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REPRESENTATION OF A POSITIVE INTEGER BY A SUM OF LARGE FOUR SQUARES

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Abstract. In this paper, we determine all positive integers which cannot be represented by a sum of four squares at least 9, and prove that for each N, there are finitely many positive integers which cannot be represented by a sum of four squares at least N^2 except $2\cdot 4^m$, $6 \cdot 4^m$ and $14 \cdot 4^m$ for $m \geq 0$. As a consequence, we prove that for each $k \geq 5$ there are finitely many positive integers which cannot be represented by a sum of k squares at least N^2.

1. Introduction

The representation of a positive integer by a sum of squares with various conditions is an interesting area of number theory whose origin goes back to Pythagorean triples. The four square theorem that every positive integer is a represented by a sum of four integer squares is a famous result on this subject. A variant of this theorem is considered by Descartes in 17th century. He conjectured that every positive integer is represented by a sum of the four non-vanishing integer squares except some integers he stated. His conjecture was proved by Dubouis in 1911. Dubouis also found out all integers represented by a sum of non-vanishing $k \geq 5$ squares of integers in the same paper.

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The author of this paper computed the totally real algebraic integers over $K = \mathbb{Q}(\sqrt{d})$ for $d = 2, 3, 5$ which is represented by a sum of non-vanishing algebraic integers over K [3,4]. Also the author with J. Y. Kim estimated the norm of a totally positive algebraic integer which is locally represented by a sum of $k \geq 5$ non-vanishing squares over $\mathbb{Q}(\sqrt{d})$ [6].

Recently the author proposed a problem representing a positive integer n by a sum of k squares at least N^2 [5]. This is equivalent to the equation

$$n = x_1^2 + x_2^2 + \cdots + x_k^2, \text{ with } x_i \geq N \text{ for all } i = 1, 2, \cdots, k.$$

The representation of a positive integer n by a sum of non-vanishing k squares is a special case $N = 1$ of this problem. The author found all positive integer n represented by a sum of $k \geq 4$ squares at least 4.

In this paper, it is determined that positive integers which cannot be represented by a sum of four squares at least 9. Naturally the list of such integers contains positive integers which cannot be represented by a sum of non-vanishing four squares given by Dubouis. All 36 numbers are added the smallest one is 21 and the largest 248. Also we prove that there are finitely many positive integers which cannot be represented by a sum of four squares at least N^2 except $2 \cdot 4^m$, $6 \cdot 4^m$ and $14 \cdot 4^m$ for $m \geq 0$ for each N. As a corollary, we obtained the finiteness of positive integers which cannot be represented by a sum of $k \geq 5$ squares at least N^2.

2. Preliminaries

We state two important previous results.

THEOREM 1 (Dubouis). *If a positive integer n is not represented by a sum of non-vanishing k squares for $k \geq 4$, then n is one of the following.*

(1) $1, 3, 5, 8, 9, 11, 17, 29, 41$ *and* $2{\cdot}4^m, 6{\cdot}4^m, 14{\cdot}4^m$ *for* $m \geq 0$ *when* $k = 4$,
(2) $1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18, 33$ *when* $k = 5$,
(3) $1, 2, \ldots, k-1, k+1, k+2, k+4, k+5, k+7, k+10, k+13$ *when* $k \geq 6$.

THEOREM 2 ([5]). *A positive integer n is represented by a sum of four squares at least* 4 *unless it satisfies one of the following conditions.*

(1) $1 \leq n \leq 15$,

(2) $17, 18, 19, 20, 22, 23, 24, 25, 27, 29, 30, 32, 34, 35, 39, 41, 44, 46, 51, 55, 56, 62, 67, 89$,
(3) $n = a \cdot 4^s$ *when* $a = 2, 6, 14$ *and* $s \geq 1$.

The following theorem is an interesting result in the opposite direction to that of this paper. Considering the title of this paper, this is a work on the representation of a positive integer by a sum of small four squares.

THEOREM 3 (Selmer). *Let* $m \geq 7$ *and* $m \neq 8, 12, 16, 17$. *Also let*

$$N_m = \begin{cases} 22, & \text{if } 2^s \leq m < 2^s + 2^{s-2} \\ 28, & \text{if } 2^s + 2^{s-2} \leq m < 2^s + 2^{s-1} \\ 47, & \text{if } 2^s + 2^{s-1} \leq m < 2^s + 2^{s-1} + 2^{s-2} \\ 48, & \text{if } 2^s + 2^{s-1} + 2^{s-2} \leq m < 2^{s+1} \end{cases}$$

where $s = \lfloor \log_2 m \rfloor$. *Then for all* n *such that* $1 \leq n < N_m 2^{2s-3}$, *the equation*

$$n = x^2 + y^2 + z^2 + w^2, \text{ with } m \geq x \geq y \geq z \geq w \geq 0$$

has an integer solution, and the equation

$$N_m 2^{2s-3} = x^2 + y^2 + z^2 + w^2, \text{ with } m \geq x \geq y \geq z \geq w \geq 0$$

does not have a solution.

The following well-known theorems will be used to prove the main theorem.

THEOREM 4 (Two Square Theorem). *A positive integer* n *is represented by a sum of two squares if and only if there is no prime* $p \equiv 3 \pmod 4$ *and positive integer* k *such that* p^{2k-1} *divides* n *and* p^{2k} *donot.*

THEOREM 5 (Three Square Theorem). *A positive integer* n *is represented by a sum of three squares if and only if* n *is not of the form* $(8\ell + 7)4^m$ *for* $\ell, m \geq 0$.

LEMMA 1. *For each* $i \in \mathbb{Z}$ *different from* -1, *there are infinitely many prime numbers* p *such that* i *is a square modulo* p.

Proof. Let $\left(\frac{a}{b}\right)$ be the Jacobi symbol for odd a and b. If i is odd and $i > 0$, then since $\left(\frac{i}{p}\right) = \left(\frac{p}{i}\right)$ for a prime $p \equiv 1 \pmod 4$, i is a square modulo p for all primes p such that $p \equiv 1 \pmod 4$ and $p \equiv 1 \pmod i$. Then since $p \equiv 1 \pmod 4$, $-i$ is also a square modulo p. Also $\pm 2i$ is a square modulo p for all prime p such that $p \equiv 1 \pmod 8$ and

$p \equiv 1 \pmod{i}$. The existence of such primes follows from the Dirichlet Theorem. $\square$

The following lemma is a consequence of Hensel's Lemma.

Lemma 2. *If $i \in \mathbb{Z}$ is a square modulo an odd prime p and $(p, i) = (p, k) = 1$, then i is a square modulo p^k.*

From Euler's Two Square Theorem, $p = a^2 + b^2$ for some nonzero integers a and b. Thus $p^2 = c^2 + d^2$ where $c = a^2 - b^2 \neq 0$ and $d = 2ab \neq 0$.

The following lemma follows from Two Square Theorem and Three Square Theorem.

Lemma 3. *(1). If $k^2 m$ is represented by a sum of two squares, then m is represented by a sum of two squares.*
(2). If $k^2 m$ is represented by a sum of three squares, then m is represented by a sum of three squares.

3. Main Theorem

Theorem 6. *If a positive integer n is not represented by a sum of four squares at least 9, then n is one of the following.*

$$1, 3, 4, 5, 7, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27, 28,$$
$$29, 30, 31, 33, 34, 35, 37, 38, 39, 40, 41, 42, 44, 45, 46, 47, 48, 49, 51, 53, 54,$$
$$55, 58, 60, 61, 62, 65, 67, 69, 71, 72, 74, 78, 80, 81, 85, 87, 88, 89, 94, 101,$$
$$109, 120, 136, 176, 184, 248$$

and

$$2 \cdot 4^m, 6 \cdot 4^m, 14 \cdot 4^m \text{ for } m \geq 0.$$

Proof. If $n \equiv 2, 3 \pmod{4}$ and $n > 41^2 + 12$, then since $n - 41^2 > 12$ and $n - 41^2 \equiv 1 \pmod{4}$, n is represented by a sum of three squares. Thus $n - 41^2 = x^2 + y^2 + z^2$ for some $x \geq y \geq z \geq 0$. If $y = z = 0$, then

$$n = x^2 + 41^2 = x^2 + 32^2 + 24^2 + 9^2.$$

If $y = 1$ and $z = 0$, then

$$n = x^2 + 1^2 + 41^2 = x^2 + 29^2 + 21^2 + 20^2.$$

If $y = z = 1$, then

$$n = x^2 + 1^2 + 1^2 + 41^2 = x^2 + 39^2 + 9^2 + 9^2.$$

If $y = 2$ and $z = 0$, then

$$n = x^2 + 2^2 + 41^2 = x^2 + 39^2 + 10^2 + 8^2.$$

If $y = 2$ and $z = 1$, then

$$n = x^2 + 2^2 + 1^2 + 41^2 = x^2 + 38^2 + 11^2 + 11^2.$$

If $y = z = 2$, then

$$n = x^2 + 2^2 + 2^2 + 41^2 = x^2 + 40^2 + 6^2 + 5^2.$$

If $y \geq 3$ and $z = 0$, then

$$n = x^2 + y^2 + 41^2 = x^2 + y^2 + 40^2 + 9^2.$$

If $y \geq 3$ and $z = 1$, then

$$n = x^2 + y^2 + 1^2 + 41^2 = x^2 + y^2 9^2 + 29^2.$$

If $y \geq 3$ and $z = 2$, then

$$n = x^2 + y^2 + 2^2 + 41^2 = x^2 + y^2 + 34^2 + 23^2.$$

If $z \geq 3$, then

$$n = x^2 + y^2 + z^2 + 41^2.$$

If $n \equiv 1 \pmod 4$ and $n > 41^2 + 12$, then since $n - 25^2 \equiv n - 29^2 \equiv 0 \pmod 4$, $n - 25^2 \equiv n - 29^2 + 8 \pmod{16}$ and $n - 29^2 > 41^2 + 12 - 29^2 = 852$, either $n - 25^2$ or $n - 29^2$ is represented by a sum of three squares. If $n - 25^2$ is represented by a sum of three squares, then $n - 25^2 = x^2 + y^2 + z^2$ for some even $x \geq y \geq z \geq 0$. If $y = z = 0$, then

$$n = x^2 + 25^2 = x^2 + 20^2 + 12^2 + 9^2.$$

If $y = 2$ and $z = 0$, then

$$n = x^2 + 2^2 + 25^2 = x^2 + 18^2 + 16^2 + 7^2.$$

If $y = z = 2$, then

$$n = x^2 + 2^2 + 2^2 + 25^2 = x^2 + 22^2 + 10^2 + 7^2.$$

If $y \geq 3$ and $z = 0$, then

$$n = x^2 + y^2 + 25^2 = x^2 + y^2 + 20^2 + 15^2.$$

If $y \geq 3$ and $z = 2$, then

$$n = x^2 + y^2 + 2^2 + 25^2 = x^2 + y^2 + 23^2 + 10^2.$$

If $z \geq 3$, then

$$n = x^2 + y^2 + z^2 + 25^2.$$

If $n - 29^2$ is represented by a sum of three squares, then $n - 29^2 = x^2 + y^2 + z^2$ for some even $x \geq y \geq z \geq 0$. If $y = z = 0$, then

$$n = x^2 + 29^2 = x^2 + 21^2 + 16^2 + 12^2.$$

If $y = 2$ and $z = 0$, then

$$n = x^2 + 2^2 + 29^2 = x^2 + 20^2 + 18^2 + 11^2.$$

If $y = z = 2$, then

$$n = x^2 + 2^2 + 2^2 + 29^2 = x^2 + 20^2 + 20^2 + 7^2.$$

If $y \geq 3$ and $z = 0$, then

$$n = x^2 + y^2 + 29^2 = x^2 + y^2 + 21^2 + 20^2.$$

If $y \geq 3$ and $z = 2$, then

$$n = x^2 + y^2 + 2^2 + 29^2 = x^2 + y^2 + 26^2 + 13^2.$$

If $z \geq 3$, then

$$n = x^2 + y^2 + z^2 + 29^2.$$

Thus any positive integer which cannot be represented by a sum of four squares at least 9 is less than or equal to $412 + 12 = 1693$. If n is a multiple of 4 and n is not represented by a sum of four squares at least 9, then $\frac{n}{4}$ is not represented by a sum of four squares at least 4, which is given in Theorem 5. By computation, we can find, among these candidates, all positive integers which is not represented by a sum of four squares at least 9 given in the statement of this theorem. □

THEOREM 7. *For each $N \geq 1$, there are finitely many positive integers n represented by a sum of four squares at least N^2 except $n = 2 \cdot 4^m, 6 \cdot 4^m, 14 \cdot 4^m$ for some nonnegative integer m.*

We first prove the following special case.

PROPOSITION 1. *For each $N \geq 1$, there are finitely many positive integers n such that n is not a multiple of 4 and n is represented by a sum of four squares at least N^2.*

Proof. By Lemmas 1 and 2, there are distinct odd primes $p_1, p_2, \cdots, p_{N-1}, q_1, q_2, \cdots, q_{2(N-1)^2}$ such that $p_i \equiv q_j \equiv 1 \pmod 4$, $p_i, q_j \geq N$ and j is a square modulo q_j^4 for all $i = 1, 2, \cdots, N-1$ and $j = 1, 2, \cdots, 2(N-1)^2$. Let M be the maximum of p_i^2 and q_j^4 for all i, j.

If $n \equiv 2, 3 \pmod 4$, then choose an odd integer ℓ such that $\ell \geq MN^2$, $\ell^2 \equiv -i^2 \pmod{p_i^2}$ and $\ell^2 \equiv -j \pmod{q_j^4}$. If $n > \ell^2 + 3(N-1)^2$, then since $n - \ell^2 \equiv 1, 2 \pmod 4$ and $n - \ell^2 \geq 3(N-1)^2 > 0$, $n - \ell^2$ is represented by a sum of three squares. Thus $n - \ell^2 = x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{Z}$ such that $x \geq y \geq z \geq 0$. If $z \geq N$, then we have $n = x^2 + y^2 + z^2 + \ell^2$. Thus n is a sum of four squares at least N^2.

If $y \geq N > z$, then since $z \leq N - 1$, $\ell^2 + z^2 \equiv 0 \pmod{p_z^2}$. Thus $\ell^2 + z^2 = p_z^2 m$ for some integer $m > 0$. Since m is a positive integer represented by the sum of two squares of rational numbers, m is a sum of two integer squares. Thus $m = u^2 + v^2$ for some integers u, v such that $u > 0$ and $v \geq 0$. If $v > 0$, then we have

$$n = x^2 + y^2 + z^2 + \ell^2 = x^2 + y^2 + p_z^2 m = x^2 + y^2 + (p_z u)^2 + (p_z v)^2.$$

Since $p_z \geq N$, n is a sum of four squares at least N^2. If $v = 0$, then since $p_z^2 m \geq \ell^2 \geq MN^2 \geq (p_z N)^2$, we have $m = u^2 \geq N^2$. Since $p_z \equiv 1 \pmod 4$, $p_z^2 = c^2 + d^2$ for some integers $c, d > 0$. Thus

$$\begin{aligned} n &= x^2 + y^2 + z^2 + \ell^2 = x^2 + y^2 + p_z^2 m = x^2 + y^2 + (c^2 + d^2)u^2 \\ &= x^2 + y^2 + (cu)^2 + (du)^2. \end{aligned}$$

Hence n is a sum of four squares at least N^2.

If $y < N$, then since $n = x^2 + y^2 + z^2 \geq 3(N-1)^2$, we have $x \geq N$. Let $h = y^2 + z^2$. Since $h \leq 2(N-1)^2$, $\ell^2 + h \equiv 0 \pmod{q_h^4}$. Thus $\ell^2 + h = q_h^4 b$ for some integer $b > 0$. By Lemma 3, b is a positive integer represented by a sum of three squares. Thus $b = s_1^2 + s_2^2 + s_3^2$ for some integers s_1, s_2, s_3 such that $s_1 \geq s_2 \geq s_3 \geq 0$. If $s_3 > 0$, then we have

$$\begin{aligned} n &= x^2 + y^2 + z^2 + \ell^2 = x^2 + h + \ell^2 = x^2 + q_h^4 b = x^2 + q_h^4(s_1^2 + s_2^2 + s_3^2) \\ &= x^2 + (q_h^2 s_1)^2 + (q_h^2 s_2)^2 + (q_h^2 s_3)^2. \end{aligned}$$

Thus n is a sum of four squares at least N^2.

If $s_3 = 0$ and $s_2 > 0$, then since $q_h^2 = c^2 + d^2$ for some positive integers c, d, we have

$$\begin{aligned} n &= x^2 + y^2 + z^2 + \ell^2 = x^2 + q_h^4(s_1^2 + s_2^2) = x^2 + (q_h^2 s_1)^2 + q_h^2 s_2^2(c^2 + d^2) \\ &= x^2 + (q_h^2 s_1)^2 + (q_h s_2 c)^2 + (q_h s_2 d)^2. \end{aligned}$$

Thus n is a sum of four squares at least N^2. If $s_2 = s_3 = 0$, then $b = s_1^2 > 0$. From $q_h^2 = c^2 + d^2$, we have

$$\begin{aligned} n &= x^2 + y^2 + z^2 + \ell^2 = x^2 + q_h^4 s_1^2 = x^2 + q_h^2 s_1^2(c^2 + d^2) \\ &= x^2 + (q_h s_1 c)^2 + s_1^2 d^2(c^2 + d^2) = x^2 + (q_h^2 s_1)^2 + (s_1 cd)^2 + (s_1 d^2)^2. \end{aligned}$$

Since $q_h^4 b = q_h^4 s_1^2 \geq \ell^2 \geq MN^2 \geq q_h^4 N^2$, we have $s_1 \geq N$. Thus n is a sum of four squares at least N^2.

If $n \equiv 1 \pmod 4$, then choose an even integer $\tilde{\ell}$ such that $\tilde{\ell} \geq MN^2$, $\tilde{\ell}^2 \equiv -i^2 \pmod{p_i^2}$ and $\tilde{\ell}^2 \equiv -j \pmod{q_j^4}$. If $n > \tilde{\ell}^2 + 3(N-1)^2$, then since $n - \tilde{\ell}^2 \equiv 1 \pmod 4$ and $n - \tilde{\ell}^2 \geq 3(N-1)^2 > 0$. By the same method used in above, we can show that n is a sum of four squares at least N^2. As a consequence, if n is not a multiple of 4 and $n > \max\{\ell^2, \tilde{\ell}^2\} + 3(N-1)^2$, then n is a sum of four squares at least N^2. $\square$

Proof. (Theorem 7) Let $r \in \mathbb{Z}^+$ such that $2^r \geq N$. By Proposition 1, there is $M > 0$ such that if $n > M$ and n is not a multiple of 4, then n is represented by a sum of four squares at least N^2. If $n > 4^{r+3}M$ and $n = 4^t n'$ with $n' \neq 2, 6, 14$, then we have $t \geq r+3$ or $n' > M$. If $t \geq r+3$, then since $4^{t-r} \geq 4^3 = 64$ and $n' \neq 2, 6, 14$, $4^{t-r}n'$ is represented by a sum of four nonvanishing squares. Thus $4^{t-r}n' = x^2 + y^2 + z^2 + u^2$ for some $x, y, z, u > 0$. Thus

$$n = 4^r(4^{t-r}n') = 4^r(x^2 + y^2 + z^2) = (2^t x)^2 + (2^t y)^2 + (2^t z)^2 + (2^t u)^2.$$

since $2^t \geq N$, n is a sum of four squares at least N^2.

If $n' > M$, then by Proposition 1, n' is a sum of four squares at least N^2. Hence n is a sum of four squares at least N^2. $\square$

Corollary. *For each $N \geq 1$ and $k \geq 5$, there are finitely many positive integers n not represented by a sum of k squares at least N^2.*

Proof. Theorem 7 implies that there is $M > 0$ such that for all $n > M$ different from $a \cdot 4^m$ for $m \geq 0$ is represented by a sum of four squares at least N^2. Especially if $n > M$ and n is odd, then n is represented by a sum of four squares at least N^2.

If $n > M + (k-5)N^2 + (N+1)^2$, then both $n - (k-4)N^2$ and $n - (k-5)N^2 - (N+1)^2$ are larger than M and one of them is odd. By Theorem 7, at least one of them is represented by a sum of four squares at least N^2. As a consequence, n is represented by a sum of k squares at least N^2. $\square$

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