Korean J. Math. **24** (2016), No. 1, pp. 65–70
http://dx.doi.org/10.11568/kjm.2016.24.1.65

RIESZ PROJECTIONS FOR A NON-HYPONORMAL OPERATOR

Jae Won Lee† and In Ho Jeon*

Abstract. J. G. Stampfli proved that if a bounded linear operator T on a Hilbert space $\mathscr{H}$ satisfies (G_1) property, then the Riesz projection P_λ associated with $\lambda \in \text{iso}\sigma(T)$ is self-adjoint and $P_\lambda \mathscr{H} = (T-\lambda)^{-1}(0) = (T^* - \bar{\lambda})^{-1}(0)$.

In this note we show that Stampfli's result is generalized to an nilpotent extension of an operator having (G_1) property.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators on a Hilbert space $\mathscr{H}$. Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$ and *normaloid* if $||T|| = r(T)$, the spectral radius of T. It is well known that a hyponormal operator is normaloid. Recall that a projection $P \in \mathscr{L}(\mathscr{H})$ is called an *orthogonal projection* if the range of P, denoted by $\text{ran}(P)$, and the kernel of P, denoted by $\ker(P)$, are orthogonal complements. It is well known [7, Proposition 63.1] that a projection is orthogonal if and only if it is self-adjoint. For an operator $T \in \mathscr{L}(\mathscr{H})$, if λ is an isolated point of the spectrum of T, $\lambda \in \text{iso}\sigma(T)$, *the Riesz projection* P_λ associated with λ is defined by

Received February 6, 2016. Revised February 23, 2016. Accepted February 29, 2016.

2010 Mathematics Subject Classification: 47A10, 47B20.

Key words and phrases: Riesz projections, (G_1) property.

† This paper was supported by Research Fund, Kumoh National Institute of Technology.

* Corresponding author.

Cauchy integral

$$P_\lambda = \frac{1}{2\pi i}\int_{\partial D}(T-z)^{-1}dz, \tag{1.1}$$

where D is a closed disk centered at λ and $D \cap \sigma(T) = \{\lambda\}$. The Riesz projection P_λ for λ is generally not orthogonal, that's not self-adjoint.

Stampfli([11, Theorem 2]) proved that if T is hyponormal, then P_λ is self-adjoint and

$$P_\lambda \mathscr{H} = (T-\lambda)^{-1}(0) = (T^* - \bar{\lambda})^{-1}(0) \text{ for } \lambda \in \text{iso}\sigma(T). \tag{1.2}$$

This result has since been generalized by many mathematicians ([1], [6], [4], [13]). In particular we should recall Duggal's result ([2]; [3]) for an extended class of non-hyponormal operators.

A part of an operator is its restriction to an invariant subspace. We say that $T \in \mathscr{L}(\mathscr{H})$ is *totally hereditarily normaloid*, denoted $T \in \mathscr{T}\mathscr{H}\mathscr{N}$, if every part of T, and (also) invertible part of T, is normaloid.

PROPOSITION 1.1. [2, Theorem 1.1] *Suppose that an operator $T \in \mathscr{L}(\mathscr{H})$ has a representation*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{pmatrix} \tag{1.3}$$

such that T_3 is nilpotent and $\sigma(T_1) \subset \sigma(T) \subset \sigma(T_1) \cup \{0\}$. If $T_1 \in \mathscr{T}\mathscr{H}\mathscr{N}$, non-zero isolated eigenvalues of T_1 are normal and $(T_1-\lambda)^{-1}(0) \oplus 0 \subseteq (T^* - \bar{\lambda})^{-1}(0)$, then the Riesz projection P_λ associated with λ is self-adjoint and $P_\lambda \mathscr{H} = (T-\lambda)^{-1}(0) = (T^* - \bar{\lambda})^{-1}(0)$ for every non-zero $\lambda \in \text{iso}\sigma(T)$.

We say that $T \in \mathscr{L}(\mathscr{H})$ has (G_1) *property* if

$$||(T-\lambda)^{-1}|| = r((T-\lambda)^{-1}) \text{ for } \lambda \notin \sigma(T).$$

It is well known that a hyponormal operator satisfies (G_1) property, but an operator satisfying (G_1) property is generally not normaloid.

In [12, Theorem C], Stampfli also proved that if T has (G_1) property, then for $\lambda \in \text{iso}\sigma(T)$

$$P_\lambda \text{ is self-adjoint and } P_\lambda \mathscr{H} = (T-\lambda)^{-1}(0) = (T^* - \bar{\lambda})^{-1}(0). \tag{1.4}$$

In this note we show that Stampfli's result is generalized to an nilpotent extension of an operator having (G_1) property.

2. Main results

In [8] M. Mbekhta introduced two important subspaces of $\mathscr{H}$. For an operator $T \in \mathscr{L}(\mathscr{H})$, the *quasi-nilpotent part of* T is the set

$$H_0(T) = \{x \in \mathscr{H} : \lim_{n\to\infty} ||T^n x||^{\frac{1}{n}} = 0\} \tag{2.1}$$

and the *analytic core* of T is the set

$$\begin{aligned} K(T) = \{x \in \mathscr{H} :\ &\text{there exist a sequence } \{x_n\} \subset \mathscr{H} \text{ and } \delta > 0 \\ &\text{for which } x = x_0, Tx_{n+1} = x_n \text{ and } ||x_n|| \leq \delta^n ||x|| \\ &\text{for all } n = 1, 2, \cdots\}, \end{aligned}$$

which are generally not closed subspaces of $\mathscr{H}$ such that

$$(T)^{-n}(0) \subseteq H_0(T) \text{ and } TK(T) = K(T). \tag{2.2}$$

It is well known ([8], [9], [10]) that

$$\lambda \in \text{iso}\sigma(T) \iff \mathscr{H} = H_0(T-\lambda) \oplus K(T-\lambda), \tag{2.3}$$

where $H_0(T-\lambda)$ and $K(T-\lambda)$ are closed subspaces. Moreover, (2.2) and (2.3) implies that if $H_0(T-\lambda) = (T-\lambda)^{-d}(0)$, then $\lambda \in \text{iso}\sigma(T)$ is a pole of the resolvent of T of order d.

Lemma 2.1. *If $T \in \mathscr{L}(\mathscr{H})$ has (G_1) property, then $\lambda \in \text{iso}\sigma(T)$ is a pole of the resolvent of T of order* 1.

Proof. From (1.4) we observe that

$$P_\lambda \mathscr{H} = H_0(T-\lambda) = (T-\lambda)^{-1}(0) \text{ for } \lambda \in \text{iso}\sigma(T).$$

Thus, from the above arguments, $\lambda \in \text{iso}\sigma(T)$ is a pole of the resolvent of T of order 1. □

To prove the following Lemmas we fully adopt Duggal's arguments ([2], [3]).

Lemma 2.2. *Suppose that an operator $T \in \mathscr{L}(\mathscr{H})$ has a representation*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{pmatrix} \tag{2.4}$$

such that $T_1 \in \mathscr{L}(\mathscr{H}_1)$ has (G_1) property and T_3 is nilpotent. Then every non-zero $\lambda \in \text{iso}\sigma(T)$ is a simple pole(i.e., order one pole) of the resolvent of T.

Proof. Assume that $\lambda(\neq 0) \in \operatorname{iso}\sigma(T)$. Then $\lambda(\neq 0) \in \operatorname{iso}\sigma(T_1)$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$ by [5, Corollary 8]. Since, by Stampfli's result (1.4), $(T_1 - \lambda)^{-1}(0)$ reduces T, it follows that

$$T_1 - \lambda = \begin{bmatrix} 0 & 0 \\ 0 & T_{11} - \lambda \end{bmatrix} \text{ on } \mathscr{H}_1 = (T_1 - \lambda)^{-1}(0) \oplus (T_1 - \lambda)\mathscr{H}.$$

Set

$$(T_1 - \lambda)^{-1}(0) = \mathscr{H}_1', \ \mathscr{H}_1 \ominus \mathscr{H}_1' = \mathscr{H}_3' \text{ and } \mathscr{H}_3' \oplus \mathscr{H}_2 = \mathscr{H}_2'.$$

Then it follows that

$$T - \lambda = \begin{bmatrix} 0 & 0 & T_{21} \\ 0 & T_{11} - \lambda & T_{22} \\ 0 & 0 & T_3 - \lambda \end{bmatrix} \begin{pmatrix} \mathscr{H}_1' \\ \mathscr{H}_3' \\ \mathscr{H}_2 \end{pmatrix} = \begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix} \begin{pmatrix} \mathscr{H}_1' \\ \mathscr{H}_2' \end{pmatrix},$$

where $A = \begin{bmatrix} 0 & T_{21} \end{bmatrix}$, and where

$$B = \begin{bmatrix} T_{11} - \lambda & T_{22} \\ 0 & T_3 - \lambda \end{bmatrix}$$

is invertible. Since

$$\begin{aligned} H_0(T - \lambda) &= \left\{ x \in \mathscr{H} : \lim_{n \to \infty} ||(T - \lambda)^n x||^{1/n} = 0 \right\} \\ &= \left\{ x = x_1 \oplus x_2 \in \mathscr{H} : \lim_{n \to \infty} \left\| \begin{bmatrix} AB^{n-1} x_2 \\ B^n x_2 \end{bmatrix} \right\|^{1/n} = 0 \right\}, \end{aligned}$$

the invertibility of B implies that

$$||x_2||^{1/n} \leq ||B^{-1}|| ||B^n x_2||^{1/n} \to 0 \text{ as } n \to \infty.$$

Hence, $x_2 = 0$, and

$$H_0(T - \lambda) = (T_1 - \lambda)^{-1}(0) \oplus \{0\} = (T - \lambda)^{-1}(0).$$

Therefore we have that

$$\mathscr{H} = (T - \lambda)^{-1}(0) \oplus (T - \lambda)\mathscr{H} \text{ for } \lambda(\neq 0) \in \operatorname{iso}\sigma(T).$$

□

The following result is a slight improvement of [3, Theorem 2.7].

LEMMA 2.3. *If $\lambda \in \operatorname{iso}\sigma(T)$ is a simple pole of the resolvent of T, then the Riesz projection P_λ is self-adjoint if and only if*

$$(T - \lambda)^{-1}(0) \subseteq (T^* - \bar{\lambda})^{-1}(0). \qquad (2.5)$$

Proof. Since $\lambda \in \mathrm{iso}\sigma(T)$ is a simple pole of the resolvent of T,

$$\mathscr{H} = (T-\lambda)^{-1}(0) \oplus (T-\lambda)\mathscr{H}. \tag{2.6}$$

Observe that

$$P_\lambda \mathscr{H} = H_0(T-\lambda) = (T-\lambda)^{-1}(0) \text{ and } P_\lambda^{-1}(0) = (T-\lambda)\mathscr{H}.$$

If P_λ is self-adjoint, then

$$[P_\lambda^{-1}(0)]^\perp = P_\lambda \mathscr{H}.$$

Since

$$[P_\lambda^{-1}(0)]^\perp = [(T-\lambda)\mathscr{H}]^\perp = (T^* - \bar{\lambda})^{-1}(0),$$

it immediately implies that $(T-\lambda)^{-1}(0) \subseteq (T^* - \bar{\lambda})^{-1}(0)$. Conversely, assuming that

$$(T-\lambda)^{-1}(0) \subseteq (T^* - \bar{\lambda})^{-1}(0),$$

$P_\lambda \mathscr{H} = H_0(T-\lambda) = (T-\lambda)^{-1}(0)$ is a reducing subspace of T. From (2.6), we have

$$[P_\lambda \mathscr{H}]^\perp = [(T-\lambda)^{-1}(0)]^\perp = (T-\lambda)\mathscr{H} = P_\lambda^{-1}(0).$$

Therefore P_λ is self-adjoint. □

THEOREM 2.4. *Suppose that an operator $T \in \mathscr{L}(\mathscr{H})$ has a representation*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{pmatrix} \tag{2.7}$$

such that $T_1 \in \mathscr{L}(\mathscr{H}_1)$ has (G_1) property and T_3 is nilpotent. Then $\lambda(\neq 0) \in \mathrm{iso}\sigma(T)$ is a simple pole of the resolvent of T and the Riesz projection P_λ is self-adjoint if and only if

$$(T-\lambda)^{-1}(0) \subseteq (T^* - \bar{\lambda})^{-1}(0). \tag{2.8}$$

Proof. Combining Lemma 2.2 and Lemma 2.3 completes the proof. □

References

[1] M. Cho and K. Tanahashi, *Isolated points of the spectrum of p-hyponormal and log-hyponormal operators*, Integr. Equat. Oper. Th. **43** (2002), 379–384.
[2] B. P. Duggal, *Riesz projections for a class of Hilbert space operators*, Linear Algebra Appl. **407** (2005), 140–148.
[3] B. P. Duggal, *Hereditarily normaloid operators*, Extracta Math. **20** (2005), 203–217.
[4] Y. M. Han and A.-H. Kim, *A note on $\star$-paranormal operators*, Integr. Equat. Oper. Th. **49** (2004), 435–444.
[5] J. K. Han, H. Y. Lee and W. Y. Lee, *Invertible completions of 2 ×2 upper triangular operator matrices*, Proc. Amer. Math. soc. **128** (2000), 119–123.
[6] Y. M. Han, J. I. Lee and D. Wang, *Riesz idempotents and Weyl's theorem for w-hyponormal operators*, Integr. Equat. Oper. Th. **53** (2005), 51–60.
[7] H. G. Heuser, Functional Analysis, John Wiley, New York, 1982.
[8] M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasgow Math. J. **29** (1987), 159–175.
[9] M. Oudghiri, *Weyl's and Browder's theorem for operators satisfying the SVEP*, Studia Math. **163**(2004) 85–101.
[10] C. Schmoeger, *On isolated points of the spectrum of a bounded linear operator*, Proc. Amer. Math. soc. **117** (1993), 715–719.
[11] J. G. Stampfli, *Hyponormal operators*, Pacific J. Math. **12** (1962), 1453–1458.
[12] J. G. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc. **117** (1965), 469–476.
[13] A. Uchiyama, *On the isolated points of the spectrum of paranormal operators*, Integr. Equat. Oper. Th. **55** (2006), 145–151.

Jae Won Lee
Department of Applied Mathematics
Kumoh National Institute of Technology
Gumi 730-701Korea
E-mail: ljaewon@mail.kumoh.ac.kr

In Ho Jeon
Department of Mathematics Education
Seoul National University of Education
Seoul 137-742 Korea
E-mail: jihmath@snue.ac.kr