

THE HYERS-ULAM STABILITY OF A QUADRATIC FUNCTIONAL EQUATION WITH INVOLUTION IN PARANORMED SPACES

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ABSTRACT. In this paper, using fixed point method, we prove the Hyers-Ulam stability of the following functional equation

$$\begin{aligned} & f(x + y + z) + f(\sigma(x) + y + z) + f(x + \sigma(y) + z) + f(x + y + \sigma(z)) \\ &= 4f(x) + 4f(y) + 4f(z) \end{aligned}$$

with involution in paranormed spaces.

1. Introduction and Preliminaries

In 1940, Ulam [19] proposed the following problem concerning the stability of group homomorphism: *Let G_1 be a group and let G_2 a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?*

Hyers [7] solved the Ulam's problem for the case of approximately additive functions in Banach spaces. Since then, the stability of several functional equations have been extensively investigated by several mathematicians [2, 3, 6, 8, 9, 12, 14–16].

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Let X and Y be real vector spaces. For an additive mapping $\sigma : X \rightarrow X$ with $\sigma(\sigma(x)) = x$ for all $x \in X$, σ is called an *involution* of X [1,18]. Stetkær [18] introduced the following quadratic functional equation with involution

$$(1.1) \quad f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y)$$

and solved the general solution and Belaid et al. [1] established generalized Hyers-Ulam stability in Banach spaces for this functional equation. Jung and Lee [11] investigated the Hyers-Ulam-Rassias stability of (1.1) in a complete β -normed space, using fixed point method.

The concept of statistical convergence for sequences of real numbers was introduced by Fast [5] and Steinhaus [17] independently and since then several generalizations and applications of this notion have been investigated by various authors. This notion was defined in normed spaces by Kolk [13].

DEFINITION 1.1. Let X be a vector space. A paranorm $P : X \rightarrow [0, \infty)$ is a function on X such that

- (1) $P(0) = 0$;
- (2) $P(-x) = P(x)$;
- (3) $P(x+y) \leq P(x) + P(y)$ (triangle inequality);
- (4) If $\{t_n\}$ is a sequence of scalars with $t_n \rightarrow t$ and $\{x_n\} \subset X$ with $P(x_n - x) \rightarrow 0$, then $P(t_n x_n - tx) \rightarrow 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X .

The paranorm is called *total* if, in addition, we have

- (5) $P(x) = 0$ implies $x = 0$. A *Fréchet space* is a total and complete paranormed space.

Let (X, d) be a generalized metric space. An operator $T : X \rightarrow X$ satisfies a Lipschitz condition with the Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a *strictly contractive operator*. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

We recall a fundamental result in fixed point theories.

THEOREM 1.2. [4] *Let (X, d) be a complete generalized metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$,*

either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) the sequence $\{J^n x\}$ converges to a fixed point x^* of J ;
- (3) x^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$ and
- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, Issac and Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorem with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors.

In this paper, using fixed point method, we prove the generalized Hyers-Ulam stability of the following functional equation

$$(1.2) \quad \begin{aligned} f(x + y + z) + f(\sigma(x) + y + z) + f(x + \sigma(y) + z) \\ + f(x + y + \sigma(z)) = 4f(x) + 4f(y) + 4f(z) \end{aligned}$$

Throughout this paper, we assume that X is a Fréchet space and Y is a Banach space.

2. The generalized Hyers-Ulam stability for (1.2)

Using the fixed point methods, we will prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.2) with involution σ in paranormed spaces. For a given mapping $f : X \rightarrow Y$, we define the difference operator $Df : X^3 \rightarrow Y$ by

$$\begin{aligned} Df(x, y, z) &= f(x + y + z) + f(\sigma(x) + y + z) \\ &+ f(x + \sigma(y) + z) + f(x + y + \sigma(z)) - 4f(x) - 4f(y) - 4f(z) \end{aligned}$$

for all $x, y, z \in X$.

LEMMA 2.1. *Let $f : X \rightarrow Y$ be a mapping. Then f satisfies (1.2) if and only if f satisfies (1.1).*

Proof. Suppose that f satisfies (1.2). Letting $x = y = z = 0$ in (1.2), we have $f(0) = 0$. Letting $y = z = 0$ in (1.2), we have $f(x) = f(\sigma(x))$ for all $x \in X$. Letting $z = 0$ in (1.2), we have

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

Assume that f satisfies (1.1). We have

$$\begin{aligned} & f(x+y+z) + f(\sigma(x)+y+z) + f(x+\sigma(y)+z) + f(x+y+\sigma(z)) \\ = & 2f(x+y) + 2f(z) + 2f(x+\sigma(y)) + 2f(z) \\ = & 4f(x) + 4f(y) + 4f(z) \end{aligned}$$

for all $x, y, z \in X$. □

THEOREM 2.2. *Assume that $\phi : X^3 \rightarrow [0, \infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$(2.1) \quad \begin{aligned} \phi(2x, 2y, 2z) &\leq 2L\phi(x, y, z), \\ \phi(x + \sigma(x), y + \sigma(y), z + \sigma(z)) &\leq 2L\phi(x, y, z) \end{aligned}$$

for all $x, y, z \in X$. Let $f : X \rightarrow Y$ be a mapping such that $f(0) = 0$ and

$$(2.2) \quad \|Df(x, y, z)\| \leq \phi(x, y, z)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ with involution such that

$$(2.3) \quad \|f(x) - Q(x)\| \leq \frac{1}{8(1-L)}\phi(x, x, 0)$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \rightarrow Y\}$ and the generalized metric d in S defined by

$$d(g, h) = \inf\{c \in [0, \infty) \mid \|g(x) - h(x)\| \leq c\phi(x, x, 0) \text{ for all } x \in X\}$$

for all $g, h \in S$. Then (S, d) is a complete metric space(See [11]). Define a mapping $J : S \rightarrow S$ by

$$Jg(x) = \frac{1}{4}\{g(2x) + g(x + \sigma(x))\}$$

for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some non-negative real number c . Then by (2.1), we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \frac{1}{4} \|g(2x) + g(x + \sigma(x)) - h(2x) - h(x + \sigma(x))\| \\ &\leq \frac{1}{4} [\|g(2x) - h(2x)\| + \|g(x + \sigma(x)) - h(x + \sigma(x))\|] \\ &\leq cL\phi(x, x, 0) \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $y = x$ and $z = 0$ in (2.2) and dividing both sides by 8, we get

$$\|Jf(x) - f(x)\| = \left\| \frac{1}{4} \{f(2x) + f(x + \sigma(x))\} - f(x) \right\| \leq \frac{1}{8} \phi(x, x, 0)$$

for all $x \in X$ and hence

$$(2.4) \quad d(Jf, f) \leq \frac{1}{8} < \infty.$$

By Theorem 1.2, there exists a mapping $Q : X \rightarrow Y$ which is a fixed point of J such that $d(J^n f, Q) \rightarrow 0$ as $n \rightarrow \infty$. By induction, we can easily show that

$$(J^n f)(x) = \frac{1}{2^{2n}} \left\{ f(2^n x) + (2^n - 1) f(2^{n-1}(x + \sigma(x))) \right\}$$

for all $x \in X$ and $n \in \mathbb{N}$. Thus for each fixed $x \in X$, we have

$$(2.5) \quad Q(x) = \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \left\{ f(2^n x) + (2^n - 1) f(2^{n-1}(x + \sigma(x))) \right\}.$$

From (2.2) and (2.5), we have

$$\begin{aligned} & \|DQ(x, y, z)\| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \left[\phi(2^n x, 2^n y, 2^n z) \right. \\ & \quad \left. + (2^n - 1) \phi(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y)), 2^{n-1}(z + \sigma(z))) \right] \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \left[2^n L^n \phi(x, y, z) + (2^n - 1) 2^n L^n \phi(x, y, z) \right] = 0 \end{aligned}$$

for all $x, y \in X$. Hence Q satisfies (1.2) and by Lemma 2.1, Q is a quadratic mapping with involution. By (4) in Theorem 1.2 and (2.4), Q satisfies (2.3).

Assume that $Q_1 : X \rightarrow Y$ is a quadratic mapping with involution satisfying (2.3). We know that Q_1 is a fixed point of J . Due to (3) in Theorem 1.2, we get $Q = Q_1$. This proves the uniqueness of Q . \square

COROLLARY 2.3. *Let r be a positive real number $r < 1$, and let $f : X \rightarrow Y$ be a mapping such that*

$$(2.6) \quad \|Df(x, y, z)\| \leq P(x)^r + P(y)^r + P(z)^r$$

for all $x, y, z \in X$ and suppose that $x + \sigma(x) \leq 2x$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ with involution such that

$$(2.7) \quad \|f(x) - Q(x)\| \leq \frac{P(x)^r}{2(2 - 2^r)}$$

for all $x \in X$.

Proof. Taking $\phi(x, y, z) = P(x)^r + P(y)^r + P(z)^r$ for all $x, y, z \in X$ and $L = 2^{r-1}$ in Theorem 2.2, we get desired result. \square

THEOREM 2.4. Assume that $\phi : Y^3 \rightarrow [0, \infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that

$$(2.8) \quad \begin{aligned} \phi(x, y, z) &\leq \frac{L}{8}\phi(2x, 2y, 2z), \\ \phi(x + \sigma(x), y + \sigma(y), z + \sigma(z)) &\leq 2\phi(2x, 2y, 2z) \end{aligned}$$

for all $x, y, z \in Y$. Let $f : Y \rightarrow X$ be a mapping such that $f(0) = 0$ and

$$(2.9) \quad P(Df(x, y, z)) \leq \phi(x, y, z)$$

for all $x, y, z \in Y$. Then there exists a unique quadratic mapping $Q : Y \rightarrow X$ with involution such that

$$(2.10) \quad P(4f(x) - Q(x)) \leq \frac{L}{2(1-L)}\phi(x, x, 0)$$

for all $x \in Y$.

Proof. Consider the set $S = \{g \mid g : Y \rightarrow X\}$ and the generalized metric d in S defined by $d(g, h) = \inf\{c \in [0, \infty) \mid P(g(x) - h(x)) \leq c\phi(x, x, 0) \text{ for all } x \in Y\}$. Then (S, d) is a complete metric space. Define a mapping $J : S \rightarrow S$ by

$$Jg(x) = 2\left\{2g\left(\frac{x}{2}\right) - g\left(\frac{x + \sigma(x)}{4}\right)\right\}$$

for all $x \in Y$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some non-negative real number c . Then by (2.8), we have

$$\begin{aligned} &P(Jg(x) - Jh(x)) \\ &\leq 2\left[2P\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right) + P\left(g\left(\frac{x + \sigma(x)}{4}\right) - h\left(\frac{x + \sigma(x)}{4}\right)\right)\right] \\ &\leq 2\left[2c\phi\left(\frac{x}{2}, \frac{x}{2}, 0\right) + c\phi\left(\frac{x + \sigma(x)}{4}, \frac{x + \sigma(x)}{4}, 0\right)\right] \\ &\leq cL\phi(x, x, 0) \end{aligned}$$

for all $x \in Y$. Hence $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $x = \frac{x}{2}$, $y = \frac{x}{2}$ and $z = 0$ in (2.9), we get

$$(2.11) \quad P\left(2f(x) + 2f\left(\frac{x + \sigma(x)}{2}\right) - 8f\left(\frac{x}{2}\right)\right) \leq \phi\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in Y$. By (2.11), we get

$$(2.12) \quad P\left(4f(x) + 4f\left(\frac{x + \sigma(x)}{2}\right) - 16f\left(\frac{x}{2}\right)\right) \leq 2\phi\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

and putting $x = \frac{x + \sigma(x)}{4}$ and $y = \frac{x + \sigma(x)}{4}$, $z = 0$ in (2.9), we get

$$(2.13) \quad \begin{aligned} & P\left(4f\left(\frac{x + \sigma(x)}{2}\right) - 8f\left(\frac{x + \sigma(x)}{4}\right)\right) \\ & \leq \phi\left(\frac{x + \sigma(x)}{4}, \frac{x + \sigma(x)}{4}, 0\right) \end{aligned}$$

for all $x \in Y$. Combining (2.12) and (2.13), by (2.8), we deduce that

$$\begin{aligned} P(J4f(x) - 4f(x)) &= P\left(16f\left(\frac{x}{2}\right) - 8f\left(\frac{x + \sigma(x)}{4}\right) - 4f(x)\right) \\ &\leq \frac{L}{2}\phi(x, x, 0) \end{aligned}$$

for all $x \in Y$ and hence

$$(2.14) \quad d(J4f, 4f) \leq \frac{L}{2} < \infty.$$

By Theorem 1.2, there exists a mapping $Q : Y \rightarrow X$ which is a fixed point of J such that $d(J^n 4f, Q) \rightarrow 0$ as $n \rightarrow \infty$. By induction, we can easily show that

$$(J^n 4f)(x) = 4^n f\left(\frac{x}{2^n}\right) + 2^n(1 - 2^n)f\left(\frac{x + \sigma(x)}{2^{n+1}}\right)$$

for each $n \in \mathbb{N}$. Thus for each fixed $x \in Y$, we have

$$(2.15) \quad Q(x) = \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right) + 2^n(1 - 2^n)f\left(\frac{x + \sigma(x)}{2^{n+1}}\right).$$

It follows from (2.9) and (2.15) that

$$\begin{aligned} P(DQ(x, y, z)) &\leq \lim_{n \rightarrow \infty} 4^n \left\{ \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \right. \\ &\quad \left. + \left(\frac{1}{2^n} - 1\right) \phi\left(\frac{x + \sigma(x)}{2^{n+1}}, \frac{y + \sigma(y)}{2^{n+1}}, \frac{z + \sigma(z)}{2^{n+1}}\right) \right\} = 0 \end{aligned}$$

for all $x, y, z \in Y$. Hence Q satisfies (1.2) and Q is a quadratic mapping with involution. By (4) in Theorem 1.2 and (2.14), Q satisfies (2.10).

Assume that $Q_1 : Y \rightarrow X$ is a quadratic mapping with involution satisfying (2.10). We know that Q_1 is a fixed point of J . Due to (3) in Theorem 1.2, we get $Q = Q_1$. This proves the uniqueness of Q . \square

COROLLARY 2.5. *Let r, θ be positive real numbers with $r > 3$, and let $f : Y \rightarrow X$ be a mapping such that*

$$(2.16) \quad P(Df(x, y, z)) \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \text{ and } \|x + \sigma(x)\| \leq 2\|x\|$$

for all $x, y, z \in Y$. Then there exists a unique quadratic mapping $Q : Y \rightarrow X$ such that

$$(2.17) \quad P(4f(x) - Q(x)) \leq \frac{2^{3-r}}{1 - 2^{3-r}} \theta \|x\|^r$$

for all $x \in Y$.

Proof. Taking $\phi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in Y$ and $L = 2^{3-r}$ in Theorem 2.4, we get desired result. \square

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