

AN IMPROVED LOWER BOUND FOR SCHWARZ LEMMA AT THE BOUNDARY

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ABSTRACT. In this paper, a boundary version of the Schwarz lemma for the holomorphic function satisfying $f(a) = b$, $|a| < 1$, $b \in \mathbb{C}$ and $\Re f(z) > \alpha$, $0 \leq \alpha < |b|$ for $|z| < 1$ is investigated. Also, we estimate a modulus of the angular derivative of $f(z)$ function at the boundary point c with $\Re f(c) = \alpha$. The sharpness of these inequalities is also proved.

1. INTRODUCTION

The classical Schwarz lemma gives information about the behavior of a holomorphic function on the unit disc $D = \{z : |z| < 1\}$ at the origin, subject only to the relatively mild hypotheses that the function map the unit disc to the disc and the origin to the origin. This lemma, named after Hermann Amandus Schwarz, is a result in complex analysis about holomorphic functions defined on the unit disc. In its most basic form, the familiar Schwarz lemma says this ([5], p.329):

Let D be the unit disc in the complex plane \mathbb{C} . Let $f : D \rightarrow D$ be a holomorphic function with $f(0) = 0$. Under these circumstances $|f(z)| \leq |z|$ for all $z \in D$, and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$ then f is a rotation, that is, $f(z) = ze^{i\theta}$, θ real.

For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [1], [19]).

Let $f(z)$ be holomorphic function in D , $f(a) = b$, $|a| < 1$, $b \in \mathbb{C}$ and $\Re f(z) > \alpha$, $0 \leq \alpha < |b|$ for $|z| < 1$.

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Let

$$\omega(z) = f\left(\frac{z+a}{1+\bar{a}z}\right)$$

and

$$F(z) = \frac{\omega(z) - \alpha}{b - \alpha}.$$

$F(z)$ is holomorphic and $\Re f(z) > 0$ for $|z| < 1$ and hence

$$\phi(z) = \frac{1 - F(z)}{1 + F(z)}$$

is holomorphic, $|\phi(z)| < 1$ for $|z| < 1$ and $\phi(0) = 0$. Thus, by Schwarz lemma, we obtain

$$(1.1) \quad |f'(a)| \leq 2 \frac{|b - \alpha|}{1 - |a|^2}.$$

The inequality in (1.1) is sharp with equality for the function

$$f(z) = \alpha + (b - \alpha) \frac{1 - \bar{a}z - z + a}{1 - \bar{a}z + z + a},$$

where $-1 < a \leq 0$ and b is any integer ≥ 1 .

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point to with $|c| = 1$, and if $|f(c)| = 1$ and $f'(c)$ exists, then $|f'(c)| \geq 1$. This result of Schwarz lemma and its generalization are described as Schwarz lemma at the boundary in the literature. This improvement was obtained in [20] by Helmut Unkelbach, and rediscovered by R. Osserman in [15] 60 years later.

In the last 15 years, there have been tremendous studies on Schwarz lemma at the boundary (see, [1], [3], [4], [6], [7], [9], [10], [15], [16], [17], [19] and references therein). Some of them are about the below boundary of modulus of the functions derivation at the points (contact points) which satisfies $|f(c)| = 1$ condition of the boundary of the unit circle.

In [15], R. Osserman offered the following boundary refinement of the classical Schwarz lemma. It is very much in the spirit of the sort of result.

Lemma 1.1. *Let $f : D \rightarrow D$ be holomorphic function with $f(0) = 0$. Assume that there is a $c \in \partial D$ so that f extends continuously to c , $|f(c)| = 1$ and $f'(c)$ exists. Then*

$$(1.2) \quad |f'(c)| \geq \frac{2}{1 + |f'(0)|}.$$

Inequality (1.2) is sharp, with equality possible for each value of $|f'(0)|$.

Corollary 1.2. *Under the hypotheses Lemma 1.1, we have*

$$(1.3) \quad |f'(c)| \geq 1$$

and

$$|f'(c)| > 1 \text{ unless } f(z) = ze^{i\theta}, \theta \text{ real.}$$

Moreover, if $f(z) = a_p z^p + a_{p+1} z^{p+1} \dots$, then

$$(1.4) \quad |f'(c)| \geq p + \frac{1 - |a_p|}{1 + |a_p|}.$$

The equality in (1.4) occurs for the function $f(z) = z^p (z + \gamma) / (1 + \gamma z)$, $0 \leq \gamma \leq 1$.

Lemma 1.3 (Julia-Wolff lemma). *Let f be a holomorphic function in D , $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$ (see [18]).*

D. M. Burns and S. G. Krantz [8] and D. Chelst [2] studied the uniqueness part of the Schwarz lemma. In M. Mateljević's papers, for more general results and related estimates, see also ([11], [12], [13] and [14]).

Also, M. Jeong [7] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [6] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. MAIN RESULTS

Theorem 2.1. *Let $f(z)$ be holomorphic function in D , $f(a) = b$, $|a| < 1$, $b \in \mathbb{C}$ and $\Re f(z) > \alpha$, $0 \leq \alpha < |b|$ for $|z| < 1$. Assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c , $\Re f(c) = \alpha$. Then*

$$(2.1) \quad |f'(c)| \geq \frac{|b - \alpha|}{2} \frac{1 - |a|}{1 + |a|}.$$

The inequality (2.1) is sharp, with equality for the function

$$f(z) = \alpha + (b - \alpha) \frac{1 - \bar{a}z - z + a}{1 - \bar{a}z + z + a},$$

where $-1 < a \leq 0$ and b is any integer ≥ 1 .

Proof. Consider the function

$$\phi(z) = \frac{1 - F(z)}{1 + F(z)}.$$

Then $\phi(z)$ is holomorphic function in the unit disc D , $\phi(0) = 0$ and $|\phi(z)| < 1$. In addition, for $z_0 = \frac{c-a}{1-\bar{a}c} \in \partial D$,

$$\omega(z_0) = f\left(\frac{z_0 + a}{1 + \bar{a}z_0}\right)$$

and since $\Re f(c) = \alpha$, we take

$$|\phi(z_0)| = \left| \frac{1 - F(z_0)}{1 + F(z_0)} \right| = 1$$

From (1.3), we obtain

$$1 \leq |\phi'(z_0)| = 2 \frac{|F'(z_0)|}{|1 + F(z_0)|^2} = 2 \frac{|\omega'(z_0)|}{|b - \alpha| |1 + F(z_0)|^2}.$$

Since $|1 + F(z_0)|^2 \geq (\Re(1 + F(z_0)))^2 = (1 + \Re F(z_0))^2 = \left[1 + \Re\left(\frac{\omega(z_0) - \alpha}{b - \alpha}\right)\right]^2 = 1$, we have

$$1 \leq \frac{2}{|b - \alpha|} \frac{1 - |a|^2}{(1 + \bar{a}z)^2} \left| f' \left(\frac{z_0 + a}{1 + \bar{a}z_0} \right) \right| \leq \frac{2}{|b - \alpha|} \frac{1 + |a|}{1 - |a|} \left| f' \left(\frac{z_0 + a}{1 + \bar{a}z_0} \right) \right|.$$

Thus, we get

$$|f'(c)| \geq \frac{|b - \alpha|}{2} \frac{1 - |a|}{1 + |a|}.$$

Now, we shall that the inequality (2.1) is sharp. Let

$$f(z) = \alpha + (b - \alpha) \frac{1 - \bar{a}z - z + a}{1 - \bar{a}z + z + a}.$$

Then, we take

$$|f'(1)| = \frac{|b - \alpha|}{2} \frac{1 + a}{1 - a} = \frac{|b - \alpha|}{2} \frac{1 + a}{1 - a}.$$

Since $-1 < a \leq 0$ and b is any integer ≥ 1 , (2.1) is satisfied with equality. \square

Theorem 2.2. *Let $f(z)$ be holomorphic function in D , $f(a) = b$, $|a| < 1$, $b \in \mathbb{C}$ and $\Re f(z) > \alpha$, $0 \leq \alpha < |b|$ for $|z| < 1$. Assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c , $\Re f(c) = \alpha$. Then*

$$(2.2) \quad |f'(c)| \geq \frac{1 - |a|}{1 + |a|} \frac{2|b - \alpha|^2}{2|b - \alpha| + (1 - |a|^2)|f'(a)|}.$$

The equality in (2.2) occurs for the function

$$f(z) = \alpha + (b - \alpha) \frac{1 - \frac{z-a}{1-\bar{a}z}}{\left(\frac{z-a}{1-\bar{a}z}\right)^2 + 2m \left(\frac{z-a}{1-\bar{a}z}\right) + 1},$$

where $-1 < a \leq 0$, b is any integer ≥ 1 and $m = \frac{|f'(a)|(1-|a|^2)}{2|b-\alpha|}$ is an arbitrary number from $[0, 1]$ (see (1.1)).

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1. From (1.2), for $z_0 = \frac{c-a}{1-\bar{a}c} \in \partial D$, we obtain

$$\begin{aligned} \frac{2}{1+|\phi'(0)|} &\leq |\phi'(z_0)| = 2 \frac{|F'(z_0)|}{|1+F(z_0)|^2} = 2 \frac{|\omega'(z_0)|}{|b-\alpha|} \frac{1}{|1+F(z_0)|^2} \\ &\leq \frac{2}{|b-\alpha|} \frac{1+|a|}{1-|a|} \left| f' \left(\frac{z_0+a}{1+\bar{a}z_0} \right) \right|. \end{aligned}$$

Since

$$|\phi'(0)| = \frac{(1-|a|^2)|f'(a)|}{2|b-\alpha|},$$

we have

$$\frac{2}{1 + \frac{(1-|a|^2)|f'(a)|}{2|b-\alpha|}} \leq \frac{2}{|b-\alpha|} \frac{1+|a|}{1-|a|} \left| f' \left(\frac{z_0+a}{1+\bar{a}z_0} \right) \right|.$$

Therefore, we take the inequality (2.2).

To show that the inequality (2.2) is sharp, take the holomorphic function

$$f(z) = \alpha + (b-\alpha) \frac{1 - \frac{z-a}{1-\bar{a}z}}{\left(\frac{z-a}{1-\bar{a}z}\right)^2 + 2m \left(\frac{z-a}{1-\bar{a}z}\right) + 1}.$$

Then

$$|f'(1)| = \frac{1+|a|}{1-|a|} \frac{2(b-\alpha)^2}{2(b-\alpha) + (1-|a|^2)|f'(a)|}.$$

Since $-1 < a \leq 0$ and b is any integer ≥ 1 , (2.2) is satisfied with equality. \square

If $f(z) = b + a_p(z-a)^p + a_{p+1}(z-a)^{p+1} + \dots$ is a holomorphic function in D $f(a) = b$, $|a| < 1$, $b \in \mathbb{C}$ and $\Re f(z) > \alpha$, $0 \leq \alpha < |b|$ for $|z| < 1$, then

$$(2.3) \quad |a_p| \leq 2 \frac{|b-\alpha|}{(1-|a|^2)^p}.$$

Theorem 2.3. *Let $f(z) = b + a_p(z-a)^p + a_{p+1}(z-a)^{p+1} + \dots$ is a holomorphic function in D $f(a) = b$, $|a| < 1$, $b \in \mathbb{C}$ and $\Re f(z) > \alpha$, $0 \leq \alpha < |b|$ for $|z| < 1$. Assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c , $\Re f(c) = \alpha$. Then*

$$(2.4) \quad |f'(c)| \geq \frac{|b-\alpha|}{2} \frac{1-|a|}{1+|a|} \left[p + \frac{2|b-\alpha| - (1-|a|^2)^p |a_p|}{2|b-\alpha| + (1-|a|^2)^p |a_p|} \right].$$

Equality in (2.4) occurs for the function

$$f(z) = \alpha + (b - \alpha) \frac{1 + d \left(\frac{z-a}{1-\bar{a}z} \right) - \left(\frac{z-a}{1-\bar{a}z} \right)^{p+1} - d \left(\frac{z-a}{1-\bar{a}z} \right)^p}{1 + d \left(\frac{z-a}{1-\bar{a}z} \right) + \left(\frac{z-a}{1-\bar{a}z} \right)^{p+1} + d \left(\frac{z-a}{1-\bar{a}z} \right)^p},$$

where $-1 < a \leq 0$, b is any integer ≥ 1 and $d = \frac{|a_p|(1-|a|^2)^p}{2|b-\alpha|}$ is an arbitrary number from $[0, 1]$ (see (2.3)).

Proof. Using the inequality (1.4) for the function $\phi(z)$, for $z_0 = \frac{c-a}{1-\bar{a}c} \in \partial D$, we obtain

$$p + \frac{1 - |n_p|}{1 + |n_p|} \leq |\phi'(z_0)| \leq \frac{2}{|b-\alpha|} \frac{1 + |a|}{1 - |a|} \left| f' \left(\frac{z_0 + a}{1 + \bar{a}z_0} \right) \right|.$$

Also, since

$$|n_p| = \frac{|a_p| (1 - |a|^2)^p}{2|b-\alpha|},$$

we may write

$$p + \frac{1 - \frac{|a_p|(1-|a|^2)^p}{2|b-\alpha|}}{1 + \frac{|a_p|(1-|a|^2)^p}{2|b-\alpha|}} \leq \frac{2}{|b-\alpha|} \frac{1 + |a|}{1 - |a|} \left| f' \left(\frac{z_0 + a}{1 + \bar{a}z_0} \right) \right|.$$

Thus, we take the inequality (2.4).

The equality (2.4) is obtained for the function

$$f(z) = \alpha + (b - \alpha) \frac{1 + d \left(\frac{z-a}{1-\bar{a}z} \right) - \left(\frac{z-a}{1-\bar{a}z} \right)^{p+1} - d \left(\frac{z-a}{1-\bar{a}z} \right)^p}{1 + d \left(\frac{z-a}{1-\bar{a}z} \right) + \left(\frac{z-a}{1-\bar{a}z} \right)^{p+1} + d \left(\frac{z-a}{1-\bar{a}z} \right)^p},$$

as show simple calculations. \square

In Theorem 2.3, the inequality (2.4) is obtained by adding the term a_p of $f(z)$ function. In the following theorem, the inequality (2.4) is obtained by adding a_p and a_{p+1} that are consecutive terms of $f(z)$ function.

Theorem 2.4. *Let $f(z) = b + a_p(z-a)^p + a_{p+1}(z-a)^{p+1} + \dots$, $p \geq 1$ is a holomorphic function in D $f(a) = b$, $|a| < 1$, $b \in \mathbb{C}$ and $\Re f(z) > \alpha$, $0 \leq \alpha < |b|$ for $|z| < 1$. Assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c , $\Re f(c) = \alpha$. Then*

$$(2.5) \quad |f'(c)| \geq \frac{|b-\alpha|}{2} \frac{1-|a|}{1+|a|} \left(p + \frac{2(2|b-\alpha| - (1-|a|^2)^p |a_p|)^2}{4|b-\alpha|^2 - (1-|a|^2)^{2p} |a_p|^2 + 2|b-\alpha|(1-|a|^2)^p |(1-|a|^2)a_{p+1} - \bar{a}pa_p|} \right).$$

In addition, the equality in (2.5) occurs for the function

$$f(z) = \alpha + (b-\alpha) \left[1 - \frac{2 \left(\frac{z-a}{1-\bar{a}z} \right)^p}{1 + \left(\frac{z-a}{1-\bar{a}z} \right)^p} \right],$$

where $-1 < a \leq 0$ and b is any integer ≥ 1 .

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1. Also, let

$$k(z) = \frac{\phi(z)}{z^p}.$$

We know that, from the maximum principle for each $z \in D$, we have $|\phi(z)| \leq |z|^p$. Thus, $k(z)$ is holomorphic function in D and $|k(z)| < 1$ for $|z| < 1$. In particular, we have

$$(2.6) \quad |k(0)| = \frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \leq 1$$

and

$$|k'(0)| = \frac{(1-|a|^2)^p}{2|b-\alpha|} \left| (1-|a|^2) a_{p+1} - \bar{a}pa_p \right|.$$

Moreover, we can see that, for $z_0 = \frac{c-a}{1-\bar{a}c} \in \partial D$,

$$\frac{z_0 \phi'(z_0)}{\phi(z_0)} = |\phi'(z_0)| \geq |\hbar'(z_0)| = \frac{z_0 \hbar'(z_0)}{\hbar(z_0)},$$

where $\hbar(z) = z^p$.

The composite function

$$\mathfrak{F}(z) = \frac{k(z) - k(0)}{1 - \overline{k(0)}k(z)}$$

satisfies the assumptions of the Schwarz lemma on the boundary, whence we obtain

$$\begin{aligned} \frac{2}{1 + |\mathfrak{F}'(0)|} &\leq |\mathfrak{F}'(z_0)| = \frac{1 - |k(0)|^2}{|1 - \overline{k(0)}k(z_0)|^2} |k'(c)| \leq \frac{1 + |k(0)|}{1 - |k(0)|} |k'(z_0)| \\ &= \frac{1 + |k(0)|}{1 - |k(0)|} \{ |\phi'(z_0)| - |\hbar'(z_0)| \}. \end{aligned}$$

Because

$$\mathfrak{F}'(z) = \frac{1 - |k(0)|^2}{\left(1 - \overline{k(0)}k(z)\right)^2} k'(z)$$

and

$$\begin{aligned} |\mathfrak{F}'(0)| &= \frac{|k'(0)|}{1 - |k(0)|^2} = \frac{\frac{(1-|a|^2)^p}{2|b-\alpha|} \left| (1-|a|^2) a_{p+1} - \bar{a} p a_p \right|}{1 - \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right)^2} \\ &= \frac{2|b-\alpha| (1-|a|^2)^p \left| (1-|a|^2) a_{p+1} - \bar{a} p a_p \right|}{4|b-\alpha|^2 - (1-|a|^2)^{2p} |a_p|^2}, \end{aligned}$$

we take

$$\frac{2}{1 + \frac{2|b-\alpha|(1-|a|^2)^p \left| (1-|a|^2) a_{p+1} - \bar{a} p a_p \right|}{4|b-\alpha|^2 - (1-|a|^2)^{2p} |a_p|^2}} \leq \frac{1 + \frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|}}{1 - \frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|}} \left(\frac{2}{|b-\alpha|} \frac{1+|a|}{1-|a|} \left| f' \left(\frac{z_0+a}{1+\bar{a}z_0} \right) \right| - p \right).$$

Therefore, we have

$$|f'(c)| \geq \frac{|b-\alpha|}{2} \frac{1-|a|}{1+|a|} \left(p + \frac{2(2|b-\alpha| - (1-|a|^2)^p |a_p|)^2}{4|b-\alpha|^2 - (1-|a|^2)^{2p} |a_p|^2 + 2|b-\alpha|(1-|a|^2)^p \left| (1-|a|^2) a_{p+1} - \bar{a} p a_p \right|} \right).$$

Now, we shall that the inequality (2.5) is sharp.

Consider the function

$$f(z) = \alpha + (b-\alpha) \left[1 - \frac{2 \left(\frac{z-a}{1-\bar{a}z} \right)^p}{1 + \left(\frac{z-a}{1-\bar{a}z} \right)^p} \right].$$

Then

$$\begin{aligned} f \left(\frac{z+a}{1+\bar{a}z} \right) &= \alpha + (b-\alpha) \left[1 - \frac{2z^p}{1+z^p} \right] = \alpha + (b-\alpha) \frac{1-z^p}{1+z^p}, \\ \frac{(1-|a|^2)}{(1+\bar{a}z)^2} f' \left(\frac{z+a}{1+\bar{a}z} \right) &= (b-\alpha) \left(\frac{-2pz^{p-1}}{(1+z^p)^2} \right) \end{aligned}$$

and

$$\frac{(1-|a|^2)}{(1+\bar{a})^2} f' \left(\frac{1+a}{1+\bar{a}} \right) = (b-\alpha) \left(\frac{-2p}{(1+1)^2} \right) = -p \frac{b-\alpha}{2}.$$

Since $-1 < a \leq 0$ and b is any integer ≥ 1 , we take

$$|f'(1)| = p \frac{b-\alpha}{2} \frac{1+a}{1-a}.$$

Also, because $|a_p| = 2 \frac{|b-\alpha|}{(1-|a|^2)^p}$, (2.5) is satisfied with equality. \square

If $f(z) - b$ has no zeros different from $z = a$ in Theorem 2.4, the inequality (2.5) can be further strengthened. This is given by the following Theorem.

Theorem 2.5. *Let $f(z) = b + a_p(z-a)^p + a_{p+1}(z-a)^{p+1} + \dots$ $p \geq 1$, $a_p > 0$ is a holomorphic function in D $f(a) = b$, $|a| < 1$, $b \in \mathbb{C}$ and $\Re f(z) > \alpha$, $0 \leq \alpha < |b|$ for $|z| < 1$ and $f(z) - b$ has no zeros in D except $z = a$. Assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c , $\Re f(c) = \alpha$. Then*

$$(2.7) \quad |f'(c)| \geq \frac{|b-\alpha|}{2} \frac{1-|a|}{1+|a|} \left(p - \frac{2|a_p| \ln^2 \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right)}{2|a_p| \ln \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right) - \left| (1-|a|^2) a_{p+1} - \bar{a} p a_p \right|} \right).$$

The equality in (2.7) occurs for the function

$$f(z) = \alpha + (b-\alpha) \left[1 - \frac{2 \left(\frac{z-a}{1-\bar{a}z} \right)^p}{1 + \left(\frac{z-a}{1-\bar{a}z} \right)^p} \right],$$

where $-1 < a \leq 0$ and b is any integer ≥ 1 .

Proof. Let $a_p > 0$. Having in the mind inequality (2.6), we denote by $\ln k(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln k(0) = \ln \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right) < 0.$$

The auxiliary function

$$\Phi(z) = \frac{\ln k(z) - \ln k(0)}{\ln k(z) + \ln k(0)}$$

satisfies the assumptions of the Schwarz lemma on the boundary and so, for $z_0 = \frac{c-a}{1-\bar{a}c} \in \partial D$, we obtain

$$\begin{aligned} \frac{2}{1+|\Phi'(0)|} &\leq |\Phi'(z_0)| = \frac{|2 \ln k(0)|}{|\ln k(z_0) + \ln k(0)|^2} \left| \frac{k'(z_0)}{k(z_0)} \right| \\ &= \frac{-2 \ln k(0)}{\ln^2 k(0) + \arg^2 k(z_0)} \{ |\phi'(z_0)| - p \}. \end{aligned}$$

Since

$$\begin{aligned} |\Phi'(0)| &= \frac{|k'(0)|}{|2k(0) \ln k(0)|} \\ &= \frac{\frac{(1-|a|^2)^p}{2|b-\alpha|} \left| (1-|a|^2) a_{p+1} - \bar{a} p a_p \right|}{2 \frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \ln \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right)} = \frac{\left| (1-|a|^2) a_{p+1} - \bar{a} p a_p \right|}{-2 |a_p| \ln \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right)}, \end{aligned}$$

and replacing $\arg^2 k(z_0)$ by zero, we take

$$\frac{2}{1 - \frac{|(1-|a|^2) a_{p+1} - \bar{a} p a_p|}{2 |a_p| \ln \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right)}} \leq \frac{-2}{\ln \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right)} \left[\frac{2}{|b-\alpha|} \frac{1+|a|}{1-|a|} \left| f' \left(\frac{z_0+a}{1+\bar{a}z_0} \right) \right| - p \right].$$

Thus, we obtain the inequality (2.7).

The equality in (2.7) is obtained for the function

$$f(z) = \alpha + (b-\alpha) \left[1 - \frac{2 \left(\frac{z-a}{1-\bar{a}z} \right)^p}{1 + \left(\frac{z-a}{1-\bar{a}z} \right)^p} \right],$$

by simple calculations. □

Theorem 2.6. *Under the same assumptions as in Theorem 2.5, we have*

$$(2.8) \quad |f'(c)| \geq \frac{|b-\alpha|}{2} \frac{1-|a|}{1+|a|} \left[p - \frac{1}{2} \ln \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right) \right].$$

The equality in (2.8) occurs for the function

$$f(z) = \alpha + (b-\alpha) \left[1 - \frac{2 \left(\frac{z-a}{1-\bar{a}z} \right)^p}{1 + \left(\frac{z-a}{1-\bar{a}z} \right)^p} \right],$$

where $-1 < a \leq 0$ and b is any integer ≥ 1 .

Proof. Using the inequality (1.3) for the function $\Phi(z)$, we obtain

$$1 \leq |\Phi'(z_0)| = \frac{-2 \ln k(0)}{\ln^2 k(0) + \arg^2 k(z_0)} \{ |\phi'(z_0)| - p \}.$$

Replacing $\arg^2 k(z_0)$ by zero, we take

$$1 \leq \frac{-2}{\ln \left(\frac{(1-|a|^2)^p |a_p|}{2|b-\alpha|} \right)} \left[\frac{2}{|b-\alpha|} \frac{1+|a|}{1-|a|} \left| f' \left(\frac{z_0+a}{1+\bar{a}z_0} \right) \right| - p \right].$$

Therefore, we obtain the inequality (2.8) with an obvious equality case. □

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