $\int_0^1 \sigma_s^2 ds$ and $\sum_{j=1}^{N_1} c_j^2$, are estimated from a discrete sample of Y with a sampling interval $h \in (0, 1)$. Let $n = \lfloor 1/h \rfloor$, integer part of $1/h$. Let $r_i = Y_{ih} - Y_{(i-1)h}$ be the log-return for the period $[(i-1)h, ih]$ for $i = 1, 2, \ldots, n$.

It is well-known that the realized quadratic variation (RQV) defined by $\hat{Q}_h := \sum_{i=1}^n r_i^2$ is a consistent estimator of QV. According to Barndorff-Nielsen and Shephard (2006), if $a = 0$ and σ is independent of W, then the $(1, 1)$-order bipower variation (BV) of Y defined as $\mathrm{BV} := p\lim_{n\to\infty} \sum_{i=2}^n |r_{i-1}||r_i|$ is given by

$$\mathrm{BV} = \mu_1^2 \int_0^1 \sigma_s^2\, ds,$$

where $\mu_1 = E|Z| = \sqrt{2/\pi}$ and $Z \sim N(0, 1)$, and hence, the realized bipower variation (RBV), defined by $\hat{B}_h := \sum_{i=2}^n |r_{i-1}||r_i|$, satisfies

$$\mu_1^{-2} \hat{B}_h \xrightarrow{p} \int_0^1 \sigma_s^2\, ds.$$

Consequently, $\hat{Q}_h - \mu_1^{-2} \hat{B}_h$ is a consistent estimator of purely discontinuous component $\sum_{j=1}^{N_1} c_j^2$ of the QV. Deviation of the ratio $\mu_1^{-2} \hat{B}_h / \hat{Q}_h$ below from 1 is a measure for the jump. The following Proposition 1 of Barndorff-Nielsen and Shephard (2006), which is given under assumptions (A1) and (A2), characterizes the null asymptotic distribution of the ratio:

(A1) the volatility process σ^2 is pathwise bounded away from zero.

(A2) the joint process (a, σ) is independent of the Brownian motion W.

Proposition 1. (Barndorff-Nielsen and Shephard, 2006) *Consider model (2.1) with no jump component, i.e., $\sum_{j=1}^{N_1} c_j^2 = 0$, and assume (A1) and (A2).*

Then as $h \to 0$, we have

$$H := \frac{h^{-\frac{1}{2}} \left(\mu_1^{-2} \hat{B}_h / \hat{Q}_h - 1 \right)}{\sqrt{\vartheta \int_0^1 \sigma_s^4\, ds \Big/ \left(\int_0^1 \sigma_s^2\, ds \right)^2}} \xrightarrow{d} N(0, 1),$$

where $\vartheta = (\pi^2/4) + \pi - 5$.

Since the distribution depends on the unknown integrated quarticity $\int_0^1 \sigma_u^4 du$, using the RQV, $\hat{U}_h = h^{-1} \sum_{i=4}^n |r_i r_{i-1} r_{i-2} r_{i-3}|$, as a consistent estimator of $\mu_1^4 \int_0^1 \sigma_s^4\, ds$, Barndorff-Nielsen and Shephard

(2006) proposed the following test

$$\hat{H} := \frac{h^{-\frac{1}{2}}\left(\mu_1^{-2}\hat{B}_h/\hat{Q}_h - 1\right)}{\sqrt{\vartheta\hat{U}_h/\hat{B}_h^2}},$$

which is called the ratio test. The test $\hat{H}$ rejects the null hypothesis of no jump if negatively large. According to Proposition 1, critical value of $\hat{H}$ can be approximated by quantiles of the standard normal distribution.

3. Consistency of the stationary bootstrap jump test

In this section, we describe the stationary bootstrap (SB) procedure and the main results. A SB ratio test is constructed via SB realized quadratic variation (SB-RQV) and SB realized bipower variation (SB-RBV), and its first order asymptotic validity is established.

The SB method originally proposed by Politis and Romano (1994) is characterized by resampling blocks of geometrically distributed random length and the pseudo time series generated by the bootstrapping are stationary, conditionally on the original data.

SB has been actively studied by authors as a nonparametric inference technique in statistics and econometric financial time series analysis. See Hwang and Shin (2013a, 2013b), Shin and Hwang (2013) and references therein.

3.1. Stationary bootstrap sample

Let $r_1, \ldots, r_n$ be observed. First we define a new time series $\{r_{ni} : i \geq 1\}$ by a periodic extension of the observed data set as follows. For each $i \geq 1$, define $r_{ni} := r_j$ where j is such that $i = qn + j$ for some q. The sequence $\{r_{ni} : i \geq 1\}$ is obtained by wrapping the data $r_1, \ldots, r_n$ around a circle, and relabelling them as $r_{n1}, r_{n2}, \ldots$. Next, for a positive integer ℓ, define the blocks $B(i, \ell)$, $i \geq 1$ as $B(i, \ell) = \{r_{ni}, \ldots, r_{n(i+\ell-1)}\}$ consisting of ℓ observations starting from r_{ni}. Bootstrap observations under the stationary bootstrap method are obtained by selecting a random number of blocks from collection $\{B(i, \ell) : i \geq 1, \ \ell \geq 1\}$. To do this, we generate random variables $I_1, \ldots, I_n$ and $L_1, \ldots, L_n$ as follows: (i) $I_1, \ldots, I_n$ are i.i.d. discrete uniform on $\{1, \ldots, n\} : P(I_1 = i) = 1/n$, for $i = 1, \ldots, n$, (ii) $L_1, \ldots, L_n$ are i.i.d. random variables having the geometric distribution with a parameter $p \in (0, 1) : P(L_1 = \ell) = p(1-p)^{\ell-1}$ for $\ell = 1, 2, \ldots$, where $p = p_n$ depends on the sample size n, and (iii) the collections $\{I_1, \ldots, I_n\}$ and $\{L_1, \ldots, L_n\}$ are independent.

For notational simplicity, we suppress dependence of the variables $I_1, \ldots, I_n, L_1, \ldots, L_n$ and of the parameter p on n. We assume that $p = p_n$ goes to 0 as $n \to \infty$. Under the stationary bootstrap the block length variables $L_1, \ldots, L_n$ are random and the expected block length EL_1 is p^{-1}, which tends to ∞ as $n \to \infty$. Now, a pseudo-time series $r_1^*, \ldots, r_n^*$ is generated in the following way. Let $\tau = \inf\{k \geq 1 : L_1 + \cdots + L_k \geq n\}$. Then select τ blocks $B(I_1, L_1), \ldots, B(I_\tau, L_\tau)$. Note that there are $L_1 + \cdots + L_\tau$ elements in the resampled blocks $B(I_1, L_1), \ldots, B(I_\tau, L_\tau)$. Arranging these elements in a series and deleting the last $L_1 + \cdots + L_\tau - n$ elements, we get the bootstrap observations $r_1^*, \ldots, r_n^*$. Conditionally on $\{r_1, \ldots, r_n\}$, the process $\{r_i^* : i = 1, 2, \ldots\}$ is stationary. In the following, P^*, E^*, Var^* denote the conditional probability, the conditional expectation, and the conditional variance, respectively, given $r_1, \ldots, r_n$.

3.2. Asymptotic results

The SB-RQV, SB-RBV, and SB realized quadpower variation are respectively given by

$$\hat{Q}_h^* := \sum_{i=1}^{n} r_i^{*2}, \qquad \hat{B}_h^* := \sum_{i=2}^{n} \left|r_{i-1}^*\right|\left|r_i^*\right|, \qquad \hat{U}_h^* := h^{-1}\sum_{i=4}^{n} \left|r_i^* r_{i-1}^* r_{i-2}^* r_{i-3}^*\right|.$$

The next theorems establish consistency of the SB distribution of the ratio test. We make the following conditions, which are presented in earlier works of the stationary bootstrap.

(A3) Assume $E|r_i|^{8+4\delta} < \infty$, for some $\delta > 0$.

Note that the condition $E|r_i|^{8+4\delta} < \infty$ implies $E|r_i r_{i+1}|^{4+2\delta} < \infty$ by Hölder inequality. These two finite moment conditions are used to show the limiting of conditional variance of the stationary bootstrap estimators. Consistency of $\hat{Q}_h^*$ can be seen in Hwang and Shin (2013a, 2013b).

Theorem 1. *We consider model (2.1) with no jump component and assume (A1), (A2) and (A3). If parameter p of geometric distribution of random block length in the stationary bootstrap procedure is chosen so that $h/p^2 \to 0$ as $h \to 0$. Then we have*

$$(i)\quad \sup_{x\in\mathbb{R}}\left|P^*\left(\frac{h^{-\frac{1}{2}}\left[\mu_1^{-2}\hat{B}_h^*/\hat{Q}_h^* - \mu_1^{-2}E^*\left(\hat{B}_h^*\right)/E^*\left(\hat{Q}_h^*\right)\right]}{\sqrt{h^{-1}Var^*\left(\mu_1^{-2}\hat{B}_h^*/\hat{Q}_h^*\right)}} \le x\right) - P\left(\frac{h^{-\frac{1}{2}}\left[\mu_1^{-2}\hat{B}_h/\hat{Q}_h - 1\right]}{\sqrt{h^{-1}Var\left(\mu_1^{-2}\hat{B}_h/\hat{Q}_h\right)}} \le x\right)\right| \xrightarrow{p} 0.$$

$$(ii)\quad \sup_{x\in\mathbb{R}}\left|P^*\left(\frac{h^{-\frac{1}{2}}\left[\mu_1^{-2}\hat{B}_h^*/\hat{Q}_h^* - \mu_1^{-2}E^*\left(\hat{B}_h^*\right)/E^*\left(\hat{Q}_h^*\right)\right]}{\sqrt{\vartheta\hat{U}_h^*/\left(\hat{B}_h^*\right)^2}} \le x\right) - P\left(\frac{h^{-1/2}\left[\mu_1^{-2}\hat{B}_h/\hat{Q}_h - 1\right]}{\sqrt{\vartheta\hat{U}_h/\left(\hat{B}_h\right)^2}} \le x\right)\right| \xrightarrow{p} 0.$$

Theorem 1(ii) yields approximation of critical values of the ratio test by quantiles of the bootstrap distribution. Bootstrap values of $\hat{B}_h^{*(k)}$, $\hat{Q}_h^{*(k)}$ and $\hat{U}_h^{*(k)}$, $k = 1, \ldots, m$, are simulated. Let $\bar{B}_h^*$, $\bar{Q}_h^*$ be the averages of m values of $\hat{B}_h^{*(k)}$, $\hat{Q}_h^{*(k)}$, $k = 1, \ldots, m$, respectively. Given level $\alpha \in (0, 1)$, the critical value $H^*(\alpha)$ of $\hat{H}$ is the empirical α-th quantile of the m bootstrap values of

$$\frac{h^{-\frac{1}{2}}\left[\mu_1^{-2}\hat{B}_h^{*(k)}/\hat{Q}_h^{*(k)} - \mu_1^{-2}\bar{B}_h^*/\bar{Q}_h^*\right]}{\sqrt{\vartheta\hat{U}_h^{*(k)}/\left(\hat{B}_h^{*(k)}\right)^2}}, \quad k = 1, \ldots, m. \tag{3.1}$$

The bootstrap test of $\hat{H}$ rejects the null hypothesis of no jump if $\hat{H} \le H^*(\alpha)$. The following theorem establishes asymptotic validity of the test under the alternative of jumps, i.e., consistency of the stationary bootstrap test.

Theorem 2. *We consider model (2.1) with positive jump component $\sum_{j=1}^{N_1} c_j^2 > 0$ and assume (A1), (A2) and (A3). If $h/p^2 \to 0$ as $h \to 0$, then for $\alpha \in (0, 1)$, we have*

$$P\left(\hat{H} \le H^*(\alpha)\right) \to 1.$$

Note that Proposition 1, which was established by Barndorff-Nielsen and Shephard (2006), assumed the no-leverage condition of (A2), and also our asymptotic bootstrap results in Theorems 1 and 2 are given under the same condition. Extension of Proposition 1 and the subsequent stationary bootstrap results to the leverage cases would be a good topic for a future study.

Table 1: Rejection rates (%) the normal test (N) and the stationary bootstrap test (B)

	n	$\text{Jump}_0(0)$		$\text{Jump}_1(.2)$		$\text{Jump}_1(.4)$		$\text{Jump}_2(.4)$	
		N	B	N	B	N	B	N	B
GARCH(1, 1) diffusion	12	18.3	6.8	22.7	10.1	28.4	14.8	25.4	12.8
	48	9.9	5.5	24.4	16.1	37.6	27.3	37.0	25.8
	288	6.8	5.0	46.6	25.5	59.9	40.0	70.2	48.3
	1152	6.1	4.9	62.3	25.0	72.6	40.0	86.3	55.2
log-normal diffusion	12	18.3	6.9	22.7	10.0	28.5	14.9	25.3	12.8
	48	9.9	5.4	24.4	16.0	37.6	27.3	37.0	25.8
	288	6.8	5.1	46.6	25.5	59.9	40.1	70.2	48.4
	1152	6.1	4.9	62.3	25.0	72.6	39.9	86.3	55.2

4. A Monte Carlo study

Finite sample size and power of the stationary bootstrap ratio test (B) are compared with those of the normal ratio test (N). We consider the data generating process $Y_t = \int_0^t \sigma_s dW_{1s} + \sum_{j=1}^{N_t} c_j$ where W_{1t} is a standard Brownian motion. For the volatility process, we consider two models: $d\sigma_t^2 = 0.04(0.64 - \sigma_t^2)dt + 0.14\sigma_t^2 dW_{2t}$; $d\log\sigma_t^2 = -0.014(0.84 + \log\sigma_t^2)dt + 0.11dW_{2t}$ which are called GARCH(1, 1) diffusion; log-normal diffusion, respectively, where W_{2t} is a standard Brownian motion independent of W_{1t}. These volatility processes are found in the empirical studies of Bollerslev and Zhou (2002), Andersen *et al.* (2002), Gonçalves and Meddahi (2009), and Hwang and Shin (2013b, 2014).

We consider $n = 12, 48, 288$ and 1152 which correspond to $h = 2$ hour, 30 minute, 5 minute, and 1.25 minute, respectively. The discrete observations $Y_{i/n}$, $i = 1, \ldots, n$ are simulated by using normal errors $W_{1\,i/n}$, $W_{2\,i/n}$, $i = 1, \ldots, n$ generated by RNNOA, a FORTRAN subroutine in IMSL with initial volatilities $\sigma_0^2 = 0.64$, $\log\sigma_0^2 = -0.84$ for GARCH(1, 1) and log-normal diffusion, respectively,

The jump part is specified so that no jump occurs; one jump of size c_1 occurs at time K_1 distributed uniformly over the interval $\{1, \ldots, n\}$ where $c_1 \sim N(0, \lambda\text{IV}), \lambda = 0.2, 0.4$ and $\text{IV} = \int_0^1 \sigma_s^2 ds$ is the integrated variance; or two independent jumps of sizes c_1, c_2 occur at times K_1, K_2 independently distributed uniformly over the interval $\{1, \ldots, n\}$ where $c_1, c_2 \sim N(0, \lambda\text{IV}), \lambda = 0.2$. These jump cases are denoted by $\text{Jump}_0(.0)$, $\text{Jump}_1(.2)$, $\text{Jump}_1(.4)$, $\text{Jump}_2(.4)$, respectively. The subscript denotes the number of jumps and the numbers in the parenthesis denote $E[\sum_i c_i^2]/\text{IV}$, the ratio of the expected jump component over the continuous component.

Table 1 reports empirical rejection probabilities of level 5% tests for which $m = 1000$ is used for the number of bootstrap replications and $p = 0.4(n/100)^{-1/3}$ is used for the block length parameter. The rejection probabilities are computed using 10,000 independent replications. For each replication, the continuous component IV is approximated by $\sum_{i=1}^n r_i^2$ with $n = 100{,}000$ for the jump-free process $Y_t = \int_0^t \sigma_s dW_{1s}$. We see that size and power performances of the tests under the GARCH(1, 1) diffusion are very similar to those under the log-normal diffusion.

The values in blocks under the column $\text{Jump}_0(0)$ are sizes. We see very stable size for the stationary bootstrap test B but some over size for the normal test N. Relative size advantage of the stationary bootstrap test B over the normal test N is more conspicuous for smaller n: for $n = 12$, the test N has severe over-size over 18% while the test B has size 6.8% or 6.9%; for $n = 48$, N has still unsatisfactory size values around 9.9% while the test B has size 5.5% or 5.4%. As n increases to 288 and 1152, N still has over-size around 6.1% while size of B is 5.1% or 4.9%.

The values in blocks under the column $\text{Jump}_1(.2)$, $\text{Jump}_1(.4)$, $\text{Jump}_2(.4)$ are powers, showing that both tests N and B have powers. Powers of both tests increase as n increases or $E[\sum_i c_i^2]/\text{IV}$,

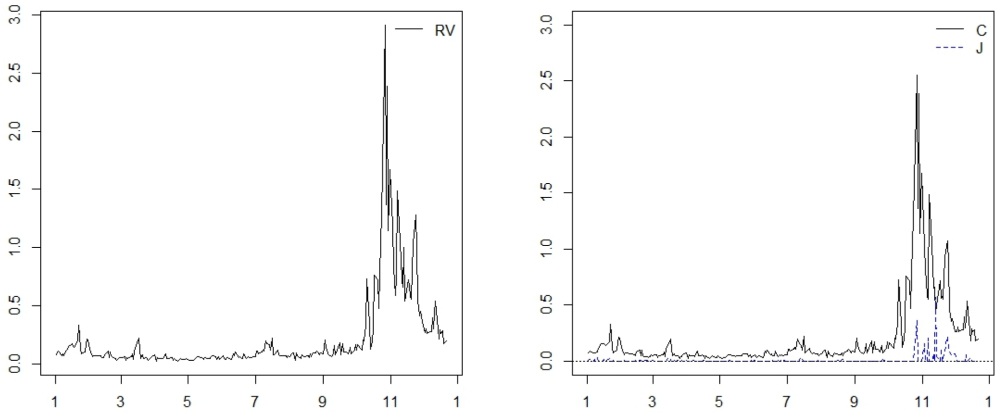

Figure 1: *Daily realized variances (RV) and their continuous-jump decompositions (C, J) of the KOSPI for Jan 1, 2008–Dec 31, 2008, multiplied by 1,000.*

Table 2: Continuous-jump decomposition of realized variance

Date	$\hat{H}$	$H^*(0.05)$	Jump?	$1000 \times \hat{Q}_h$	$1000 \times \mu_1^{-2}\hat{B}_h$	$1000 \times C_h$	$1000 \times J_h$
2008/10/27	−2.201	−2.124	yes	2.913	2.548	2.548	.365
2008/10/28	−.557	−4.556	no	1.364	1.329	1.364	.000

the number in the parenthesis, increases. The normal test N has seemingly higher power than the stationary bootstrap test B. The seemingly higher power of the normal test in large part due to the over-size of the normal test. The seemingly higher power of N over B will then be substantially reduced if the over-size of the normal test N is adjusted.

5. Example

The proposed method is applied to a continuous jump (CJ) decomposition of realized variance of the KOSPI for the period Jan 1, 2008–Dec 31, 2008, of the world-wide financial crisis, which is display in the left part of Figure 1. Let a working day in Jan 1, 2008–Dec 31, 2008, be given. Let $\hat{Q}_h$ be the 1 minute realized variation of the day. The stationary bootstrap test H is applied to determine significance of the jump component. Recall that the QV is decomposed into the continuous part $\int_0^1 \sigma_s^2 ds$ and purely discontinuous part $\sum_{j=1}^{N_1} c_j^2$ so that QV $= \int_0^1 \sigma_s^2 ds + \sum_{j=1}^{N_1} c_j^2$ and that the realized variance $\hat{Q}_h$ is a consistent estimator of QV.

The latter term $\sum_{j=1}^{N_1} c_j^2$ is the jump component of the QV which is consistently estimated by $(\hat{Q}_h - \mu_1^{-2}\hat{B}_h)$ if this term is statistically significant. Therefore, the jump component of $\hat{Q}_h$ is $J_h = (\hat{Q}_h - \mu_1^{-2}\hat{B}_h)I(\hat{H} < H^*(\alpha))$, where $I(A)$ is the indicator function of an event A and $H^*(\alpha)$ is the bootstrap critical value of level α. The continuous component of $\hat{Q}_h$ is $C_h = \hat{Q}_h - J_h$. We then have the CJ-decomposition $\hat{Q}_h = C_h + J_h$ of $\hat{Q}_h$.

Time series plots of C_h, J_h are displayed in the right part of Figure 1. Values of C_h, J_h are given in Table 2 for two selected days. For day Oct 27, 2008, $\hat{H} = -2.201$ is significant at 5% level because it is smaller than the stationary bootstrap critical value $H^*(0.05) = -1.985$ which is computed by 10,000 bootstrap replications. Therefore, $J_h = \hat{Q}_h - \mu_1^{-2}\hat{B}_h = 0.365 \times 10^{-3}$ and $C_h = \mu_1^{-2}\hat{B}_h = 2.548 \times 10^{-3}$. For day Oct 28, 2008, $\hat{H} = -0.557 > H^*(0.05) = -4.213$ is not significant at 5% level, giving us

$J_h = 0$ and $C_h = \hat{Q}_h = 1.364 \times 10^{-3}$.

6. Conclusion

This paper applied stationary bootstrapping to the ratio jump test of Barndorff-Nielsen and Shephard (2006) based on high-frequency data for a jump diffusion process. Asymptotic null validity of the stationary bootstrap ratio jump test was proved by establishing the bivariate normality of the stationary bootstrapping realized quadratic variation and stationary bootstrapping realized bipower variation as well as by using a delta method for the ratio. Consistency of the stationary bootstrap ratio jump test was proved under the alternative of jumps. A Monte Carlo experiment shows that the proposed bootstrap test has a more stable size than the normal test based on the central limit theorem. The proposed bootstrap test is illustrated by continuous-jump decomposition of the daily realized variance of the KOSPI for the year 2008.

7. Proofs

In the proofs, $X_n \xrightarrow{p^*} X$ and $X_n \xrightarrow{d^*} X$ mean that X_n converges to X in probability and in distribution, conditionally given $r_1, \ldots, r_n$, respectively.

According to Theorem 3 of Barndorff-Nielsen and Shephard (2006), we have

$$h^{-\frac{1}{2}}\left[\hat{Q}_h - \int_0^1 \sigma_s^2 ds,\ \mu_1^{-2}\hat{B}_h - \mu_1^{-2}\int_0^1 \sigma_s^2 ds\right]' \xrightarrow{d} N(0, \Sigma), \tag{7.1}$$

where

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 2\int_0^1 \sigma_s^4 ds, & 2\int_0^1 \sigma_s^4 ds \\ 2\int_0^1 \sigma_s^4 ds, & \left(\frac{\pi^2}{4} + \pi - 3\right)\int_0^1 \sigma_s^4 ds \end{pmatrix}. \tag{7.2}$$

Proposition 1 follows from the result in (7.1) above.

We first present the (bivariate) normalities of two SB estimators $\hat{Q}_h^*$ and $\hat{B}_h^*$ in the following three theorems below.

Theorem 3. *We consider model (2.1) with no jump component and assume (A1), (A2) and (A3). As $h \to 0$, we have*

$$h^{-\frac{1}{2}}\left[\hat{Q}_h^* - E^*\left(\hat{Q}_h^*\right)\right] \xrightarrow{d^*} N\left(0, \sigma_1^2\right),$$

where $\sigma_1^2 = 2\int_0^1 \sigma_s^4 ds$.

Theorem 4. *We consider model (2.1) with no jump component and assume (A1), (A2) and (A3). As $h \to 0$, we have*

$$h^{-\frac{1}{2}}\mu_1^{-2}\left[\hat{B}_h^* - E^*\left(\hat{B}_h^*\right)\right] \xrightarrow{d^*} N\left(0, \sigma_2^2\right),$$

where $\sigma_2^2 = (\vartheta + 2)\int_0^1 \sigma_s^4 ds$.

Theorem 5. *We consider model (2.1) with no jump component and assume (A1), (A2) and (A3). As* $h \to 0$*, we have*

$$h^{-\frac{1}{2}}\left[\hat{Q}_h^* - E^*\left(\hat{Q}_h^*\right), \mu_1^{-2}\hat{B}_h^* - \mu_1^{-2}E^*\left(\hat{B}_h^*\right)\right]' \xrightarrow{d^*} N(0,\Sigma),$$

where Σ *is given in (7.2).*

The proof of Theorem 3 is given in Hwang and Shin (2013b). Once we have the result of Theorem 4, by the Cramér-Wold device, the bivariate normality result in Theorem 5 is obvious. Thus here we prove Theorem 4. The proof of our main theory in Theorem 1 follows from Theorem 5.

Proof of Theorem 4: Let

$$V_\tau^* = \mu_1^{-2}\sum_{i=1}^{s_\tau-1}\left|r_i^*\right|\left|r_{i+1}^*\right| \quad \text{and} \quad U_\tau = h\mu_1^{-2}\sum_{j=1}^{\tau} T_{I_j,L_j},$$

where $s_\tau = L_1 + \cdots + L_\tau$ and $T_{i,\ell} = h^{-1}\sum_{j=i}^{i+\ell-1}|r_{nj}||r_{n(j+1)}|$.

In order to prove Theorem 4, we will show the following five asymptotic results in Lemmas 1–5 below:

$$h^{-\frac{1}{2}}\left|V_\tau^* - U_\tau\right| \xrightarrow{p^*} 0, \qquad h^{-\frac{1}{2}}\left(V_\tau^* - \mu_1^{-2}\hat{B}_h^*\right) \xrightarrow{p^*} 0,$$
$$h^{-\frac{1}{2}}\left(U_\tau - \mu_1^{-2}\hat{B}_h^*\right) \xrightarrow{d^*} N\left(0, (\vartheta+2)\int_0^t \sigma_s^4 ds\right),$$

and

$$h^{-\frac{1}{2}}\left(E^*\left(\hat{B}_h^*\right) - \hat{B}_h\right) \xrightarrow{p} 0, \qquad \left|\mathrm{Var}^*\left(h^{-\frac{1}{2}}\mu_1^{-2}\hat{B}_h^*\right) - \mathrm{Var}\left(h^{-\frac{1}{2}}\mu_1^{-2}\hat{B}_h\right)\right| \xrightarrow{p} 0.$$

The following five lemmas are given under the same assumptions as in Theorem 4.

Lemma 1.

$$h^{-\frac{1}{2}}\left|V_\tau^* - U_\tau\right| \xrightarrow{p^*} 0.$$

Proof of Lemma 1: We observe that, denoting $I_{\tau+1} = 0$,

$$\begin{aligned}
h^{-\frac{1}{2}}\left|V_\tau^* - U_\tau\right| &= h^{-\frac{1}{2}}\mu_1^{-2}\left|\sum_{i=1}^{s_\tau-1}\left|r_i^*\right|\left|r_{i+1}^*\right| - \sum_{j=1}^{\tau}\left(\sum_{i=I_j}^{I_j+L_j-1}|r_{ni}|\left|r_{n(i+1)}\right|\right)\right| \\
&= h^{-\frac{1}{2}}\mu_1^{-2}\left|\sum_{j=1}^{\tau}\left(\sum_{i=I_j}^{I_j+L_j-2}|r_{ni}|\left|r_{n(i+1)}\right| + \left|r_{n(I_j+L_j-1)}\right|\left|r_{nI_{j+1}}\right|\right) - \sum_{j=1}^{\tau}\left(\sum_{i=I_j}^{I_j+L_j-1}|r_{ni}|\left|r_{n(i+1)}\right|\right)\right| \\
&\le h^{-\frac{1}{2}}\mu_1^{-2}\sum_{j=1}^{\tau}\left|\left|r_{n(I_j+L_j-1)}\right|\cdot\left|\left|r_{nI_{j+1}}\right| - \left|r_{n(I_j+L_j)}\right|\right|\right|.
\end{aligned}$$

Let $Y_j = |r_{n(I_j+L_j-1)}|\cdot\left||r_{nI_{j+1}}| - |r_{n(I_j+L_j)}|\right|$. Then $\{Y_j : j = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables since $\{(I_j, L_j) : j = 1, 2, \ldots\}$ are i.i.d. Note that $\tau = np + O_p(\sqrt{np})$ by Politis and Romano

(1994). It suffices to show that for any sequence m with $m/(np) \to 1$, $h^{-1/2}\sum_{j=1}^{m}|Y_j| \xrightarrow{p^*} 0$. For any $\epsilon > 0$ and $\delta > 0$, we have

$$P^*\left(h^{-\frac{1}{2}}\sum_{j=1}^{m}|Y_j| > \epsilon\right) \le \frac{1}{h^{1+\frac{\delta}{2}}\epsilon^{2+\delta}}E^*\left|\sum_{j=1}^{m}Y_j\right|^{2+\delta} = C\left(\frac{m}{h}\right)^{1+\frac{\delta}{2}}E^*|Y_j|^{2+\delta} \tag{7.3}$$

and observe

$$|Y_j| \le \left|r_{n(I_j+L_j-1)}\right|\left|r_{nI_{j+1}}\right| + \left|r_{n(I_j+L_j-1)}\right|\left|r_{n(I_j+L_j)}\right| \le 2\max_{1\le i,j\le n}|r_{ni}|\left|r_{nj}\right| \le 2\left(\max_{1\le i\le n}|r_{ni}|\right)^2 \text{ a.s.}$$

$$E^*|Y_j|^{2+\delta} \le C\left(\max_{1\le i\le n}|r_{ni}|\right)^{4+2\delta} = O_p\left(n^{-2-\delta}\right).$$

Therefore (7.3) is equal to $O_p((m/n)^{1+\delta/2})$, which tends to zero in probability as $n \to \infty$, and thus the desired convergence holds. □

Lemma 2.

$$h^{-\frac{1}{2}}\left(V_\tau^* - \mu_1^{-2}\hat{B}_h^*\right) \xrightarrow{p^*} 0.$$

Proof of Lemma 2: We observe $h^{-1/2}\left(V_\tau^* - \mu_1^{-2}\hat{B}_h^*\right) = h^{-1/2}\mu_1^{-2}\sum_{i=n+1}^{s_\tau}|r_{i-1}^*||r_i^*|$. Write

$$h^{-\frac{1}{2}}\sum_{i=n+1}^{s_\tau}\left|r_{i-1}^*\right|\left|r_i^*\right| = h^{\frac{1}{2}}\left[\sum_{i=n+1}^{s_\tau}\left(h^{-1}\left|r_{i-1}^*\right|\left|r_i^*\right| - \hat{B}_h\right)\right] + h^{\frac{1}{2}}(s_\tau - n)\hat{B}_h.$$

Let $\eta_1 = n - s_{\tau-1}$ and $\eta = L_\tau - \eta_1$, where $s_{\tau-1} = L_1 + \cdots + L_{\tau-1}$. Note that η, conditional on $(\eta_1, s_{\tau-1})$, has a geometric distribution with mean $1/p$ because of the memoryless property of the geometric distribution. Hence, $h^{1/2}[\sum_{i=n+1}^{s_\tau}(h^{-1}|r_{i-1}^*||r_i^*| - \hat{B}_h)]$ is equal in distribution to $h^{1/2}[T_{I-1,\eta} - \eta\hat{B}_h]$, where I is uniform on $\{1,\ldots,n\}$. It is enough to show that,

$$h^{\frac{1}{2}}\sum_{j=I}^{I+\eta-1}\left(h^{-1}\left|r_{n(j-1)}\right|\left|r_{nj}\right| - \hat{B}_h\right) \xrightarrow{p} 0 \quad\text{and}\quad h^{\frac{1}{2}}\eta\hat{B}_h \xrightarrow{p} 0. \tag{7.4}$$

For any $\epsilon > 0$,

$$P\left(h^{\frac{1}{2}}\sum_{j=I}^{I+\eta-1}\left(h^{-1}\left|r_{n(j-1)}\right|\left|r_{nj}\right| - \hat{B}_h\right) > \epsilon\right) = \frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=1}^{\infty}p(1-p)^{\ell-1}P\left(h^{\frac{1}{2}}\sum_{j=i}^{i+\ell-1}\left(h^{-1}\left|r_{n(j-1)}\right|\left|r_{nj}\right| - \hat{B}_h\right) > \epsilon\right).$$

We note that, for $\delta > 0$,

$$P\left(h^{\frac{1}{2}}\sum_{j=i}^{i+\ell-1}\left(h^{-1}\left|r_{n(j-1)}\right|\left|r_{nj}\right| - \hat{B}_h\right) > \epsilon\right) \le \frac{h^{1+\frac{\delta}{2}}}{\epsilon^{2+\delta}}E\left|T_{i-1,\ell} - \ell\hat{B}_h\right|^{2+\delta}.$$

It can be shown that $E|T_{i-1,\ell} - \ell\hat{B}_h|^{2+\delta} = C\ell^{1+\delta/2}$ by applying Minkowski inequality, (see Hwang and Shin, (2013b), Appendix (i), for a similar but detailed argument). Thus,

$$P\left(h^{\frac{1}{2}}\sum_{j=I}^{I+\eta-1}\left(h^{-1}\left|r_{n(j-1)}\right|\left|r_{nj}\right| - \hat{B}_h\right) > \epsilon\right) \le C\sum_{\ell=1}^{\infty}p(1-p)^{\ell-1}h^{1+\frac{\delta}{2}}\ell^{1+\frac{\delta}{2}} = C\left(\frac{h}{p}\right)^{1+\frac{\delta}{2}}$$

since $\sum_{\ell=1}^{\infty}(1-p)^{\ell-1}\ell^a = O(1/p^{a+1})$ for $a \geq 1$. The last term tends to zero since $h/p \to 0$, and thus the first convergence in probability in (7.4) holds. Now for the second in (7.4), we have

$$P\left(h^{\frac{1}{2}}\eta\hat{B}_h > \epsilon\right) = \sum_{\ell=1}^{\infty} p(1-p)^{\ell-1} P\left(h^{\frac{1}{2}}\ell\hat{B}_h > \epsilon\right) \leq \sum_{\ell=1}^{\infty} p(1-p)^{\ell-1}\frac{h\,\ell^2}{\epsilon^2}\mathrm{Var}\left(\hat{B}_h\right)$$
$$= \sum_{\ell=1}^{\infty} p(1-p)^{\ell-1}\frac{h\,\ell^2}{\epsilon^2}\left(h\mu_1^4(\vartheta+2)\int_0^1 \sigma_s^4 ds + o(1)\right) = C\frac{h^2}{p^2} + o\left(\frac{h}{p^2}\right) \to 0$$

since $h/p^2 \to 0$, where identity $\sum_{r=1}^{\infty}(1-p)^{r-1}r^2 = 2(1-p)/p^3$ is used. Thus the second convergence in probability of (7.4) holds, and the proof of Lemma 2 is completed. □

Lemma 3.

$$h^{-\frac{1}{2}}\left(U_\tau - \mu_1^{-2}\hat{B}_h\right) \xrightarrow{d^*} N\left(0, (\vartheta+2)\int_0^t \sigma_s^4\, ds\right).$$

Proof of Lemma 3: It suffices to show that for any sequence m with $m/(np) \to 1$,

$$h^{-\frac{1}{2}}\left(U_m - \frac{m}{np}\mu_1^{-2}\hat{B}_h\right) \xrightarrow{d^*} N\left(0, (\vartheta+2)\int_0^t \sigma_s^4\, ds\right) \tag{7.5}$$

since $\tau = np + O_p(\sqrt{np})$ by Politis and Romano (1994).

For $j = 1, \ldots, m$, let $Z_{n,j} = \sqrt{mh}\mu_1^{-2}T_{I_j,L_j}$. Note that $\bar{Z}_m = (1/m)\sum_{j=1}^m Z_{n,j}$ is the average of i.i.d. variables since $\{I_j\}$ and $\{L_j\}$ are i.i.d. Also, observing

$$E^*\left[T_{I_j,L_j}|L_j = \ell\right] = h^{-1}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=i}^{i+\ell-1}\left|r_{nj}\right|\left|r_{n(j+1)}\right| = h^{-1}\frac{\ell}{n}\sum_{i=1}^{n}|r_i|\,|r_{i+1}| \quad \text{with} \quad r_{n+1} = r_1,$$

we have

$$E^*\left[Z_{n,j}|L_j\right] = \frac{\sqrt{m}}{n\sqrt{h}}\mu_1^{-2}L_j\sum_{i=1}^{n}|r_i|\,|r_{i+1}| = \frac{\sqrt{m}}{n\sqrt{h}}\mu_1^{-2}L_j\hat{B}_h + \frac{\sqrt{m}}{n\sqrt{h}}\mu_1^{-2}L_j\,|r_1|\,|r_n|$$

and thus $E^*[\bar{Z}_m] = \sqrt{m}/(pn\sqrt{h})\mu_1^{-2}\hat{B}_h + \sqrt{m}/(pn\sqrt{h})\mu_1^{-2}|r_1||r_n|$, of which the second term is $O_p(1/\sqrt{nm})$ tending to 0 in probability as $n \to \infty$. Thus the left term of (7.5) is equal to $\sqrt{m}[\bar{Z}_m - E^*\bar{Z}_m] + o_p(1)$.

Now let $Z_{n,j}^* = Z_{n,j} - E^*Z_{n,j}$, and then

$$\sqrt{m}\left[\bar{Z}_m - E^*\bar{Z}_m\right] = \frac{1}{\sqrt{m}}\sum_{j=1}^{m} Z_{n,j}^*.$$

$\{Z_{n,j}^* : 1 \leq j \leq m\}$ are i.i.d. variables with mean zero under P^*. Thus we obtain, (for $\iota = \sqrt{-1}$),

$$E^*\left[e^{\iota\left(\frac{t}{\sqrt{m}}\right)\sum_{j=1}^m Z_{n,j}^*}\right] = \left(E^*\left[e^{\iota\left(\frac{t}{\sqrt{m}}\right)Z_{n,1}^*}\right]\right)^m = \left[1 + \frac{\iota t}{\sqrt{m}}E^*Z_{n,1}^* - \frac{t^2}{2m}(1+o(1))E^*\left(Z_{n,1}^*\right)^2\right]^m. \tag{7.6}$$

By Lemma 5 below, we have

$$h^{-1}\mathrm{Var}^*\left(\mu_1^{-2}\hat{B}_h^*\right) = \mathrm{Var}^*\left(Z_{n,1}^*\right) \xrightarrow{p} (\vartheta + 2)\int_0^1 \sigma_s^4 ds.$$

Therefor (7.6) tends to $\exp(-(1/2)t^2(\vartheta + 2)\int_0^1 \sigma_s^4 ds)$ in probability, and the desired asymptotic normality result in Lemma 3 is obtained. □

Lemma 4.

$$h^{-\frac{1}{2}}\mu_1^{-2}\left(E^*\left(\hat{B}_h^*\right) - \hat{B}_h\right) \xrightarrow{p} 0.$$

Proof of Lemma 4: By the stationarity, the result in this lemma is obvious. □

Lemma 5.

$$\left|Var^*\left(h^{-\frac{1}{2}}\mu_1^{-2}\hat{B}_h^*\right) - Var\left(h^{-\frac{1}{2}}\mu_1^{-2}\hat{B}_h\right)\right| \xrightarrow{p} 0.$$

Proof of Lemma 5: We show that $h^{-1}\mathrm{Var}^*(\hat{B}_h^*)$ and $h^{-1}\mathrm{Var}(\hat{B}_h)$ have the same limit. Let $X_i = h^{-1}|r_i||r_{i+1}|$ and $X_i^* = h^{-1}|r_i^*||r_{i+1}^*|$ for $i = 1, \ldots, n-1$. Then $\hat{B}_h^* = h\sum_{i=1}^{n-1} X_i^*$. Observe that

$$h^{-1}\mathrm{Var}^*\left(\hat{B}_h^*\right) = h^{-1}\mathrm{Var}^*\left(h\sum_{i=1}^{n-1} X_i^*\right) = v_X^*(0) + 2\sum_{i=1}^{n-2}\left(1 - \frac{i}{n-1}\right)v_X^*(i) + o_p(1),$$

where $v_X^*(i) = \mathrm{Cov}^*(X_1^*, X_{1+i}^*) = h^{-2}\mathrm{Cov}^*(|r_1^*||r_2^*|, |r_{1+i}^*||r_{2+i}^*|)$. □

Note that by the normality in (7.1) of Barndorff-Nielsen and Shephard (2006),

$$\lim_{n\to\infty} h^{-1}\mathrm{Var}\left(\mu_1^{-2}\hat{B}_h\right) = (\vartheta + 2)\int_0^1 \sigma_s^4 ds,$$

which can be expressed as the limiting of $\mu_1^{-4}[v_X(0) + 2\sum_{i=1}^{n-2} v_X(i)]$ where $v_X(i) = \mathrm{Cov}(X_1, X_{1+i}) = h^{-2}\mathrm{Cov}(|r_1||r_2|, |r_{1+i}||r_{2+i}|)$. That is, $h^{-1}\mathrm{Var}(\hat{B}_h) = v_X(0) + 2\sum_{i=1}^{n-2} v_X(i) + o(1)$. Thus, it suffices to show that

$$v_X^*(0) + 2\sum_{i=1}^{n-2}\left(1 - \frac{i}{n-1}\right)v_X^*(i) = v_X(0) + 2\sum_{i=1}^{n-2} v_X(i) + o_p(1). \tag{7.7}$$

By Politis and Romano (1994) or Nordman (2009) the left term of (7.7) is equal to

$$\hat{\gamma}(0) + 2\sum_{j=1}^{n-2}\left(\left(1 - \frac{i}{n-1}\right)(1-p)^i + \frac{i}{n-1}(1-p)^{n-1-i}\right)\hat{\gamma}(i) \tag{7.8}$$

where $\hat{\gamma}(i) = \{1/(n-1)\}\sum_{j=1}^{n-1}(X_j - \bar{X})(X_{j+i} - \bar{X})$ with $\bar{X} = \{1/(n-1)\}\sum_{j=1}^{n-1} X_j$. By similar arguments to those in Politis and Romano (1994) or Hwang and Shin (2012), the limit of (7.8) can be shown to be the same as the limit of the right term in (7.7) under the finite moment condition of (A3). Therefore, the result in Lemma 5 holds. □

Now we are ready to prove our main result in Theorem 1 of this paper, whose proof is given by Taylor's expansion with the bivariate normality result in Theorem 5.

Proof of Theorem 1: We apply Taylor's expansion to $\psi : \mathbb{R}^2 \to \mathbb{R}$, $\psi(x, y) = y/x$ and obtain

$$\begin{aligned} h^{-\frac{1}{2}}\left[\frac{\mu_1^{-2}\hat{B}_h^*}{\hat{Q}_h^*} - \frac{\mu_1^{-2}E^*\left(\hat{B}_h^*\right)}{E^*\left(\hat{Q}_h^*\right)}\right] &= -\frac{\mu_1^{-2}E^*\left(\hat{B}_h^*\right)}{\left(E^*\hat{Q}_h^*\right)^2}h^{-\frac{1}{2}}\left[\hat{Q}_h^* - E^*\left(\hat{Q}_h^*\right)\right] + \frac{1}{E^*\left(\hat{Q}_h^*\right)}h^{-\frac{1}{2}}\mu_1^{-2}\left[\hat{B}_h^* - E^*\left(\hat{B}_h^*\right)\right] \\ &\quad + o_p\left(\left[h^{-\frac{1}{2}}\left(\hat{Q}_h^* - E^*\left(\hat{Q}_h^*\right)\right)\right]^2 + \left[h^{-\frac{1}{2}}\mu_1^{-2}\left(\hat{B}_h^* - E^*\left(\hat{B}_h^*\right)\right)\right]^2\right) \\ &= -\frac{\mu_1^{-2}E^*\left(\hat{B}_h^*\right)}{\left(E^*\hat{Q}_h^*\right)^2}Z_1 + \frac{1}{E^*\left(\hat{Q}_h^*\right)}Z_2 + o_p(1), \end{aligned}$$

where$(Z_1, Z_2)'$ is the random vector with $N(0, \Sigma)$ distribution by Theorem 5. Observe $E^*(\hat{Q}_h^*) = \hat{Q}_h$ and $E^*(\hat{B}_h^*) = (n-1)E^*[|r_1^*||r_2^*|]$. Since $E^*[|r_1^*||r_2^*|] = E^*[|r_1^*||r_2^*||L_1 > 1]P(L_1 > 1) + E[|r_1^*||r_2^*||L_1 \le 1]P(L_1 \le 1) = (1/n)\sum_{j=1}^n |r_{nj}||r_{n(j+1)}|(1-p) + ((1/n)\sum_{j=1}^n |r_j|)^2 p = \{(1-p)/n\}\hat{B}_h + O_p(p/n)$, $E^*(\hat{B}_h^*) = \hat{B}_h + O_p(1/n) + O_p(p)$. Thus the limit of the conditional variance exists as:

$$\begin{aligned} &\mathrm{Var}^*\left(h^{-\frac{1}{2}}\left[\frac{\mu_1^{-2}\hat{B}_h^*}{\hat{Q}_h^*} - \frac{\mu_1^{-2}E^*\left(\hat{B}_h^*\right)}{E^*\left(\hat{Q}_h^*\right)}\right]\right) \\ &= \mathrm{Var}^*\left(-\frac{\mu_1^{-2}E^*\left(\hat{B}_h^*\right)}{\left(E^*\hat{Q}_h^*\right)^2}Z_1 + \frac{1}{E^*\left(\hat{Q}_h^*\right)}Z_2 + o_p(1)\right) \\ &= \frac{\mu_1^{-4}\left(\hat{B}_h + O_p(1/n) + O_p(p)\right)^2}{\hat{Q}_h^4}\sigma_1^2 + \frac{1}{\hat{Q}_h^2}\sigma_2^2 - 2\frac{\mu_1^{-2}\left(\hat{B}_h + O_p(1/n) + O_p(p)\right)}{\hat{Q}_h^3}\sigma_{12} + o_p(1) \\ &\to \frac{\mu_1^{-4}BV^2}{QV^4}\sigma_1^2 + \frac{1}{QV^2}\sigma_2^2 - 2\frac{\mu_1^{-2}BV}{QV^3}\sigma_{12}. \end{aligned}$$

Therefore the desired asymptotic normality holds and thus the asymptotic result in (i) is shown. Now in order to show (ii), it suffices to show that $\mathrm{Var}^*(\hat{B}_h^*) \xrightarrow{p} 0$ and $\mathrm{Var}^*(\hat{U}_h^*) \xrightarrow{p} 0$, by the Chebyshev inequality. The first convergence follows from Theorem 4. For the second convergence of $\hat{U}_h^*$, let $Y_i^* = h^{-2}|r_i^* r_{i+1}^* r_{i+2}^* r_{i+3}^*|$, $i = 1, 2, \ldots, n-3$, we observe

$$\begin{aligned} \mathrm{Var}^*\left(\hat{U}_h^*\right) &= \mathrm{Var}^*\left(h^{-1}\sum_{i=4}^n \left|r_i^* r_{i-1}^* r_{i-2}^* r_{i-3}^*\right|\right) = \mathrm{Var}^*\left(h\sum_{i=1}^{n-3} Y_i^*\right) \\ &= h^2(n-3)\left[v_Y^*(0) + 2\sum_{i=1}^{n-3}\left(1 - \frac{i}{n-3}\right)v_Y^*(i)\right], \end{aligned}$$

where $v_Y^*(i) = \mathrm{Cov}^*(Y_1^*, Y_{1+i}^*)$, $j = 0, 1, \ldots, n-3$. Similarly to the argument in the proof of Lemma 5, it can be shown that $v_Y^*(0) + 2\sum_{i=1}^{n-3}[1 - \{i/(n-3)\}]v_Y^*(i)$ has the same limit of $v_Y(0) + 2\sum_{i=1}^{n-3} v_Y(i)$ where $v_Y(i) = \mathrm{Cov}(Y_1, Y_{1+i})$ with $Y_i = h^{-2}|r_i r_{i+1} r_{i+2} r_{i+3}|$. Thus the limit of $v_Y^*(0) + 2\sum_{i=1}^{n-3}(1 - i/(n-3))v_Y^*(i)$ exists and hence $\mathrm{Var}^*(\hat{U}_h^*) \to 0$ as $h \to 0$. □

Proof of Theorem 2: Its proof is given in the same way as that in Theorem 3.2 of Hwang and Shin (2014), which is consistency of the i.i.d bootstrap test for the jumps via the result of Theorem 3.1 of Hwang and Shin (2014). The same argument but the stationary bootstrap asymptotic result in Theorem 3.1 in this paper are used.

Under the alternative hypothesis with jump component $\sum_{j=1}^{N_1} c_j^2 > 0$, $\hat{Q}_h - \mu_1^{-2}\hat{B}_h \xrightarrow{p} \sum_{j=1}^{N_1} c_j^2$, and

$$P\left(\hat{H} \leq H^*(\alpha) \middle| \sum_{j=1}^{N_1} c_j^2 > 0\right) = P\left(-\frac{h^{-\frac{1}{2}} \sum_{j=1}^{N_1} c_j^2}{\hat{Q}_h \sqrt{\vartheta \hat{U}_h / \hat{B}_h^2}} \leq H^*(\alpha)\right) + o_p(1),$$

which tends to $P(-\infty \leq H^*(\alpha)) = 1$ as $h \to 0$, provided $H^*(\alpha) = O_p(1)$.

We will complete the proof by showing that $H^*(\alpha) = O_p(1)$. For the empirical α-th quantile of m bootstrap values of (3.1), we observe the numerator of the form $h^{-1/2}[\mu_1^{-2}\hat{B}_h^*/\hat{Q}_h^* - \mu_1^{-2}\bar{B}_h^*/\bar{Q}_h^*]$, which is decomposed into

$$h^{-\frac{1}{2}}\left[\mu_1^{-2}\frac{\hat{B}_h^*}{\hat{Q}_h^*} - \mu_1^{-2}\frac{E^*\left(\hat{B}_h^*\right)}{E^*\left(\hat{Q}_h^*\right)}\right] + h^{-\frac{1}{2}}\left[\mu_1^{-2}\frac{E^*\left(\hat{B}_h^*\right)}{E^*\left(\hat{Q}_h^*\right)} - \mu_1^{-2}\frac{\bar{B}_h^*}{\bar{Q}_h^*}\right] =: A_1 + A_2.$$

It is clear that $A_1 = O_p(1)$. In order to observe A_2, we use similar arguments to those in Theorem 5. Then it can be shown that $h^{-1/2}[\bar{Q}_h^* - E^*(\hat{Q}_h^*), \mu_1^{-2}\bar{B}_h^* - \mu_1^{-2}E^*(\hat{B}_h^*)]$ follows asymptotically bivariate normality. Thus, similarly to the proof of Theorem 1, it follows that

$$h^{-\frac{1}{2}}\left[\frac{\mu_1^{-2}\bar{B}_h^*}{\bar{Q}_h^*} - \frac{\mu_1^{-2}E^*\left(\hat{B}_h^*\right)}{E^*\left(\hat{Q}_h^*\right)}\right] = O_p(1),$$

and thus $A_2 = O_p(1)$. □

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References

Aït-Sahalia Y (2002). Telling from discrete data whether the underlying continuous-time model is a diffusion, *Journal of Finance*, **57**, 2075–2112.

Aït-Sahalia Y and Jacod J (2009). Testing for jumps in a discretely observed process, *Annals of Statistics*, **37**, 184–222.

Aït-Sahalia Y, Jacod J, and Li J (2012). Testing for jumps in noisy high frequency data, *Journal of Econometrics*, **168**, 207–222.

Andersen TG, Benzoni L and Lund J (2002). An empirical investigation of continuous-time equity return models, *The Journal of Finance*, **57**, 1239–1284.

Barndorff-Nielsen OE and Shephard N (2004). Power and bipower variation with stochastic volatility and jumps, *Journal of Financial Econometrics*, **2**, 1–37.

Barndorff-Nielsen OE and Shephard N (2006). Econometrics of testing for jumps in financial economics using bipower variation, *Journal of Financial Econometrics*, **4**, 1–30.

Bollerslev T and Zhou H (2002). Estimating stochastic volatility diffusion using conditional moments of integrated volatility, *Journal of Econometrics*, **109**, 33–65.

Carr P and Wu L (2003). What type of process underlies options? A simple robust test, *Journal of Finance*, **58**, 2581–2610.

Dovonon P, Gonçalves S, Hounyo U, and Meddahi N (2014). Bootstrapping high-frequency jump tests, In *Proceedings of International Association for Applied Econometrics (IAAE)*, London, 1–36.

Dovonon P, Gonçalves S, and Meddahi N (2013). Bootstrapping realized multivariate volatility measures, *Journal of Econometrics*, **172**, 49–65.

Gonçalves S and Meddahi N (2009). Bootstrapping realized volatility, *Econometrics*, **77**, 283–306.

Huang X and Tauchen G (2005). The relative contribution of jumps to total price variance, *Journal of Financial Econometrics*, **3**, 456–499.

Hwang E and Shin DW (2012). Strong consistency of the stationary bootstrap under ψ-weak dependence, *Statistics & Probability Letters*, **82**, 488–495.

Hwang E and Shin DW (2013a). Stationary bootstrapping realized volatility under market microstructure noise, *Electronic Journal of Statistics*, **7**, 2032–2053.

Hwang E and Shin DW (2013b). Stationary bootstrapping realized volatility, *Statistics & Probability Letters*, **83**, 2045–2051.

Hwang E and Shin DW (2014). A bootstrap test for jumps in financial economics, *Economics Letters*, **125**, 74–78.

Jacod J and Reiss M (2014). A remark on the rates of convergence for integrated volatility estimation in the presence of jumps, *Annals of Statistics*, **42**, 1131–1144.

Jacod J and Todorov V (2014). Efficient estimation of integrated volatility in presence of infinite variation jumps, *The Annals of Statistics*, **42**, 1029–1069.

Jing BY, Liu Z, and Kong XB (2014). On the estimation of integrated volatility with jumps and microstructure noise, *Journal of Business & Economic Statistics*, **32**, 457–467.

Lee SS and Mykland PA (2008). Jumps in financial markets: a new nonparametric test and jump dynamics, *Review of Financial Studies*, **21**, 2535–2563.

Nordman DJ (2009). A note on the stationary bootstrap's variance, *Annals of Statistics*, **37**, 359–370.

Podolskij M and Vetter M (2009). Estimation of volatility functionals in the simultaneous presence of microstructure noise and jumps, *Bernoulli*, **15**, 634–658.

Politis DN and Romano JP (1994). The stationary bootstrap, *Journal of the American Statistical Association*, **89**, 1303–1313.

Shin DW and Hwang E (2013). Stationary bootstrapping for cointegrating regressions, *Statistics & Probability Letters*, **83**, 474–480.

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