

# New generalized inverse Weibull distribution for lifetime modeling

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## Abstract

This paper introduces the four parameter new generalized inverse Weibull distribution and investigates the potential usefulness of this model with application to reliability data from engineering studies. The new extended model has upside-down hazard rate function and provides an alternative to existing lifetime distributions. Various structural properties of the new distribution are derived that include explicit expressions for the moments, moment generating function, quantile function and the moments of order statistics. The estimation of model parameters are performed by the method of maximum likelihood and evaluate the performance of maximum likelihood estimation using simulation.

**Keywords:** reliability functions, moment estimation, moment generating function, order statistics, maximum likelihood estimation

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## 1. Introduction

Hundreds of lifetime distributions are developed in statistics literature and commonly used to describe real world phenomena. Due to the effectiveness of this theory, many new families of lifetime distributions are developed in statistics literature. A common feature of these new class of distributions is that they have more parameters and the model adequacy of the new generalized distribution performs better than the baseline distribution. However there are still many real world scenarios where the classical model does not adequately fit the real data; therefore, many new lifetime distributions are required to describe the real world phenomena. The inverse Weibull distribution is the lifetime probability distribution which is used in the reliability engineering, bio-engineering and many other areas of biological disciplines. Keller *et al.* (1982) introduced the inverse Weibull distribution for modelling reliability data and failures of mechanical components subject to degradation. de Gusmão *et al.* (2011) addressed the inverse Weibull model as the limiting distribution of the largest order statistics that is also known as reciprocal Weibull distribution. In recent statistical literature modified inverse Weibull distribution have been proposed by Khan and King (2012) to present a comprehensive description of mathematical properties along with reliability behavior. The cumulative distribution function (cdf) of the modified inverse Weibull distribution is given by

$$G(x) = \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\}, \quad x > 0, \quad (1.1)$$

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where  $\eta > 0$  is the shape parameters and  $\alpha, \theta > 0$  are the scale parameters. The probability density function (pdf) corresponding to (1.1) is given by

$$g(x) = \left( \alpha + \eta \theta \left( \frac{1}{x} \right)^{\eta-1} \right) \left( \frac{1}{x} \right)^2 \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\}, \quad x > 0. \quad (1.2)$$

Modified extensions of the Weibull and exponential distributions have recently been proposed in literature on statistical theory to obtain better estimates. Generalized exponential distribution was introduced by Gupta and Kundu (1999) as a generalization of the standard exponential distribution. Nadarajah and Kotz (2003) proposed an exponentiated Frechet distribution to generalize the standard Frechet distribution and studied some of its mathematical properties. Khan *et al.* (2008) studied the flexibility of the inverse Weibull distribution. de Gusmão *et al.* (2011) proposed the generalized inverse Weibull distribution and discussed several properties of this model with applications. Khan and King (2014) proposed the new class of transmuted inverse Weibull distribution with application to reliability data. Khan *et al.* (2014) studied characterizations of the transmuted Inverse Weibull distribution with an application to bladder cancer remission times data. Jiang *et al.* (1999) presented Weibull and Weibull inverse mixture models. Jiang *et al.* (2001) discussed the models involving two inverse Weibull distributions. Sultan *et al.* (2007) discussed the mixture of two inverse Weibull distributions. In recent literature modified extensions of the Weibull distributions for modelling bathtub instantaneous failure rates have been proposed by many authors. Such as, Cordeiro *et al.* (2013) introduced the beta exponentiated Weibull distribution and studied various statistical properties along with applications. The beta modified Weibull distribution studied by Silva *et al.* (2010) included 17 distributions as special cases and studied various structural properties with applications. Hemmati *et al.* (2011) proposed the three parameter Weibull-Poisson distribution as an aging class distribution. We are motivated to introduce the new generalized inverse Weibull distribution because of the above generalizations in the exponentiated family of lifetime distributions. This research introduces a new four parameter distribution, which contains nine lifetime distributions as special sub-models. This paper deliberates the comprehensive description of mathematical properties of the new model and presents a graphical analysis of some of its properties.

The article is organized as follows. In Section 2, we present the special sub-models to illustrate the analytical shapes of the probability density and hazard functions of the new generalized inverse Weibull (NGIW) distribution. In Section 3, we derived the quantile function, moment estimation and moment generating function. In Section 4, we address the order statistics and the moments of order statistics. Maximum likelihood estimations (MLEs) of the unknown parameters with asymptotic confidence intervals of the parameters are discussed in Section 5. In Section 6, we evaluate the performance of a maximum likelihood method by using Monte Carlo simulation. In Section 7, we illustrate the usefulness of the new extended model. Concluding remarks are addressed in Section 8.

## 2. New generalized inverse Weibull distribution

A non-negative random variable  $X$  is said to have NGIW distribution with four parameters  $\alpha, \eta, \theta, \beta > 0$  and  $x > 0$ , its cdf and pdf are given by

$$F(x) = 1 - \left[ 1 - \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\} \right]^\beta, \quad x > 0, \quad (2.1)$$

Table 1: Sub-models of the new generalized inverse Weibull distribution

Distribution	$\alpha$	$\eta$	$\theta$	$\beta$
Modified inverse Weibull	-	-	-	1
Modified inverse Rayleigh	-	2	-	1
Modified inverse exponential	-	1	-	1
Generalized inverse Weibull	0	-	-	-
Generalized inverse Rayleigh	0	2	-	-
Generalized inverse exponential	0	1	-	-
Inverse Weibull	0	-	-	1
Inverse Rayleigh	0	2	-	1
Inverse exponential	0	1	-	1

and

$$f(x) = \beta \left( \alpha + \eta \theta \left( \frac{1}{x} \right)^{\eta-1} \right) \left( \frac{1}{x} \right)^2 \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\} \left[ 1 - \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\} \right]^{\beta-1} \tag{2.2}$$

respectively, where  $\eta > 0$  and  $\beta > 0$  are the shape parameters and  $\alpha > 0$  and  $\theta > 0$  are the scale parameters of the subject distribution. The survival function, hazard function and cumulative hazard function of the NGIW distribution are given by

$$R(x) = \left[ 1 - \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\} \right]^\beta, \tag{2.3}$$

$$h(x) = \frac{\beta \left( \alpha + \eta \theta \left( \frac{1}{x} \right)^{\eta-1} \right) \left( \frac{1}{x} \right)^2 \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\}}{1 - \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\}}, \tag{2.4}$$

$$H(x) = -\ln \left[ 1 - \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\} \right]^\beta. \tag{2.5}$$

Let  $X$  be a random variable with density function (2.2), we write  $X \sim \text{NGIW}(x; \alpha, \theta, \eta, \beta)$ . Figure 1 shows the diverse shape of the NGIW pdf with different choice of parameters that include some well-known distributions. When  $\eta, \beta \geq 1$  the NGIW distribution becomes unimodal. As the random variable  $x \rightarrow \infty$  the density of the NGIW distribution tends to zero. Figure 1 illustrates some possible shapes of the instantaneous failure rate function for some selected choices of parameters for the NGIW model. A characteristic of the NGIW distribution shows that the distribution has upside-down bathtub shape hazard rate function for all choice of parameters. It is notable that the parameter  $\beta$  increases as the behavior of the instantaneous failure rate strictly increases then gradually decreases. The NGIW distribution contains several well-known distributions as special cases when its parameters change. Table 1 demonstrates the sub-models of the NGIW distribution.

### 3. Moments and quantiles

This section is devoted for studying statistical properties of NGIW distribution such as moments, moment generating function and quantile analysis.

**Theorem 1.** *If  $X$  has the NGIW( $x; \alpha, \theta, \eta, \beta$ ), then the  $k^{\text{th}}$  moment of  $X$ ,  $\mu_k$  is given as follows*

$$\dot{\mu}_k = \sum_{i,j=0}^{\infty} \binom{\beta-1}{j} \frac{\alpha \beta h_{i,j,\theta} \Gamma(i\eta - k + 1)}{\alpha_j^{i\eta - k + 1}} + \sum_{i,j=0}^{\infty} \binom{\beta-1}{j} \frac{\eta \beta \theta h_{i,j,\theta} \Gamma(\eta(i+1) - k)}{\alpha_j^{\eta(i+1) - k}},$$

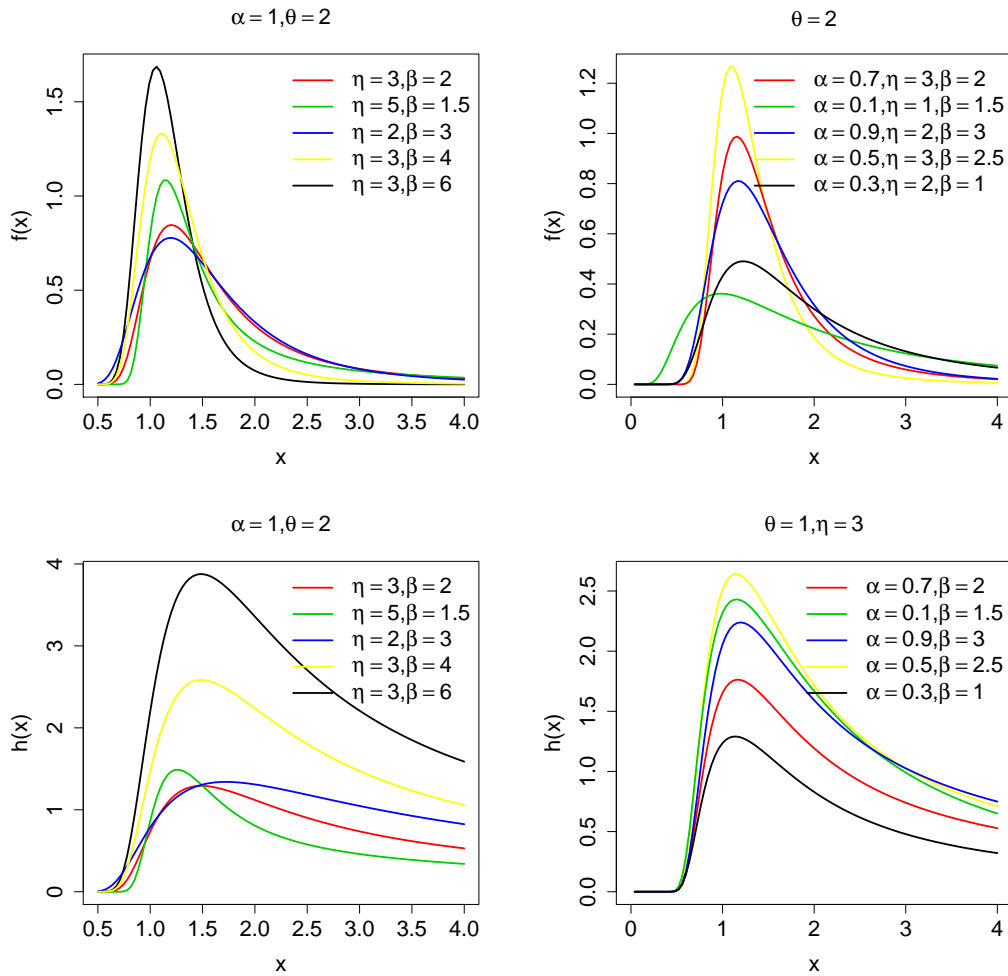


Figure 1: Plots of the NGIW pdf and hf for some parameter values (NGIW = new generalized inverse Weibull, pdf = probability density function, hf = hazard function).

where

$$h_{i,j,\theta} = \frac{(j+1)^i \theta^j (-1)^{i+j}}{i!} \quad \text{and} \quad \alpha_j = \alpha(j+1).$$

**Proof:** The  $k^{th}$  moment of the NGIW distribution as follows

$$\dot{\mu}_k = \int_0^\infty x^k \beta \left( \alpha + \eta \theta \left( \frac{1}{x} \right)^{\eta-1} \right) \left( \frac{1}{x} \right)^2 \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\} \left[ 1 - \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^\eta \right\} \right]^{\beta-1} dx$$

using the series expansion, the above equation reduces to

$$\dot{\mu}_k = \sum_{j=0}^\infty \binom{\beta-1}{j} (-1)^j \beta \int_0^\infty x^{k-2} \left( \alpha + \eta \theta \left( \frac{1}{x} \right)^{\eta-1} \right) \exp \left\{ -(j+1) \left( \frac{\alpha}{x} + \theta \left( \frac{1}{x} \right)^\eta \right) \right\} dx,$$

using the following expansions  $\exp\{-(j+1)\theta(1/x)^\eta\}$  given by

$$\exp\left\{-(j+1)\theta\left(\frac{1}{x}\right)^\eta\right\} = \sum_{i=0}^{\infty} \frac{(j+1)^i \theta^i (-1)^i}{i!} \left(\frac{1}{x}\right)^{i\eta}$$

using  $h_{i,j,\theta}$  and  $\alpha_j$ , as defined above

$$\mu_k = \sum_{i,j=0}^{\infty} \binom{\beta-1}{j} h_{i,j,\theta} \alpha_j \int_0^{\infty} x^{k-i\eta-2} \exp\left(-\frac{\alpha_j}{x}\right) dx + \sum_{i,j=0}^{\infty} \binom{\beta-1}{j} h_{i,j,\theta} \beta \eta \theta \int_0^{\infty} x^{k-\eta(i+1)-1} \exp\left(-\frac{\alpha_j}{x}\right) dx.$$

Finally, we obtain

$$\mu_k = \sum_{i,j=0}^{\infty} \binom{\beta-1}{j} \frac{\alpha_j \beta h_{i,j,\theta} \Gamma(i\eta - k + 1)}{\alpha_j^{i\eta - k + 1}} + \sum_{i,j=0}^{\infty} \binom{\beta-1}{j} \frac{\eta \beta \theta h_{i,j,\theta} \Gamma(\eta(i+1) - k)}{\alpha_j^{\eta(i+1) - k}}. \quad (3.1)$$

□

**Theorem 2.** If  $X$  has the  $NGIW(x; \alpha, \theta, \eta, \beta)$ , then the moment generating function of  $X$ , is given as follows

$$M_x(t) = \sum_{i,j=0}^{\infty} \sum_{m=0}^{\infty} \binom{\beta-1}{j} \frac{\alpha_j \beta h_{i,j,\theta} t^m \Gamma(i\eta - m + 1)}{m! \alpha_j^{i\eta - m + 1}} + \sum_{i,j=0}^{\infty} \sum_{m=0}^{\infty} \binom{\beta-1}{j} \frac{\eta \beta \theta h_{i,j,\theta} t^m \Gamma(\eta(i+1) - m)}{m! \alpha_j^{\eta(i+1) - m}},$$

where

$$h_{i,j,\theta} = \frac{(j+1)^i \theta^i (-1)^{i+j}}{i!} \quad \text{and} \quad \alpha_j = \alpha(j+1).$$

**Proof:** We have the moment generating function of the  $NGIW$  distribution as follows

$$M_x(t) = \int_0^{\infty} \frac{\beta}{x^2} \left( \alpha + \eta \theta \left(\frac{1}{x}\right)^{\eta-1} \right) \exp\left\{tx - \frac{\alpha}{x} - \theta \left(\frac{1}{x}\right)^\eta\right\} \left[1 - \exp\left\{-\frac{\alpha}{x} - \theta \left(\frac{1}{x}\right)^\eta\right\}\right]^{\beta-1} dx.$$

For mathematical tractability, using the series expansion, we obtain

$$M_x(t) = \sum_{j=0}^{\infty} \binom{\beta-1}{j} (-1)^j \int_0^{\infty} \frac{\beta}{x^2} \left( \alpha + \eta \theta \left(\frac{1}{x}\right)^{\eta-1} \right) \exp\left\{tx - (j+1) \left(\frac{\alpha}{x} + \theta \left(\frac{1}{x}\right)^\eta\right)\right\} dx,$$

the above integral reduces to

$$M_x(t) = \sum_{i,j=0}^{\infty} \binom{\beta-1}{j} h_{i,j,\theta} \alpha_j \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{\infty} x^{m-i\eta-2} \exp\left(-\frac{\alpha_j}{x}\right) dx + \sum_{i,j=0}^{\infty} \binom{\beta-1}{j} h_{i,j,\theta} \beta \eta \theta \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{\infty} x^{m-\eta(i+1)-1} \exp\left(-\frac{\alpha_j}{x}\right) dx.$$

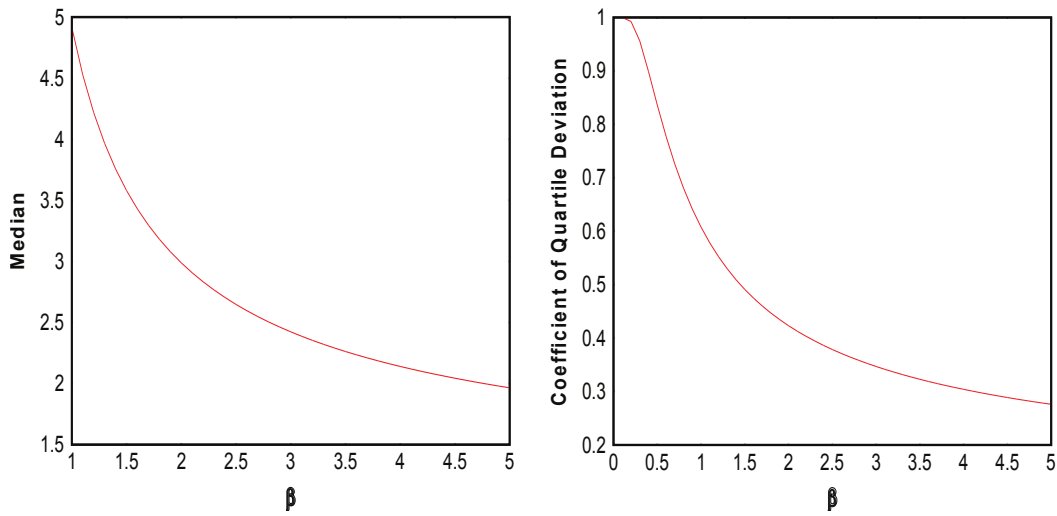


Figure 2: Median and coefficient of quartile deviation of the new generalized inverse Weibull distribution.

Finally, we obtain

$$M_x(t) = \sum_{i,j=0}^{\infty} \sum_{m=0}^{\infty} \binom{\beta-1}{j} \frac{\alpha \beta h_{i,j,\theta} t^m \Gamma(i\eta - m + 1)}{m! \alpha_j^{i\eta - m + 1}} + \sum_{i,j=0}^{\infty} \sum_{m=0}^{\infty} \binom{\beta-1}{j} \frac{\eta \beta \theta h_{i,j,\theta} t^m \Gamma(\eta(i+1) - m)}{m! \alpha_j^{\eta(i+1) - m}}. \quad (3.2)$$

□

The quantile  $x_q$  of the NGIW distribution is the real solution of the following equation

$$\theta \left( \frac{1}{x_q} \right)^{\theta} + \frac{\alpha}{x_q} + \ln \left\{ 1 - (1 - u)^{\frac{1}{\beta}} \right\} = 0. \quad (3.3)$$

By substituting  $u = 0.5$  in equation (3.3), we obtain the median of the NGIW distribution.

Figure 2 shows the median of the NGIW distribution as a function of  $\beta$  for values of the parameters  $\alpha = 3, \eta = 2$  and  $\theta = 2$ . Figure 2 also shows the coefficient of quartile deviation to illustrate the effect of  $\eta$  as a function of  $\beta$  for values of the parameters  $\alpha = 3, \eta = 2$  and  $\theta = 2$ . To illustrate the effect of shape parameter  $\eta$  as a function of  $\beta$  on skewness and kurtosis we consider the measure based on quantiles. Graphical representation of the Bowley skewness and percentile kurtosis when  $\alpha = 3, \eta = 2$  and  $\theta = 2$ , as a function of  $\beta$  are illustrated in Figure 3, respectively.

#### 4. Order statistics

Let  $X_1, X_2, \dots, X_n$  are independently identically distributed ordered random variables from the NGIW ( $x; \alpha, \theta, \eta, \beta$ ) distribution having the probability density function is given by

$$f_{r:n}(x) = \frac{(F(x))^{r-1} (1 - F(x))^{n-r} f(x)}{B(r, n - r + 1)}, \quad x > 0. \quad (4.1)$$

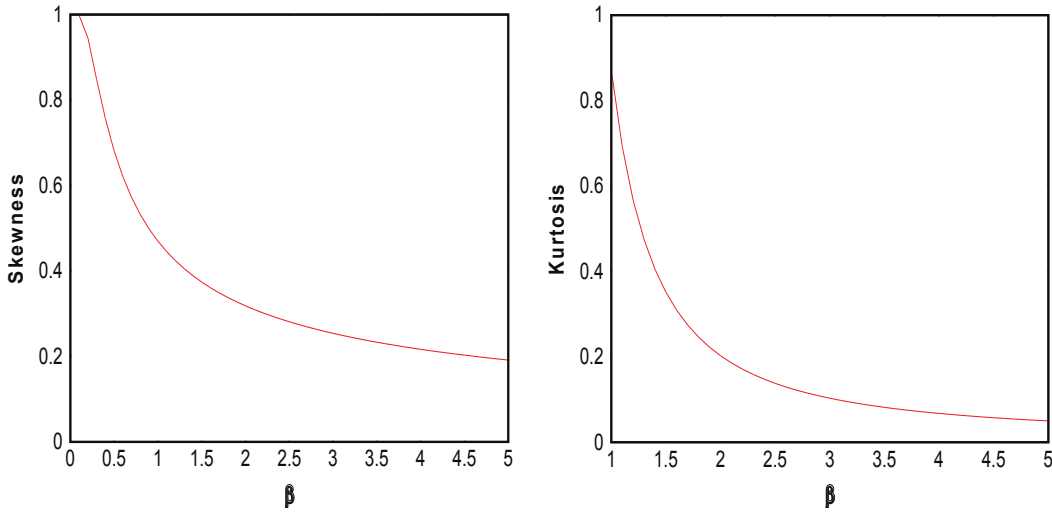


Figure 3: Bowley skewness and percentile kurtosis of the new generalized inverse Weibull distribution.

By substituting (2.1) and (2.2) in (4.1), we obtain

$$f_{r:n}(x) = n \binom{n-1}{r-1} \sum_{p=0}^{n-r} \binom{n-r}{p} (-1)^p \left[ 1 - \sum_{j=0}^{\infty} \binom{\beta-1}{j} \beta \exp \left\{ -(j+1) \left( \frac{\alpha}{x} + \theta \left( \frac{1}{x} \right)^\eta \right) \right\} \right]^{r+p-1} \times \sum_{j=0}^{\infty} \binom{\beta-1}{j} \beta \left( \alpha + \eta \theta \left( \frac{1}{x} \right)^{\eta-1} \right) \left( \frac{1}{x} \right)^2 \exp \left\{ -(j+1) \left( \frac{\alpha}{x} + \theta \left( \frac{1}{x} \right)^\eta \right) \right\}, \quad (4.2)$$

using the following expansions  $\exp \{-(j+1)\theta(1/x)^\eta\}$  given by

$$\exp \left\{ -(j+1)\theta \left( \frac{1}{x} \right)^\eta \right\} = \sum_{q=0}^{\infty} \frac{(j+1)^q \theta^q (-1)^q}{q!} \left( \frac{1}{x} \right)^{q\eta}.$$

Setting  $C_{q,j} = (j+1)^q \theta^q (-1)^q / q!$  and  $\alpha_j = \alpha(j+1)$ , we obtain

$$f_{r:n}(x) = n \binom{n-1}{r-1} \sum_{p=0}^{n-r} \sum_{m,j=0}^{\infty} \lambda_{p,m,j} \left\{ \sum_{q=0}^{\infty} \frac{\alpha C_{q,j}}{x^{\eta q+2}} \exp \left( -\frac{\alpha_j}{x} \right) + \sum_{q=0}^{\infty} \frac{\eta \theta C_{q,j}}{x^{\eta(q+1)+1}} \exp \left( -\frac{\alpha_j}{x} \right) \right\}, \quad (4.3)$$

where

$$\lambda_{p,m,j} = \binom{n-r}{p} \binom{r+p-1}{m} \binom{\beta(m+1)-1}{j} (-1)^{p+m+j}.$$

Using equation (4.3), we can obtain the statistical regular properties of the NGIW order statistics. Such as moments, mgf, mean deviation among others. Using (4.3), the  $\kappa^{th}$  moment of the  $r^{th}$  order statistics  $X_{(r)}$  is given by

$$\mu_k^{n:r} = n \binom{n-1}{r-1} \sum_{p=0}^{n-r} \sum_{m,j=0}^{\infty} \lambda_{p,m,j} \left\{ \sum_{q=0}^{\infty} \frac{\alpha C_{q,j} \Gamma(\eta q - \kappa + 1)}{\alpha_j^{\eta q - \kappa + 1}} + \sum_{q=0}^{\infty} \frac{\eta \theta C_{q,j} \Gamma(\eta(q+1) - \kappa)}{\alpha_j^{\eta(q+1) - \kappa}} \right\}. \quad (4.4)$$

### 5. Maximum likelihood estimation

Consider the random samples  $x_1, x_2, \dots, x_n$  consisting of  $n$  observations from the NGIW distribution and  $\varphi = (\alpha, \eta, \theta, \beta)^T$  be the parameter vector. The log-likelihood function of (2.2) is given by

$$\begin{aligned} \log L = & n \log \beta + \sum_{i=1}^n \log \left( \alpha + \eta \theta \left( \frac{1}{x_i} \right)^{\eta-1} \right) + \sum_{i=1}^n \log \left( \frac{1}{x_i} \right)^2 - \sum_{i=1}^n \left( \frac{\alpha}{x_i} \right) - \theta \sum_{i=1}^n \left( \frac{1}{x_i} \right)^\eta \\ & + (\beta - 1) \sum_{i=1}^n \log \left\{ 1 - \exp \left\{ -\frac{\alpha}{x_i} - \theta \left( \frac{1}{x_i} \right)^\eta \right\} \right\}, \end{aligned} \quad (5.1)$$

respectively. By taking the partial derivatives of the log-likelihood function with respect to  $\alpha, \eta, \theta$  and  $\beta$ , we obtain the components of the score vector  $U(\varphi)$  then equating it to zero, we obtain the estimating equations

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha} &= \sum_{i=1}^n \left\{ \alpha + \eta \theta \left( \frac{1}{x_i} \right)^{\eta-1} \right\}^{-1} - \sum_{i=1}^n \left( \frac{1}{x_i} \right) + (\beta - 1) \sum_{i=1}^n \frac{\exp \left\{ -\frac{\alpha}{x_i} - \theta \left( \frac{1}{x_i} \right)^\eta \right\} \left( \frac{1}{x_i} \right)}{\left\{ 1 - \exp \left\{ -\frac{\alpha}{x_i} - \theta \left( \frac{1}{x_i} \right)^\eta \right\} \right\}}, \\ \frac{\partial \log L}{\partial \eta} &= \sum_{i=1}^n \frac{\theta \left( \frac{1}{x_i} \right)^{\eta-1} \left\{ \eta \log \left( \frac{1}{x_i} \right) + 1 \right\}}{\left\{ \alpha + \eta \theta \left( \frac{1}{x_i} \right)^{\eta-1} \right\}} - \theta \sum_{i=1}^n \left( \frac{1}{x_i} \right)^\eta \log \left( \frac{1}{x_i} \right) + (\beta - 1) \sum_{i=1}^n \frac{\exp \left\{ -\frac{\alpha}{x_i} - \theta \left( \frac{1}{x_i} \right)^\eta \right\} \theta \left( \frac{1}{x_i} \right)^\eta \log \left( \frac{1}{x_i} \right)}{\left\{ 1 - \exp \left\{ -\frac{\alpha}{x_i} - \theta \left( \frac{1}{x_i} \right)^\eta \right\} \right\}}, \\ \frac{\partial \log L}{\partial \theta} &= \sum_{i=1}^n \frac{\eta \left( \frac{1}{x_i} \right)^{\eta-1}}{\left\{ \alpha + \eta \theta \left( \frac{1}{x_i} \right)^{\eta-1} \right\}} - \sum_{i=1}^n \left( \frac{1}{x_i} \right)^\eta + (\beta - 1) \sum_{i=1}^n \frac{\exp \left\{ -\frac{\alpha}{x_i} - \theta \left( \frac{1}{x_i} \right)^\eta \right\} \left( \frac{1}{x_i} \right)^\eta}{\left\{ 1 - \exp \left\{ -\frac{\alpha}{x_i} - \theta \left( \frac{1}{x_i} \right)^\eta \right\} \right\}}, \end{aligned}$$

and

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n \log \left\{ 1 - \exp \left\{ -\frac{\alpha}{x_i} - \theta \left( \frac{1}{x_i} \right)^\eta \right\} \right\}.$$

The MLEs can be determined numerically from the solution of nonlinear system of equations; subsequently, these solutions will yield the ML estimators  $\hat{\alpha}, \hat{\eta}, \hat{\theta}$  and  $\hat{\beta}$ . We required the observed information matrix for the interval estimation and hypothesis testing. For the four parameters NGIW distribution pdf all the second order derivatives exist. Thus we have the observed information matrix as

$$V^{-1} = -E \begin{pmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \eta} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \eta} & \frac{\partial^2 \log L}{\partial \eta^2} & \frac{\partial^2 \log L}{\partial \eta \partial \theta} & \frac{\partial^2 \log L}{\partial \eta \partial \beta} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \theta} & \frac{\partial^2 \log L}{\partial \eta \partial \theta} & \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta \partial \beta} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \beta} & \frac{\partial^2 \log L}{\partial \eta \partial \beta} & \frac{\partial^2 \log L}{\partial \theta \partial \beta} & \frac{\partial^2 \log L}{\partial \beta^2} \end{pmatrix} \quad (5.2)$$

respectively. Equation (5.2) is the expected information matrix of the NGIW distribution. Solving the observed information matrix will provide solutions for asymptotic variance and co-variances of these



Table 2: Mean, standard error (S.E), bias and MSE of the NGIW distribution

$n$	Parameter	Mean	S.E	Bias	MSE
100	$\alpha$	0.1360	0.3237	-0.8640	0.8512
	$\eta$	1.9230	0.3064	-0.0770	0.0998
	$\theta$	2.5948	0.4780	0.5948	0.5822
	$\beta$	0.3706	0.1183	-0.1294	0.0307
200	$\alpha$	1.4680	0.4008	0.4680	0.3796
	$\eta$	3.1461	0.7602	1.1461	1.8914
	$\theta$	1.5525	0.4483	-0.4475	0.4012
	$\beta$	0.5352	0.0617	0.0352	0.0050
300	$\alpha$	1.4213	0.5240	0.4213	0.4520
	$\eta$	2.6869	0.7230	0.6869	0.9945
	$\theta$	1.8573	0.4822	-0.1427	0.2528
	$\beta$	0.5705	0.0710	0.0705	0.0100
400	$\alpha$	0.7295	0.5741	-0.2705	0.4027
	$\eta$	2.2364	0.3782	0.2364	0.1989
	$\theta$	2.7360	0.4451	0.7360	0.7398
	$\beta$	0.4399	0.0702	-0.0601	0.0085
500	$\alpha$	0.3957	0.3176	-0.6043	0.4660
	$\eta$	2.0257	0.1673	0.0257	0.0286
	$\theta$	2.1367	0.2539	0.1367	0.0831
	$\beta$	0.4165	0.0572	-0.0835	0.0102

MSE = mean square error, NGIW = new generalized inverse Weibull.

ML estimators for  $\hat{\alpha}, \hat{\eta}, \hat{\theta},$  and  $\hat{\beta}$ . By using (5.2) the confidence intervals for the parameters  $\alpha, \eta, \theta$  and  $\beta$  are approximately  $100(1 - \gamma)\%$  can be determined as

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{11}}, \quad \hat{\eta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{22}}, \quad \hat{\theta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{33}}, \quad \hat{\beta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{44}},$$

where  $Z_{\gamma/2}$  is the upper  $\gamma^{th}$  percentile of the standard normal distribution.

### 6. Simulation

In this section we evaluate the performance of the MLEs for the NGIW distribution. We consider simulation values of a random variable  $X$  using the quantile function in equation (3.3). Let  $U$  denote the uniform random variable over the interval  $(0, 1)$ . We perform a simulation study in order to evaluate the mean estimates, standard error (S.E), bias and mean square error (MSE). We generate random samples of size  $n = 100, 200, 300, 400, 500$  using a Monte Carlo simulation. For this study we consider the fixed choice of parameter values  $\alpha = 1, \eta = 2, \theta = 2, \beta = 0.5$ .

The simulation process is repeated for 1,000 times using the BFGS optimization method in R and the parameter estimates obtained by optimum routine (Table 2). Results from Table 2 shows that the bias and MSE does not provide satisfactory estimates for small sample sizes. These simulated results suggest that as the sample size  $n$  increases the method of MLEs does provide better estimates. Figure 4, illustrates the exact densities and histogram of the NGIW distribution from two simulated data sets for some selected values of parameters.

### 7. Application

In this section we provide a data analysis in order to assess the goodness-of-fit of a model with failure times for the air conditioning system of an aircraft from a random sample of 30 observations to see how the new model works in practice. The data have been obtained from (Linhart and Zucchini, 1986,

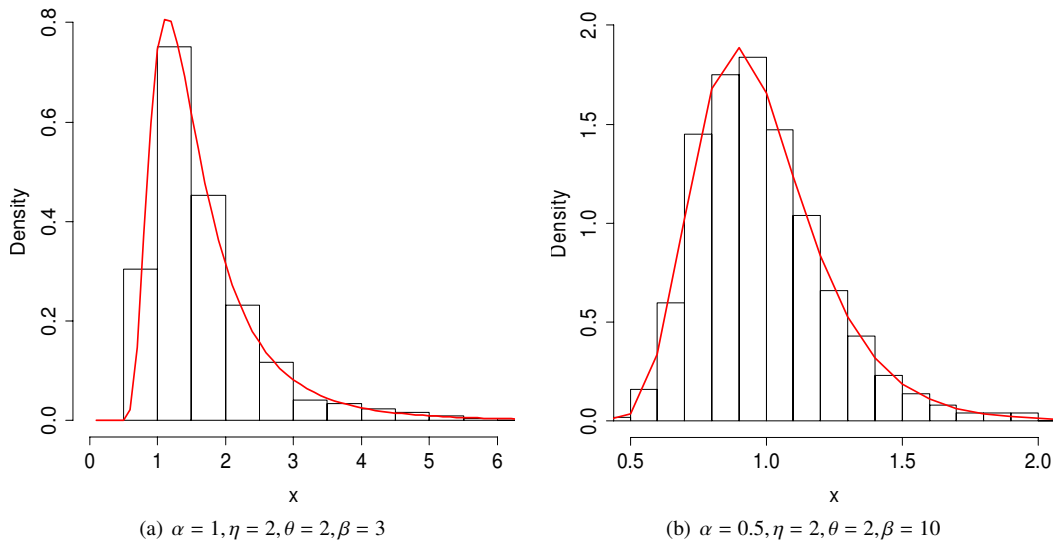


Figure 4: Plots of the new generalized inverse Weibull distribution for simulated data sets.

p.69). The data represents the failure times of the air conditioning system of an aircraft: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95. We compare the goodness of fit with several other lifetime distributions to demonstrate that the NGIW distribution can be a superior lifetime distribution

1. New generalized inverse Rayleigh (NGIR) distribution with the pdf

$$f(x) = \beta \left( \frac{\alpha}{x^2} + \frac{2\theta}{x^3} \right) \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^2 \right\} \left[ 1 - \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^2 \right\} \right]^{\beta-1}, \quad x > 0,$$

where  $\alpha, \theta > 0$  are the scale parameters and  $\beta > 0$  is the shape parameter of the NGIR distribution.

2. Modified inverse Weibull (MIW) distribution with the pdf

$$f(x) = \left( \alpha + \eta \theta \left( \frac{1}{x} \right)^{\eta-1} \right) \left( \frac{1}{x} \right)^2 \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^{\eta} \right\}, \quad x > 0,$$

where  $\alpha, \theta > 0$  are the scale parameters and  $\eta > 0$  is the shape parameter of the MIW distribution.

3. Modified inverse Rayleigh (MIR) distribution with the pdf

$$f(x) = \beta \left( \frac{\alpha}{x^2} + \frac{2\theta}{x^3} \right) \exp \left\{ -\frac{\alpha}{x} - \theta \left( \frac{1}{x} \right)^2 \right\}, \quad x > 0,$$

where  $\alpha, \theta > 0$  are the scale parameters of the MIR distribution.

4. Inverse Weibull (IW) distribution with the pdf

$$f(x) = \eta \theta \left( \frac{1}{x} \right)^{\eta+1} \exp \left\{ -\theta \left( \frac{1}{x} \right)^{\eta} \right\}, \quad x > 0,$$

Table 3: MLEs of the parameters for the air conditioning system data, the corresponding SEs (given in parentheses) and AIC, BIC, CAIC measures

Model	$\alpha$	$\eta$	$\theta$	$\beta$	AIC	BIC	CAIC
NGIW	0.9514 (2.1354)	0.1138 (0.1028)	10.7856 (5.2884)	1042.46 (6296.18)	310.490	316.090	312.090
NGIR	8.1239 (2.8177)	-	0.0001 (4.5222)	0.6493 (0.1615)	320.380	324.590	321.310
MIW	11.1798 (2.0412)	2.0056 (0.0001)	1.1E-9 (0.0014)	-	324.120	328.320	325.050
MIR	11.1799 (2.2739)	-	1.2E-6 (5.6022)	-	322.124	322.568	324.926
IW	-	0.7239 (0.0927)	6.9712 (1.8038)	-	314.229	317.031	314.673
IR	-	-	24.279 (4.4327)	-	433.491	434.892	433.634

NGIW = new generalized inverse Weibull, NGIR = new generalized inverse Rayleigh, MIW = modified inverse Weibull, MIR = modified inverse Rayleigh, IW = inverse Weibull, IR = inverse Rayleigh.

Table 4: The Log-likelihood, the values of likelihood ratio test (LRT) and  $p$ -values

Model	$H_0$	$-L(\cdot; x)$	$\Lambda$	d.f	$p$ -value
NGIR	$\eta = 2$	157.193	11.8914	1	0.0006
MIW	$\beta = 1$	159.060	15.6254	1	7.7E-5
MIR	$\eta = 2, \beta = 1$	159.062	15.6294	2	7.7E-5
IW	$\alpha = 0, \beta = 1$	155.114	7.7334	2	0.0054
IR	$\alpha = 0, \eta = 2, \beta = 1$	215.746	128.9900	3	6.8E-30

NGIR = new generalized inverse Rayleigh, MIW = modified inverse Weibull, MIR = modified inverse Rayleigh, IW = inverse Weibull, IR = inverse Rayleigh.

where  $\theta > 0$  is the scale parameter and  $\eta > 0$  is the shape parameter of the IW distribution.

5. Inverse Rayleigh (IR) distribution with the pdf

$$f(x) = 2\theta \left(\frac{1}{x}\right)^3 \exp\left\{-\theta \left(\frac{1}{x}\right)^2\right\}, \quad x > 0,$$

where  $\theta > 0$  is the scale parameter of the IR distribution.

We fitted the NGIW, NGIR, MIW, MIR, IW and IR distributions to the failure times for the 30 components of data from the air conditioning system of an aircraft data by the method of maximum likelihood. The MLEs of the parameters with their corresponding standard errors are given in parenthesis and Table 3 provides the Akaike information criteria (AIC), Bayesian information criteria (BIC) and the corrected Akaike information criteria (CAIC) for the fitted models. The AIC, BIC, CAIC indicate that the proposed distribution provides a better fit for this data. Table 4 shows a comparison of the NGIW distribution with five of its sub-models using likelihood ratio (LR) statistics. The LR statistics for testing of hypothesis with their corresponding  $p$ -values indicate that the proposed model is the most adequate model. Hence we reject the null hypothesis in favor of the NGIW distribution in all cases because the  $p$ -values are small. The AIC, BIC, CAIC and the LRT results indicate that the NGIW distribution is a capable model for fitting lifetime data. The required numerical evaluation was performed using the R program (<http://www.r-project.org/>).

Figure 5 illustrates the fitted NGIW, MIW and IW density, distribution, reliability and hazard functions for the failure times of the air conditioning system data. Figure 5 shows that the proposed

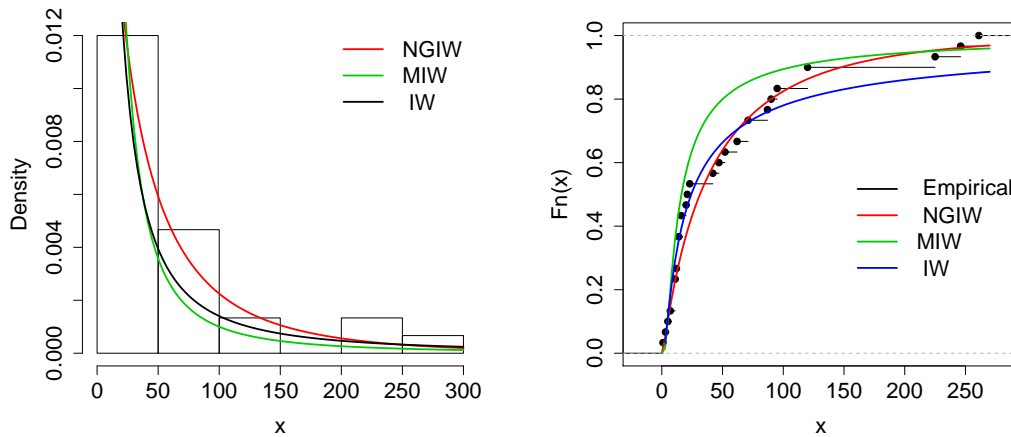


Figure 5: Plots of the estimated densities and cdfs for the failure times of air conditioning system data (NGIW = new generalized inverse Weibull, MIW = modified inverse Weibull, IW = inverse Weibull, cdf = cumulative distribution function).

Table 5: Goodness-of-fit statistics for the air conditioning data

Distribution	K-S test	$W^*$	$A^*$	$p$ -value
NGIW	0.1239	0.0744	0.4134	0.7466
NGIR	0.2108	0.1497	1.0708	0.1388
MIW	0.2329	0.1435	1.0309	0.0770
MIR	0.2330	0.1437	1.0340	0.0771
IW	0.1594	0.1028	0.7514	0.4309
IR	0.6849	0.3128	2.0274	1.2E-12

K-S = Kolmogorov-Smirnov, NGIW = new generalized inverse Weibull, NGIR = new generalized inverse Rayleigh, MIW = modified inverse Weibull, MIR = modified inverse Rayleigh, IW = inverse Weibull, IR = inverse Rayleigh.

distribution provides better fit than the sub-models of the NGIW distribution; therefore, the proposed NGIW distribution is good model for the failure times of air conditioning data. We also apply the Kolmogorov-Smirnov (K-S) test, the Cramér-von Mises and Anderson-Darling goodness-of-fit statistics to verify which model provides a better fit for the air conditioning system failure time data; the results of these statistics are displayed on Table 5. Table 5 illustrate that the NGIW distribution has the smallest values of these statistics; therefore, the proposed extended model can be chosen as the best model among the six fitted models.

We use the pp-plot to assess the failure time distribution, empirical survival function and the estimated survival function to verify the graphical goodness-of-fit of the NGIW distribution; the estimated hazard function of the NGIW distribution are displayed in Figure 6. The fitted survival plot and pp-plot shows that the NGIW distribution has a close approach to the empirical line. Figure 6 suggests that the failure times of the air conditioning system of the aircraft data has decreasing hazard function with time and follows the infant mortality period of bathtub shape failure rates. Failure is generally caused by fatigue factors during this period.

### 8. Conclusion

We introduce the NGIW distribution, (an extension of the MIW distribution) and study its theoretical

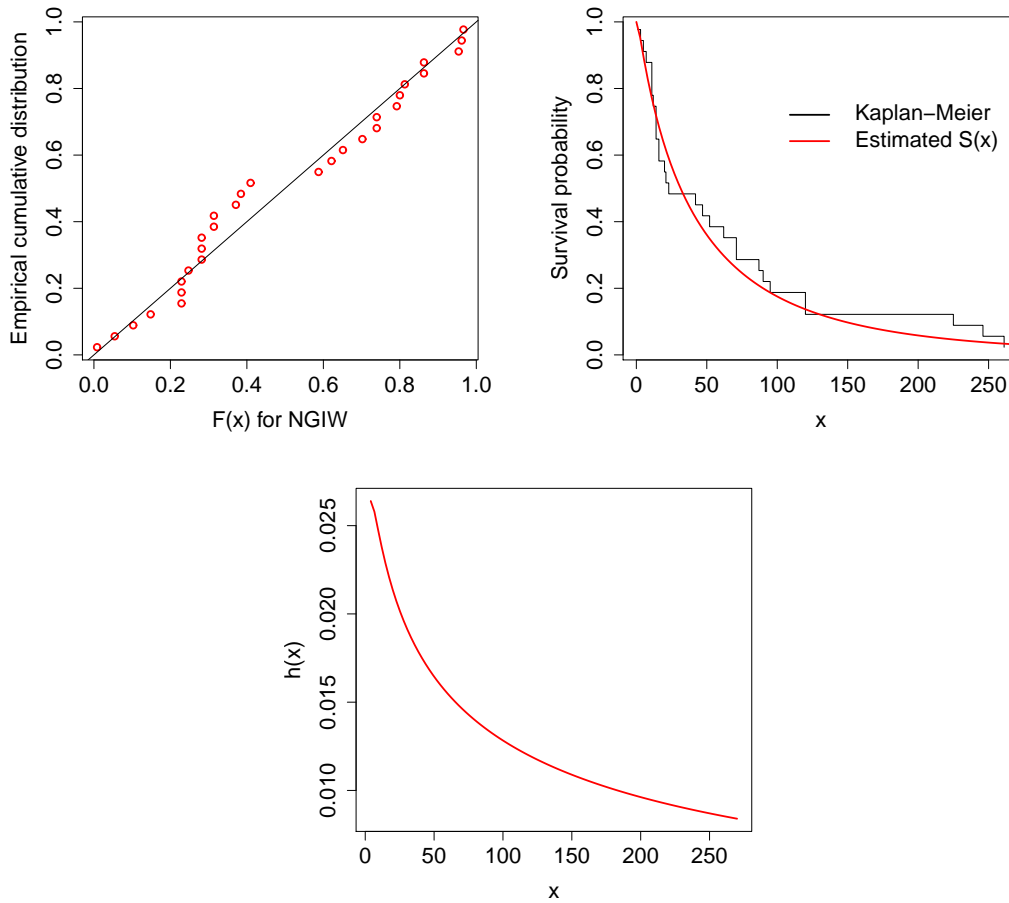


Figure 6: PP-Plot, estimated survival and hazard curves for the new generalized inverse Weibull (NGIW) distribution for air conditioning data.

properties. The new parameter  $\beta$  provides additional flexibility for fitting lifetime data. The new extended model has an upside-down hazard rate function. We study some structural properties of the NGIW distribution that include explicit expressions for the moments, moment generating function and quantile function. We also derive the moment of order statistics. We evaluate the performance of ML method to estimate NGIW parameters using Monte Carlo simulation. The usefulness of the NGIW distribution is illustrated in an application of the air conditioning system of aircraft data. Based on the seven goodness-of-fit measures, the NGIW distribution provides better fit than the other five sub-models. We hope that the proposed model may attract wider applications in many areas of real world data sets.

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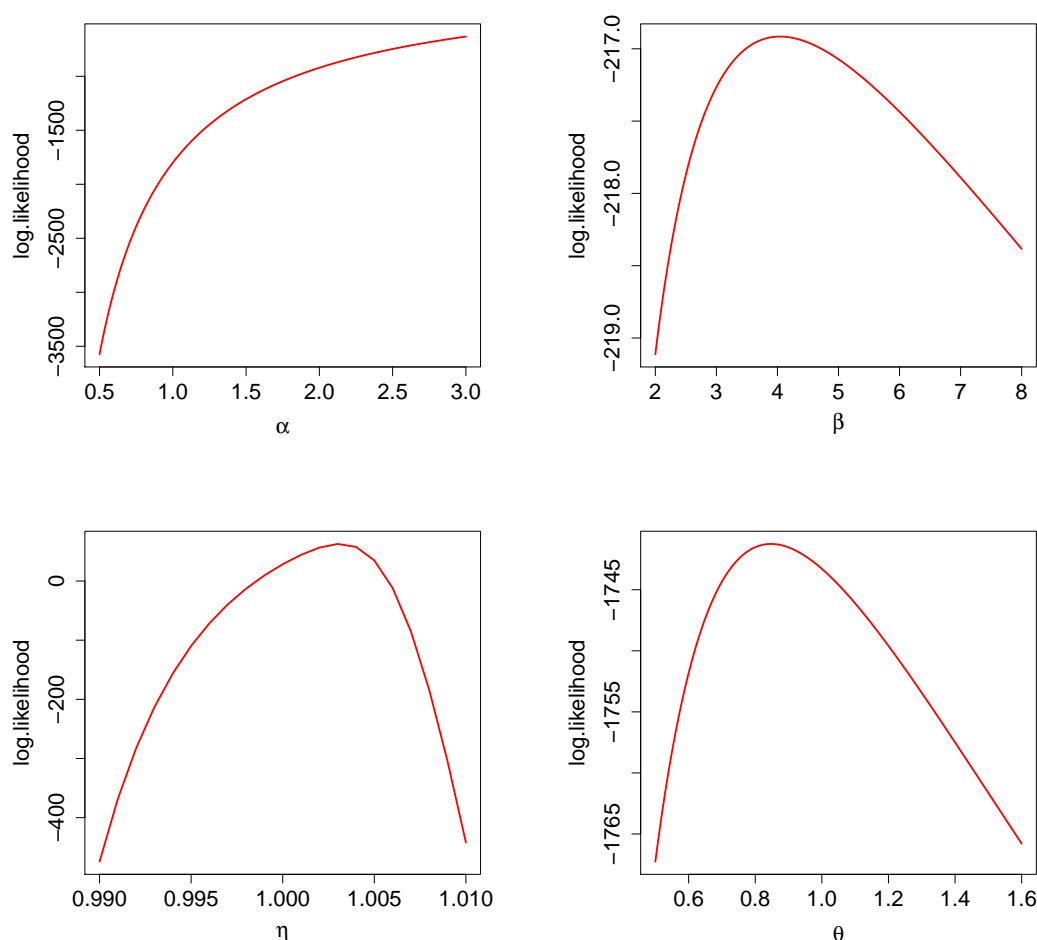


Figure 7: The profile of log-likelihood functions for  $\alpha, \beta, \eta$  and  $\theta$  for air conditioning data.

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