DUAL QUATERNIONIC REGULAR FUNCTION OF DUAL QUATERNION VARIABLES

Ji Eun Kim [a] and Kwang Ho Shon [b,∗]

Abstract. We give representations of differential operators and rules for addition and multiplication of dual quaternions. Also, we research the notions and properties of a regular function and a corresponding harmonic function with values in dual quaternions of Clifford analysis.

1. Introduction

Quaternions have been developed by Hamilton's the discovery and studies. Hamilton [6] extended the theory of functions of a quaternion variable by using the theory of functions of several real variables. Tait [17] and Joly [7] developed a special class of regular functions which had quaternion-valued functions of a quaternion variable. In 1935, Fueter [4, 5] studied the definition of regularity for quaternionic functions from an analogue of the Cauchy-Riemann equations, Cauchy theorem and Cauchy integral formula. Based Fueter's results of the theory of quaternionic analysis, Deavours [2] gave the simpler fundamentals of quaternionic analysis. Sudbery [16] researched the notation and algebraic properties of quaternions and proposed the power series representing a regular function in the algebra of quaternions. Kajiwara et al. [8, 9] applied the theory on a Hilbert space and brconvex domains and studied an inhomogeneous Cauchy-Riemann system in quaternion analysis. Kim et al. [10, 11, 12, 13] researched properties of functions with values in special quaternions such as reduced quaternion and split quaternions by using each a corresponding Cauchy-Riemann system. They [14, 15] also investigated properties of the differential operators and regular functions defined by those operators of special quaternion numbers.

Dual number was defined and developed by Clifford in 1873 and applications of dual number were studied by Kotelnikov in 1895. Yaglom [18] provided a description and basic properties of dual numbers in 1963. Deakin [1] gives the definition and theorems of analytic functions of a dual variable. Ferdinands et al. [3] obtained the theorems of that a Laguerre transformation on the space of parabolas in the dual plane.

In this paper, we give a representation of dual quaternions and their calculations. Also, we investigate the definition and properties of a regular function and a corresponding harmonic function of dual quaternion-valued functions in Clifford analysis.

2. PRELIMINARIES

We consider the notions and representations of dual-quaternions. Let 1, i, j, k be bases of the algebra of quaternions, denoted by $\mathbb{H}$, with the following rules:

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j.$$

Let the set of dual quaternions be

$$\mathbb{H}(D) := \{P \mid P = \lambda_0 + i\lambda_1 + j\lambda_2 + k\lambda_3, \ \lambda_r \in \mathbb{D} \ (r = 0, 1, 2, 3)\},$$

where $\mathbb{D}$ is the set of dual numbers. For an element $P \in \mathbb{H}(D)$, we give the forms of the real part and the vector part of a dual quaternion $P = (\lambda_r, \lambda_v)$, where $\lambda_r = \lambda_0$ is the non-pure part, $\lambda_v = i\lambda_1 + j\lambda_2 + k\lambda_3$ is the pure part of P and $\lambda_l = x_l + \varepsilon y_l$ $(l = 0, 1, 2, 3)$ are dual numbers. The addition and multiplication of elements P and Q of dual quaternions are given by

$$
\begin{aligned}
P + Q &= (\lambda_0 + \mu_0) + i(\lambda_1 + \mu_1) + j(\lambda_2 + \mu_2) + k(\lambda_3 + \mu_3), \\
PQ &= \lambda_0\mu_0 - \lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3 \\
&\quad + i(\lambda_0\mu_1 + \lambda_1\mu_0 + \lambda_2\mu_3 - \lambda_3\mu_2) \\
&\quad + j(\lambda_0\mu_2 - \lambda_1\mu_3 + \lambda_2\mu_0 + \lambda_3\mu_1) \\
&\quad + k(\lambda_0\mu_3 + \lambda_1\mu_2 - \lambda_2\mu_1 + \lambda_3\mu_0).
\end{aligned}
$$

The dual quaternions $\mathbb{H}(D)$ have a eight-dimensional algebra over the real field $\mathbb{R}$, with an identity element 1. Also, we write $\mathbb{H}(D) = \mathbb{R} \bigoplus V(D)$, where $V(D)$ is a three-dimensional dual vector space, that is, a six-dimensional vector space, and the

product of two quaternions is given by

$$(\lambda_r; \lambda_v)(\mu_r; \mu_v) = (\lambda_r\mu_r - \lambda_v\widehat{\cdot}\mu_v; \lambda_r\mu_v + \mu_r\lambda_v + \lambda_v\widehat{\times}\mu_v),$$

where $\lambda_r, \mu_r \in \mathbb{D}$, $\lambda_v, \mu_v \in V(D)$,

$$\lambda_v\widehat{\cdot}\mu_v := x_1l_1 + x_2l_2 + x_3l_3 + \varepsilon(x_1m_1 + x_2m_2 + x_3m_3 + y_1l_1 + y_2l_2 + y_3l_3)$$

is the inner product of the elements of $V(D)$, and

$$\begin{aligned}
\lambda_v\widehat{\times}\mu_v \quad := \quad & (x_2l_3 - x_3l_2;\ x_3l_1 - x_1l_3;\ x_1l_2 - x_2l_1) \\
& + \varepsilon(x_2m_3 - x_3m_2 + y_2l_3 - y_3l_2;\ x_3m_1 - x_1m_3 + y_3l_1 - y_1l_3; \\
& x_1m_2 - x_2m_1 + y_1l_2 - y_2l_1)
\end{aligned}$$

is the vector product on $V(D)$.

The conjugate of the dual quaternion P is given by

$$P^* = \lambda_0 - i\lambda_1 - j\lambda_2 - k\lambda_3.$$

For every element of $\mathbb{H}(D)$, their product is

(2.1) $$PP^* = P^*P = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

The modulus, denoted by M_P, of P is the dual number

$$M_P = PP^*.$$

From (2.1), it follows for every non-zero divisor dual quaternion, it has a multiplicative inverse element of $\mathbb{H}(D)$,

$$P^{-1} = \frac{P^*}{M_P} \quad (x_t \neq 0,\ t = 0, 1, 2, 3).$$

Now, we consider the following differential operators in $\mathbb{H}(D)$:

$$D := \frac{\partial}{\partial\lambda_0} + i\frac{\partial}{\partial\lambda_1} - j\frac{\partial}{\partial\lambda_2} - k\frac{\partial}{\partial\lambda_3}$$

and

$$D^* = \frac{\partial}{\partial\lambda_0} - i\frac{\partial}{\partial\lambda_1} + j\frac{\partial}{\partial\lambda_2} + k\frac{\partial}{\partial\lambda_3},$$

where

$$\frac{\partial}{\partial\lambda_t} := \frac{1}{2}\left(\frac{\partial}{\partial x_t} + \varepsilon\frac{\partial}{\partial y_t}\right) \quad (t = 0, 1, 2, 3).$$

Consider an open subset Ω of $\mathbb{H}(D)$ and a function $F : \Omega \to \mathbb{H}(D)$ of class $C^1(\Omega, \mathbb{H}(D))$ such that

$$F(P) = f_0 + if_1 + jf_2 + kf_3,$$

called a dual quaternion-valued function, where

$$f_t : \mathbb{D}^4 \rightarrow \mathbb{D}, \quad f_t = f_t(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$$

are dual number-valued functions and $f_t = u_t + \varepsilon v_t$ with $u_t,\ v_t : \mathbb{R}^8 \rightarrow \mathbb{R}$ ($t = 0, 1, 2, 3$).

Definition 2.1. Let Ω be an open set in $\mathbb{H}(D)$. The function F is said to be *left-regular* in Ω if f_t ($t = 0, 1, 2, 3$) are continuously differentiable quaternion-valued functions and the equation $D^*F = 0$ is satisfied.

From the calculation of the equation $D^*F = 0$ in Definition 2.1, we have the equivalent equations of $D^*F = 0$:

$$
\begin{aligned}
D^*F &= \frac{\partial F}{\partial \lambda_0} - i\frac{\partial F}{\partial \lambda_1} + j\frac{\partial F}{\partial \lambda_2} + k\frac{\partial F}{\partial \lambda_3} \\
&= \left(\frac{\partial f_0}{\partial \lambda_0} + \frac{\partial f_1}{\partial \lambda_1} - \frac{\partial f_2}{\partial \lambda_2} - \frac{\partial f_3}{\partial \lambda_3}\right) \\
&\quad + i\left(-\frac{\partial f_0}{\partial \lambda_1} + \frac{\partial f_1}{\partial \lambda_0} - \frac{\partial f_2}{\partial \lambda_3} + \frac{\partial f_3}{\partial \lambda_2}\right) \\
&\quad + j\left(\frac{\partial f_0}{\partial \lambda_2} + \frac{\partial f_1}{\partial \lambda_3} + \frac{\partial f_2}{\partial \lambda_0} + \frac{\partial f_3}{\partial \lambda_1}\right) \\
&\quad + k\left(\frac{\partial f_0}{\partial \lambda_3} - \frac{\partial f_1}{\partial \lambda_2} - \frac{\partial f_2}{\partial \lambda_1} + \frac{\partial f_3}{\partial \lambda_0}\right).
\end{aligned}
$$

Hence, we have a corresponding Cauchy-Riemann system as follows:

$$
(2.2) \quad
\begin{cases}
\dfrac{\partial f_0}{\partial \lambda_0} + \dfrac{\partial f_1}{\partial \lambda_1} - \dfrac{\partial f_2}{\partial \lambda_2} - \dfrac{\partial f_3}{\partial \lambda_3} = 0, \\[2mm]
-\dfrac{\partial f_0}{\partial \lambda_1} + \dfrac{\partial f_1}{\partial \lambda_0} - \dfrac{\partial f_2}{\partial \lambda_3} + \dfrac{\partial f_3}{\partial \lambda_2} = 0, \\[2mm]
\dfrac{\partial f_0}{\partial \lambda_2} + \dfrac{\partial f_1}{\partial \lambda_3} + \dfrac{\partial f_2}{\partial \lambda_0} + \dfrac{\partial f_3}{\partial \lambda_1} = 0 \\[2mm]
\dfrac{\partial f_0}{\partial \lambda_3} - \dfrac{\partial f_1}{\partial \lambda_2} - \dfrac{\partial f_2}{\partial \lambda_1} + \dfrac{\partial f_3}{\partial \lambda_0} = 0.
\end{cases}
$$

For example, a function $F(P) = P$ is left-regular in $\mathbb{H}(D)$ since the equation $D^*F = 0$ is satisfied. In other words, functions $F(P) = P^*$ and $F(P) = P^{-1}$ are not left-regular since $D^*F \neq 0$.

Also, from the definition of the inner product of dual quaternions, we have the Laplacian operator, denoted by $\triangle_{\mathbb{H}(D)}$, as follows:

$$(2.3) \qquad \triangle_{\mathbb{H}(D)} := DD^* = \sum_{t=0}^{3} \frac{\partial^2}{\partial \lambda_t^2}.$$

Definition 2.2. Let U be an open set of $\mathbb{D}^4$ and a function f_r $(r = 0, 1, 2, 3)$ be defined on U. Then f_r is said to be *harmornic* if it satisfies the following equation:

$$\triangle_{\mathbb{H}(D)} f_r = 0 \quad (r = 0, 1, 2, 3).$$

Definition 2.3. Let Ω be an open set of $\mathbb{H}(D)$ and a function F be defined on Ω. Then F is said to be *harmornic* in $\mathbb{H}(D)$ if all components of F are harmornic.

Theorem 2.4. *Let Ω be an open set of $\mathbb{H}(D)$ and a function F be left-regular on Ω. Then the equation*

$$(2.4) \qquad DF = \frac{\partial F}{\partial \lambda_0} = i \frac{\partial F}{\partial \lambda_1} - j \frac{\partial F}{\partial \lambda_2} - k \frac{\partial F}{\partial \lambda_3}$$

is satisfied.

Proof. From the definition of D, we have

$$
\begin{aligned}
DF &= \left(\frac{\partial F}{\partial \lambda_0} + i \frac{\partial F}{\partial \lambda_1} - j \frac{\partial F}{\partial \lambda_2} - k \frac{\partial F}{\partial \lambda_3} \right) \\
&= \left(\frac{\partial f_0}{\partial \lambda_0} - \frac{\partial f_1}{\partial \lambda_1} + \frac{\partial f_2}{\partial \lambda_2} + \frac{\partial f_3}{\partial \lambda_3} \right) \\
&\quad + i \left(\frac{\partial f_0}{\partial \lambda_1} + \frac{\partial f_1}{\partial \lambda_0} + \frac{\partial f_2}{\partial \lambda_3} - \frac{\partial f_3}{\partial \lambda_2} \right) \\
&\quad + j \left(-\frac{\partial f_0}{\partial \lambda_2} - \frac{\partial f_1}{\partial \lambda_3} + \frac{\partial f_2}{\partial \lambda_0} - \frac{\partial f_3}{\partial \lambda_1} \right) \\
&\quad + k \left(-\frac{\partial f_0}{\partial \lambda_3} + \frac{\partial f_1}{\partial \lambda_2} + \frac{\partial f_2}{\partial \lambda_1} + \frac{\partial f_3}{\partial \lambda_0} \right).
\end{aligned}
$$

By rearranging the terms of the above equations and applying the equation (2.2), we obtain the result (2.4). $\qquad \square$

Remark 2.5. From the equation (2.1), we obtain that each regular function is also harmonic. And, from the Definition 2.1 and the equation (2.1), if F is harmonic, then DF is regular in $\mathbb{H}(D)$.

Theorem 2.6. *Let Ω be an open neighborhood $\Omega \subset \mathbb{D}^4$ of zero and φ be a function defined on Ω with values in dual numbers such that*

$$\varphi = \varphi_1 + \varepsilon \varphi_2, \quad \varphi_r = \varphi_r(\lambda_0, \lambda_1, \lambda_2, \lambda_3).$$

If φ is harmonic and continuously second differentiable, then there exists a regular function F defined on Ω such that

$$F(P) := \varphi + Pu\Big\{\int_0^1 l^2 D\varphi(lP)P \; dl\Big\},$$

where $0 \leq l \leq 1$ and $Pu\{\}$ is the pure part of $\{\}$.

Proof. We assume that $0 \in \Omega$ and for any $P \in \Omega$ and $0 \leq l \leq 1$ we have

$$lP_0 + (1 - l)P \in \Omega$$

with respect to P_0. We let the function

$$F(P) := \varphi + Pu\int_0^1 l^2 D\varphi(lP)P \; dl.$$

Since we have

$$NP\Big\{\int_0^1 l^2 D\varphi(lP)P \; dl\Big\}$$

$$= \int_0^1 l^2 \Big(\frac{\partial\varphi(lP)}{\partial\lambda_0}\lambda_0 + i\frac{\partial\varphi(lP)}{\partial\lambda_1}\lambda_1 - j\frac{\partial\varphi(lP)}{\partial\lambda_2}\lambda_2 - k\frac{\partial\varphi(lP)}{\partial\lambda_3}\lambda_3\Big) \; dl$$

$$= \int_0^1 l^2 \frac{d\varphi(lP)}{dl} \; dl = \varphi(P) - \int_0^1 l\varphi(lP) \; dl,$$

where $NP\{\}$ is the non-pure part of $\{\}$, we obtain the function

$$F(P) = \int_0^1 (l^2 D\varphi(lP)P + l\varphi(lP)) \; dl.$$

Since φ and $D\varphi$ have continuously differentiable in Ω, for $P \in \Omega$,

$$D^*F(P) = \int_0^1 l^2 D^*(D\varphi(lP))P \; dl$$

$$+ \int_0^1 l^2 \{D\varphi(lP) + iD\varphi(lP)i + jD\varphi(lP)j + kD\varphi(lP)k\} \; dl.$$

$$+ \int_0^1 l^2 D^*\varphi(lP) \; dl.$$

In other words,

$$D^*(D\varphi(lP)) = l\triangle_{\mathbb{H}(D)}\varphi(lP) = 0.$$

Since φ is harmonic in Ω and φ is a dual number, we have

$$D\varphi(lP) + iD\varphi(lP)i + jD\varphi(lP)j + kD\varphi(lP)k = D^*\varphi(lP).$$

Hence, $D^*F = 0$ in Ω and so F is regular in Ω. $\qquad\qquad\square$

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[a]Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea
Email address: jeunkim@pusan.ac.kr

[b]Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea
Email address: khshon@pusan.ac.kr