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A CONSTRUCTION OF STRICTLY INCREASING CONTINUOUS SINGULAR FUNCTIONS

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Abstract. In this paper, we construct a strictly increasing continuous singular function which has a simple algebraic expression.

1. Introduction

A function is called *singular* if it is not a constant function and at the same time its derivative is zero almost everywhere. It seems to be very strange that a continuous increasing function is singular. But there are even strictly increasing continuous singular functions (see, for example, [4] and [5]). It's well known that all the derivatives of the boundary functions of strictly convex divisible (or quasi-homogeneous) projective domains are such functions if the domain is not an ellipse (see [1]).

In this paper we construct another example of a strictly increasing continuous singular function. Since it is more convenient to use the binary expansion for giving its explicit formula, we'll denote all the real numbers by their binary expressions throughout this paper.

2. Definition of f

For any real number $r = 0.r_1r_2r_3\ldots$ in $[0,1]$, we define

$$f(r) = \sum_{i=1}^{\infty} r_i (0.1)^{2i+1-\sum_{j=1}^{i} r_j} (1.1)^{-1+\sum_{j=1}^{i} r_j}.$$

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If we denote the number of 0's and 1's among $\{r_1, \dots, r_i\}$ by n_{i0} and n_{i1} respectively, that is, $n_{i1} = r_1 + r_2 + \cdots + r_i$ and $n_{i0} = i - n_{i1}$, then $f(r)$ can be expressed like this :

$$
\begin{aligned}
f(r) &= \sum_{i=1}^{\infty} r_i (0.1)^{2i+1-n_{i1}} (1.1)^{-1+n_{i1}} \\
&= \sum_{i=1}^{\infty} (0.1)^i r_i (0.1)^{n_{i0}+1} (1.1)^{n_{i1}-1}.
\end{aligned}
$$

Lemma 1. *(well-defined)*

(i) *For each* $r = 0.r_1 r_2 r_3 \cdots \in [0,1]$, *the series*

$$
\sum_{i=1}^{\infty} r_i (0.1)^{2i+1-\sum_{j=1}^{i} r_j} (1.1)^{-1+\sum_{j=1}^{i} r_j}
$$

converges.

(ii) *If* $0.r_1 r_2 r_3 \cdots = 0.r_1' r_2' r_3' \dots$, *then* $f(0.r_1 r_2 r_3 \dots) = f(0.r_1' r_2' r_3' \dots)$.

Proof. To prove (i), it suffices to show that $\sum_{i=1}^{\infty} r_i (0.1)^{2i+1-\sum_{j=1}^{i} r_i} (1.1)^{-1+\sum_{j=1}^{i} r_i}$ is bounded by 1. This is an immediate consequence of

$$
\sum_{i=1}^{\infty} r_i (0.1)^{2i+1-\sum_{j=1}^{i} r_i} (1.1)^{-1+\sum_{j=1}^{i} r_i} \le \sum_{i=1}^{\infty} (0.1)^{i+1} (1.1)^{-1+i},
$$

and

$$
\sum_{i=1}^{\infty} (0.1)^{i+1} (1.1)^{-1+i} = (0.1)^2 \sum_{i=1}^{\infty} \{(0.1)(1.1)\}^{i-1} = 1.
$$

To prove (ii), we show that for $r = 0.r_1 r_2 r_3 \dots r_k (r_k = 1)$, $f(r)$ is equal to $f(0.r_1 r_2 r_3 \dots r_{k-1} 0\dot{1})$.

$$
\begin{aligned}
&f(0.r_1 r_2 r_3 \dots r_{k-1} 0\dot{1}) \\
&= \sum_{i=1}^{k-1} r_i (0.1)^{2i+1-\sum_{j=1}^{i} r_i} (1.1)^{-1+\sum_{j=1}^{i} r_i} + \sum_{i=1}^{\infty} (0.1)^{2(k+i)+1-(n_{k1}+i-1)} (1.1)^{-1+n_{k1}+i-1} \\
&= f(0.r_1 r_2 r_3 \dots r_k) - (0.1)^{2k+1-n_{k1}} (1.1)^{-1+n_{k1}} [1 - \sum_{i=1}^{\infty} (0.1)^{i+1} (1.1)^{i-1}] \\
&= f(0.r_1 r_2 r_3 \dots r_k) - (0.1)^{2k+1-n_{k1}} (1.1)^{-1+n_{k1}} [1 - \frac{(0.1)^2}{1-(0.1)(1.1)}] \\
&= f(0.r_1 r_2 r_3 \dots r_k)
\end{aligned}
$$

□

3. Properties of f

From the definition of f, we get the following:

Lemma 2. *Let $r = 0.r_1r_2r_3\dots r_k \in [0,1]$, $n_1 = \sum_{i=1}^{k} r_i$, and $n_0 = k - n_1$. Then $(r, f(r))$ lies on the graph of the linear function passing through*

$$(0.r_1r_2r_3\dots r_{k-1}, f(0.r_1r_2r_3\dots r_{k-1}))$$

with the slope $(0.1)^{n_0+1}(1.1)^{n_1-1}$, that is,

$$y = f(0.r_1r_2r_3\dots r_{k-1}) + (0.1)^{n_0+1}(1.1)^{n_1-1}(x - 0.r_1r_2r_3\dots r_{k-1}).$$

Proof.

$$\begin{aligned} f(0.r_1r_2r_3\dots r_k) =& f(0.r_1r_2r_3\dots r_{k-1}) + r_k(0.1)^{2k+1-\sum_{i=1}^{k} r_i}(1.1)^{-1+\sum_{i=1}^{k} r_i} \\ =& f(0.r_1r_2r_3\dots r_{k-1}) + r_k(0.1)^{k+n_0+1}(1.1)^{n_1-1} \\ =& f(0.r_1r_2r_3\dots r_{k-1}) + (0.1)^k r_k(0.1)^{n_0+1}(1.1)^{n_1-1}. \end{aligned}$$

□

Lemma 3. *f has the following properties:*

(i) $f(0) = 0, f(1) = 1$, and $0 < f(r) < 1$ if $0 < r < 1$,
(ii) $f((0.1)^k r) = (0.1)^{2k} f(r)$,
(iii) $f(0.r_1r_2\dots) = f(0.r_1r_2\dots r_k) + (\frac{1.1}{0.1})^{\sum_{j=1}^{k} r_j} f(0.0\dots 0r_{k+1}r_{k+2}\dots)$,
(iv) f is not convex,
(v) $f(z) \le z$ for all $z \in [0,1]$.

Proof. (i) and (ii) are immediate from the definition of f.

The equality (iii) is easily proved by calculation :

$$\begin{aligned} & f(0.r_1r_2\dots) - f(0.r_1r_2\dots r_k) \\ =& \sum_{i=k+1}^{\infty} r_i(0.1)^{2i+1-\sum_{j=1}^{i} r_j}(1.1)^{-1+\sum_{j=1}^{i} r_j} \\ =& (0.1)^{-\sum_{j=1}^{k} r_j}(1.1)^{\sum_{j=1}^{k} r_j} f(0.0\dots 0r_{k+1}r_{k+2}\dots) \\ =& (\frac{1.1}{0.1})^{\sum_{j=1}^{k} r_j} f(0.0\dots 0r_{k+1}r_{k+2}\dots) \end{aligned}$$

Non-convexity of f is proved by comparing the points $(0.01, f(0.01))$, $(0.1, f(0.1))$, and $(0.101, f(0.101))$. Actually one can check

$$f(0.1) > f(0.01) + \frac{f(0.101) - f(0.01)}{0.101 - 0.01}(0.1 - 0.01).$$

The inequality (v) is an immediate consequence of (5.1) and lemma 6 of the next section. □

Corollary 4. *For a rational number* $r = 0.\dot{r_1}\dots\dot{r_l}$ *with* $n_1 = \sum_{i=1}^{l} r_i$ *and* $n_0 = l - n_1$,

$$f(r) = \frac{f(0.r_1\dots r_l)}{1-(0.1)^{2l-n_1}(1.1)^{n_1}} = \frac{10^{2l}}{10^{2l}-11^{n_1}} f(0.r_1\dots r_l).$$

Proof. By (iii) of Lemma 3, we get

$$\begin{aligned} f(0.\dot{r_1}\dots\dot{r_l}) =& f(0.r_1\dots r_l) + (\frac{1.1}{0.1})^{n_1} f((0.1)^l 0.\dot{r_1}\dots\dot{r_l}) \\ =& f(0.r_1\dots r_l) + (\frac{1.1}{0.1})^{n_1}(0.1)^{2l} f(0.\dot{r_1}\dots\dot{r_l}) \\ =& f(0.r_1\dots r_l) + (0.1)^{2l-n_1}(1.1)^{n_1} f(0.\dot{r_1}\dots\dot{r_l}) \end{aligned}$$

and thus

$$f(0.\dot{r_1}\dots\dot{r_l}) = \frac{f(0.r_1\dots r_l)}{1-(0.1)^{2l-n_1}(1.1)^{n_1}} = \frac{10^{2l}}{10^{2l}-11^{n_1}} f(0.r_1\dots r_l).$$

□

4. f IS STRICTLY INCREASING

Lemma 5.

$$f(s) < f(t) \text{ if } s < t$$

Proof. Given $s = 0.s_1 s_2 \cdots < 0.t_1 t_2 \cdots = t$, there is $k > 0$ such that

$$s_1 = t_1, s_2 = t_2, \dots, s_k = t_k, s_{k+1} < t_{k+1} \text{ (i.e., } s_{k+1} = 0 \text{ and } t_{k+1} = 1).$$

By (iii) and (ii) of Lemma 3,

$$\begin{aligned} f(s) =& f(0.s_1 s_2 \dots s_k) + (\frac{1.1}{0.1})^{\sum_{j=1}^{k} s_j} f(0.0\dots 0 s_{k+1} s_{k+2}\dots) \\ =& f(0.s_1 s_2 \dots s_k) + (\frac{1.1}{0.1})^{\sum_{j=1}^{k} s_j} (0.1)^{2k} f(0.s_{k+1} s_{k+2}\dots), \end{aligned}$$

and similarly

$$f(t) = f(0.t_1 t_2 \dots t_k) + (\frac{1.1}{0.1})^{\sum_{j=1}^{k} t_j} (0.1)^{2k} f(0.t_{k+1} t_{k+2}\dots).$$

By (ii) of Lemma 3 and the fact $s_{k+1} = 0, t_{k+1} = 1$, we get

$$f(0.s_{k+1} s_{k+2}\dots) = f((0.1)(0.s_{k+2}\dots)) = (0.1)^2 f(0.s_{k+2} s_{k+3}\dots) \le (0.1)^2,$$

and

$$f(0.t_{k+1} t_{k+2}\dots) \ge f(0.1) = (0.1)^2,$$

which implies

$$f(0.s_{k+1}s_{k+2}\dots) \le f(0.t_{k+1}t_{k+2}\dots).$$

If we suppose $f(0.s_{k+1}s_{k+2}\dots) = f(0.t_{k+1}t_{k+2}\dots)$, then this value must be $(0.1)^2$ and

$$s_{k+1} = 0, s_{k+2} = s_{k+3} = \cdots = 1 \text{ and } t_{k+1} = 1, t_{k+2} = t_{k+3} = \cdots = 0,$$

which implies $s = t$. So we can conclude that $f(s) < f(t)$ if $s < t$. □

5. f IS CONTINUOUS

We'll see in this section that f is the limit of a uniformly converging sequence of functions $\{f_n\}$ on $[0, 1]$, which are piecewise linear strictly increasing continuous functions. They are geometrically constructed in the following way: First, we define $f_0(x) \equiv x$. Then f_1 is constructed so that $f_1(0) = f_0(0) = 0, f_1(1) = f_0(1) = 1, f_1(0.1) = 0.1f_0(0.1)$ and f_1 is linear in both intervals $[0, 0.1]$ and $[0.1, 1]$. Graphically, we get the graph of f_1 from the graph of f_0 by bending at the midpoint 0.1 with lowering the height by half. Now f_2 is constructed by applying the same process on each interval $[0, 0.1]$ and $[0.1, 1]$, that is, $f_2(0.01) = 0.1f_1(0.01), f_2(0.11) = f_1(0.1) + 0.1(f_1(0.11)-f_1(0.1))$ and f_2 is linear in all four intervals $[0, 0.01], [0.01, 0.1], [0.1, 0.11]$ and $[0.11, 1]$ (actually, f_2 is linear in $[0.01, 0.11]$, so the graph of f_2 consists of three line segments). Repeating this procedure, we get strictly increasing, piecewise linear, continuous functions f_n's. Note that

$$0 < \cdots \le f_{n+1}(x) \le f_n(x) \le \cdots \le f_1(x) \le f_0(x) = x, \tag{5.1}$$

and thus $f_n(x)$ converges for all $x \in [0, 1]$. If we define a function F on $[0, 1]$ by

$$F(x) = \lim_{n\to\infty} f_n(x), \text{ for all } x \in [0, 1],$$

then F is continuous because $\{f_n\}$ is a uniformly converging sequence.[a)]

Lemma 6. $F \equiv f$.

Proof. First, we show that for any natural number k and any element $(r_1, \dots, r_k), r_i \in \{0, 1\}$,

$$F(0.r_1r_2\dots r_k) = F(0.r_1\dots r_{k-1}) + r_k(0.1)^{k+n_{k0}+1}(1.1)^{n_{k1}-1}$$

[a)] This geometric constuction is exactly the same as the method of performing the transform $T(1/4, 3/4)$ that R. Salem used in his paper [5]. H. Okamoto had also generalized Salem's method in his paper [3] and [4] to obtain more singular functions and continuous nowhere differntiable functions.

and

$$F(0.r_1r_2\dots r_k) = \sum_{i=1}^{m} r_i(0.1)^{2i+1-n_{i1}}(1.1)^{-1+n_{i1}},$$

where $n_{k1} = \sum_{i=1}^{k} r_i$ and $n_{k0} = k - n_{k1}$. This is obviously true for $k = 1$. If we assume that this holds for all $k \le m$, then

$$\begin{aligned} F(0.r_1r_2\dots r_m) =& f_m(0.r_1r_2\dots r_m) \\ =& f_{m-1}(0.r_1\dots r_{m-1}) + r_m(0.1)^{m+n_{m0}+1}(1.1)^{n_{m1}-1} \\ =& f_{m-1}(0.r_1\dots r_{m-1}) + r_m(0.1)^{2m+1-n_{m1}}(1.1)^{n_{m1}-1} \\ =& \sum_{i=1}^{m} r_i(0.1)^{2i+1-n_{i1}}(1.1)^{-1+n_{i1}} \end{aligned}$$

And from the definition of F we see

$$\begin{aligned} & F(0.r_1r_2\dots r_{m+1}) = f_{m+1}(0.r_1r_2\dots r_{m+1}) \\ =& f_m(0.r_1\dots r_m) + (0.1)r_{m+1}(f_m(0.r_1\dots r_m + (0.1)^{m+1}) - f_m(0.r_1\dots r_m)). \end{aligned}$$

We may assume $r_{m+1} = 1$. Since the slope of f_m in the interval

$$(0.r_1\dots r_m, 0.r_1\dots r_m r_{m+1}) = (0.r_1\dots r_m, 0.r_1\dots r_m 1)$$

is $(0.1)^{n_{m,0}}(1.1)^{n_{m,1}} = (0.1)^{n_{m+1,0}}(1.1)^{n_{m+1,1}-1}$, we get

$$\begin{aligned} F(0.r_1r_2\dots r_{m+1}) =& f_m(0.r_1\dots r_m) + (0.1)r_{m+1}(f_m(0.r_1\dots r_{m+1}) - f_m(0.r_1\dots r_m)) \\ =& f_m(0.r_1\dots r_m) + (0.1)r_{m+1}(0.1)^{m+1}(0.1)^{n_{m+1,0}}(1.1)^{n_{m+1,1}-1} \\ =& f_m(0.r_1\dots r_m) + r_{m+1}(0.1)^{(m+1)+n_{m+1,0}+1}(1.1)^{n_{m+1,1}-1} \\ =& \sum_{i=1}^{m+1} r_i(0.1)^{2i+1-n_{i1}}(1.1)^{-1+n_{i1}}, \end{aligned}$$

which proves our claim and implies

$$F(0.r_1r_2\dots r_k) = f(0.r_1r_2\dots r_k).$$

For an arbitrary point $r = 0.r_1r_2\dots$ in $[0,1]$, we consider the increasing sequence $\{r(k) = 0.r_1\dots r_k\}$ converging to r. Since F is continuous,

$$\begin{aligned} F(0.r_1r_2\dots) =& \lim_{k\to\infty} F(0.r_1r_2\dots r_k) \\ =& \lim_{k\to\infty} \sum_{i=1}^{k} r_i(0.1)^{2i+1-n_{i1}}(1.1)^{-1+n_{i1}} \end{aligned}$$

$$=\sum_{i=1}^{\infty} r_i(0.1)^{2i+1-n_{i1}}(1.1)^{-1+n_{i1}}$$
$$=f(0.r_1r_2\dots)$$

□

Corollary 7. *f is a strictly increasing continuous function.*

6. Differentiability of f at Rational Numbers

In this section, we'll investigate the differentiability of f at rational numbers. Each rational number r in $[0,1]$ has an infinite binary expansion $r = 0.s_1 \dots s_k\dot{r_1} \dots \dot{r_l}$. If we denote the number of 1's in $\{r_1, \dots, r_l\}$ and 0's by n_1 and n_0 respectively, that is, $n_1 = \sum_{i=1}^{l} r_i$ and $n_0 = l - n_1$, then we get a number $D(r) = (0.1)^{n_0}(1.1)^{n_1}$. For example, $D(r) = 1.1 > 1$ for any rational number r which has a finite binary expansion, since $0.r_1 \dots r_k = 0.r_1, \dots, r_{k-1}0\dot{1}$.

We'll see in this section that the number $D(r)$ is closely related to the differentiability of f at r. Actually we'll prove the following.

Theorem 8. *For a rational number r, f is differentiable at r if and only if $D(r) < 1$. Furthermore, $f'(r) = 0$ if exists.*

6.1. Differentiability at $r = 0.r_1 \dots r_k$ We can see immediately from the geometric construction of f that f is not differentiable at rational numbers which have finite binary expansions, that is, f has singular points at those points.

Lemma 9. *If r is a rational number with a finite binary expansion, f is not differentiable at r.*

Proof. Let $r = 0.r_1, \dots, r_k$ be the shortest finite binary expression of r. Then r_k must be 1 and

$$r = 0.r_1, \dots, r_{k-1}0\dot{1}.$$

Consider the following sequences $r^+(n)$ and $r^-(n)$ converging to r : $r^+(n)$ is an increasing sequence defined as

$$r^+(1) = 0.r_1, \dots, r_k1$$
$$r^+(2) = 0.r_1, \dots, r_k01$$
$$r^+(3) = 0.r_1, \dots, r_k001$$
$$\dots$$

and $r^-(n)$ is a decreasing sequence defined as

$$r^-(1) = 0.r_1, \ldots, r_{k-1}01$$
$$r^-(2) = 0.r_1, \ldots, r_{k-1}011$$
$$r^-(3) = 0.r_1, \ldots, r_{k-1}0111$$
$$\ldots$$

Then

$$\frac{f(r) - f(r^+(n))}{r - r^+(n)} = \frac{(0.1)^{2(k+n)+1-(n_1+1)}(1.1)^{-1+n(_1+1)}}{(0.1)^{k+n}}$$
$$= (0.1)^{k+n-n_1}(1.1)^{n_1} = (0.1)^{n_0}(1.1)^{n_1}(0.1)^n$$

and

$$\frac{f(r) - f(r^-(n))}{r - r^-(n)}$$
$$= \frac{(0.1)^{2k+1-n_1}(1.1)^{-1+n_1} - \sum_{i=1}^{n}(0.1)^{2(k+i)+1-(n_1+i-1)}(1.1)^{-1+(n_1+i-1)}}{(0.1)^{k+n}}$$
$$= \frac{(0.1)^{2k+1-n_1}(1.1)^{-1+n_1}(1 - \frac{0.1}{1.1}\sum_{i=1}^{n}(0.1)^i(1.1)^i)}{(0.1)^{k+n}}$$
$$= (0.1)^{k+1-n_1}(1.1)^{-1+n_1}(1.1)^n.$$

Therefore

$$\lim_{n\to\infty} \frac{f(r) - f(r^+(n))}{r - r^+(n)} = 0, \lim_{n\to\infty} \frac{f(r) - f(r^-(n))}{r - r^-(n)} = \infty,$$

and thus f is not differentiable at r. □

6.2. For each real number $r = 0.r_1r_2r_3\ldots$ in $[0,1]$, we get a sequence $\{a_k(r) = (0.1)^{n_{k0}}(1.1)^{n_{k1}}\}$. Note that for a rational number $r = 0.s_1 \ldots s_k\dot{r_1}\ldots\dot{r_l}\ldots,$

$$D(r) = \frac{a_{k+l}(r)}{a_k(r)} = \frac{a_{k+2l}(r)}{a_{k+l}(r)} = \frac{a_{k+3l}(r)}{a_{k+2l}(r)} = \ldots$$

and

$$D(r)^n = \frac{a_{k+nl}(r)}{a_k(r)} = \frac{a_{k+(n+m)l}(r)}{a_{k+ml}(r)}.$$

Lemma 10. (i) *f is differentiable at r if and only if f is differentiable at $(0.1)^k r$ and $f'((0.1)^k r) = (0.1)^k f'(r)$,*

(ii) *If two rational numbers $z = 0.z_1z_2\ldots$ and $r = 0.r_1r_2\ldots \in [0,1]$ have the same first k digits, that is, $z_1 = r_1, z_2 = r_2, \ldots, z_k = r_k$, then*

$$\frac{f(r) - f(z)}{r - z} = a_k(r)\frac{f(0.r_{k+1}r_{k+2}\ldots) - f(0.z_{k+1}z_{k+2}\ldots)}{0.r_{k+1}r_{k+2}\cdots - 0.z_{k+1}z_{k+2}\ldots}.$$

(iii) *f is differentiable at $r = 0.r_1r_2\ldots$ if and only if f is differentiable at $0.r_{k+1}r_{k+2}\ldots$ and*

$$f'(r) = a_k(r)f'(0.r_{k+1}r_{k+2}\ldots).$$

Proof. (i)

$$\begin{aligned}\frac{f((0.1)^kr+h)-f((0.1)^kr)}{h} &= \frac{f((0.1)^kr+(0.1)^kh')-f((0.1)^kr)}{(0.1)^kh'}\\ &= \frac{(0.1)^{2k}[f(r+h')-f(r)]}{(0.1)^kh'}\\ &= (0.1)^k\frac{f(r+h')-f(r)}{h'},\end{aligned}$$

where $h' = 10^kh$.

(ii) For any $z = 0.z_1z_2\cdots \in [0,1]$ such that

$$z_1 = r_1, z_2 = r_2, \ldots, z_k = r_k,$$

we get

$$\begin{aligned}&\frac{f(r)-f(z)}{r-z}\\ =&\frac{(0.1)^{2k-n_{k1}}(1.1)^{n_{k1}}[f(0.r_{k+1}r_{k+2}\ldots)-f(0.z_{k+1}z_{k+2}\ldots)]}{(0.1)^k[0.r_{k+1}r_{k+2}\cdots-0.z_{k+1}z_{k+2}\ldots]}\\ =&(0.1)^{n_{k0}}(1.1)^{n_{k1}}\frac{f(0.r_{k+1}r_{k+2}\ldots)-f(0.z_{k+1}z_{k+2}\ldots)}{0.r_{k+1}r_{k+2}\cdots-0.z_{k+1}z_{k+2}\ldots}\\ =&a_k(r)\frac{f(0.r_{k+1}r_{k+2}\ldots)-f(0.z_{k+1}z_{k+2}\ldots)}{0.r_{k+1}r_{k+2}\cdots-0.z_{k+1}z_{k+2}\ldots}.\end{aligned}$$

(iii) Suppose $\{z'(n) = 0.z_{n,k+1}z_{n,k+2}\ldots\}$ is an arbitrary sequence of real numbers in $[0,1]$ which converges to $0.r_{k+1}r_{k+2}r_{k+3}\ldots$. Then the sequence

$$\{z(n) = 0.r_1r_2\ldots r_kz_{n,k+1}z_{n,k+2}\cdots = 0.r_1r_2\ldots r_k + (0.1)^kz'(n)\}$$

converges to r, and by (ii)

$$\frac{f(r)-f(z(n))}{r-z(n)} = a_k(r)\frac{f(0.r_{k+1}r_{k+2}\ldots)-f(z'(n))}{0.r_{k+1}r_{k+2}\cdots-z'(n)}.$$

So if if f is differntiable at r, then f is differntiable at $0.r_{k+1}r_{k+2}r_{k+3}\ldots$ and $f'(0.r_{k+1}r_{k+2}\ldots) = \frac{1}{a_k(r)}f'(r)$.

Conversely, if f is differntiable at $0.r_{k+1}r_{k+2}r_{k+3}\ldots$ and $z(n) = 0.z_{n,1}z_{n,2}\ldots$ is an arbitrary sequence converging to r, then there is a natural number t

such that

$$z_{n,1} = r_1, z_{n,2} = r_2, \dots, z_{n,k} = r_k, \text{for all } n > t.$$

So we get

$$\begin{aligned} f'(r) &= \lim_{n\to\infty} \frac{f(r) - f(z(n))}{r - z(n)} \\ &= a_k(r) \lim_{n\to\infty} \frac{f(0.r_{k+1}r_{k+2}\dots) - f(0.z_{n,k+1}z_{n,k+2}\dots)}{0.r_{k+1}r_{k+2}\dots - 0.z_{n,k+1}z_{n,k+2}\dots} \\ &= a_k(r) f'(0.r_{k+1}r_{k+2}\dots). \end{aligned}$$

□

6.3. Differentiability at $r = 0.\dot{r_1}\dots\dot{r_l}$ Given a rational number $r = 0.\dot{r_1}\dots\dot{r_l}$, we define $r(nl)$ as follows:

$$r(l) = 0.r_1 \dots r_l$$

$$r(2l) = 0.r_1 \dots r_l r_1 \dots r_l$$

$$r(3l) = 0.r_1 \dots r_l r_1 \dots r_l r_1 \dots r_l$$

$$\dots$$

$$r(nl) = 0.r_1 \dots r_l r_1 \dots r_l r_1 \dots r_l \dots r_1 \dots r_l (\text{n times})$$

$$\dots$$

By Lemma 10, we see that if f is differentiable at a rational number $r = 0.\dot{r_1}\dots\dot{r_l}$, then

$$\begin{aligned} f'(r) &= f'(0.\dot{r_1}\dots\dot{r_l}) \\ &= D(r) f'(0.\dot{r_1}\dots\dot{r_l}) \\ &= D(r)^2 f'(0.\dot{r_1}\dots\dot{r_l}) \\ &= \dots \\ &= D(r)^n f'(0.\dot{r_1}\dots\dot{r_l}) = D(r)^n f'(r), \end{aligned}$$

which implies $f'(r) = 0$ because $D(r)$ cannot be 1. Actually we can prove

Lemma 11. *For any rational number $r = 0.\dot{r_1}\dots\dot{r_l}$, the following is true:*

(i) $\frac{f(r)-f(r(nl))}{r-r(nl)} = D(r)^n \frac{f(r)}{r}$,

(ii) *if f is differentiable at r, then $D(r) < 1$ and $f'(r) = 0$,*

(iii) *if $D(r) < 1$, then f is differentiable at r and $f'(r) = 0$.*

Proof. (i) Since $a_{nl}(r) = D(r)^n$, this is immediate from (ii) of Lemma 10.

(ii) By (i), if f is differentiable at r, then the sequence $\{D(r)^n\}$ must converge and

$$f'(r) = \lim_{n\to\infty} \frac{f(r) - f(r(nl))}{r - r(nl)} = \frac{f(r)}{r} \lim_{n\to\infty} D(r)^n,$$

which implies $D(r) < 1$ and $f'(r) = 0$ because $D(r) = (0.1)^{n_0}(1.1)^{n_1}$ cannot be 1.

(iii) If $\{b_k\}$ is any increasing sequence converging to r, then we can find a sequence $\{n_k\}$ of natural numbers satisfying the following:

$$r(n_k l) \le b_k < r((n_k + 1)l) < r, \lim_{k\to\infty} n_k = \infty.$$

Now we see

$$\begin{aligned} 0 \le \frac{f(r) - f(b_k)}{r - b_k} \le & \frac{f(r) - f(r(n_k l))}{r - r((n_k + 1)l)} \\ = & \frac{f(r) - f(r(n_k l))}{r - r(n_k l)} \frac{r - r(n_k l)}{r - r((n_k + 1)l)} \\ = & D(r)^{n_k} \frac{f(r)}{r} \frac{(0.1)^{n_k l} r}{(0.1)^{(n_k+1)l} r} \\ = & 10^l \frac{f(r)}{r} D(r)^{n_k}. \end{aligned}$$

Therefore $\lim_{k\to\infty} \frac{f(r)-f(b_k)}{r-b_k} = 0$ and thus the left derivative of f at r exists and should be 0 if $D(r) < 1$.

To caculate the right derivative of f at r, suppose that $\{d_k\}$ is a decreasing sequence converging to r. Then there is a sequence $\{n_k\}$ of natural numbers satisfying the following:

$$r((n_k + 1)l) + (0.1)^{(n_k+1)l} \le d_k < r(n_k l) + (0.1)^{n_k l}, \lim_{k\to\infty} n_k = \infty.$$

In fact, this inequality holds when the first $n_k l$ digits of d_k and r are identical and $(n_k + 1)l$ digits are not the same. So we get

$$d'_k = 10^{n_k l}(d_k - r(n_k l)) \ge r(l) + (0.1)^l,$$

and

$$\begin{aligned} |r - d'_k| \ \ge & r(l) + (0.1)^l - r \\ = & (0.1)^l - (0.1)^l r \\ = & (0.1)^l (1 - r). \end{aligned}$$

Using this inequality and (ii) of Lemma 10, we get

$$\begin{aligned}|\frac{f(r)-f(d_k)}{r-d_k}| =& D(r)^{n_k}|\frac{f(r)-f(d'_k)}{r-d'_k}| \\ \leq& \frac{10^l D(r)^{n_k}}{1-r}|f(r)-f(d'_k)| \\ =& \frac{10^l}{1-r}D(r)^{n_k},\end{aligned}$$

which implies that the right derivative of f at r exists and should be 0 if $D(r) < 1$.

□

6.4. Differentiability at $z = 0.s_1 \dots s_k \dot{r_1} \dots \dot{r_l}$ The lemma below completes the proof of Theorem 8.

Lemma 12. *For any rational number* $z = 0.s_1 \dots s_k \dot{r_1} \dots \dot{r_l}$*, the following is true:*

(i) f *is differentiable at* z *if and only if* $D(z) = (0.1)^{n_0}(1.1)^{n_1} < 1$,

(ii) $f'(z) = 0$ *if exists.*

Proof. By Lemma 10, we see that f is differentiable at z if and only if f is differentiable at $r = 0.\dot{r_1} \dots \dot{r_l}$ and

$$f'(0.s_1 \dots s_k \dot{r_1} \dots \dot{r_l}) = a_k(z) f'(0.\dot{r_1} \dots \dot{r_l}).$$

We also see by Lemma 11 that $D(r) < 1$ if and only if f is differentiable at r and $f'(r) = 0$ if exists. Therefore $f'(z) = 0$ if f is differentiable at z and the following three are equivalent :

(1) f is differentiable at $z = 0.s_1 \dots s_k \dot{r_1} \dots \dot{r_l}$,

(2) f is differentiable at $r = 0.\dot{r_1} \dots \dot{r_l}$,

(3) $D(z) = D(r) < 1$.

This proves (i) and (ii). □

7. f IS SINGULAR

In this section, we'll show that f is a singular function.

Definition 13. For $x \in [0, 1]$, we say that x is called *simply normal* (to the base 2) if both 0 and 1 appear with the same asymptotic frequency $\frac{1}{2}$, that is,

$$\lim_{k\to\infty} \frac{n_{k0}}{k} = \frac{1}{2}, \text{ and } \lim_{k\to\infty} \frac{n_{k1}}{k} = \frac{1}{2}.$$

It is well-known that the set of simply normal numbers in $[0,1]$ has full measure (see [2].)

Define three subsets of $[0,1]$, E_1, E_2 and E as follows :

$$\begin{aligned} E =&\{x \in [0,1] \,|\, f \text{ is differentiable at } x\},\\ E_1 =&\{x \in [0,1] \,|\, f \text{ is differentiable at } x \text{ and } \lim_{k\to\infty} a_k(r) = 0\},\\ E_2 =&\{x \in [0,1] \,|\, f \text{ is differentiable at } x \text{ and } x \text{ is simply normal }\}. \end{aligned}$$

Then $E_2 \subset E_1 \subset E$, since

$$a_k(r) = (0.1)^{n_{k0}}(1.1)^{n_{k1}} = ((0.1)^{\frac{n_{k0}}{k}}(1.1)^{\frac{n_{k1}}{k}})^k,$$

and thus E, E_1 and E_2 have all full measure, since f is strictly increasing.

Theorem 14. *f is a continuous strictly increasing singular function with $f'(x) = 0$ for all $x \in E_1$.*

Proof. Consider the sequence $r(k) = 0.r_1r_2r_3\ldots r_k$ for a real number $r = 0.r_1r_2r_3\ldots$. By Lemma 3,

$$\begin{aligned} f(r) - f(r(k)) =&(0.1)^{-n_{k1}}(1.1)^{n_{k1}} f((0.1)^k 0.r_{k+1}r_{k+2}\ldots)\\ =&(0.1)^{2k-n_{k1}}(1.1)^{n_{k1}} f(0.r_{k+1}r_{k+2}\ldots) \end{aligned}$$

and

$$\begin{aligned} \frac{f(r) - f(r(k))}{r - r(k)} =&\frac{(0.1)^{2k-n_{k1}}(1.1)^{n_{k1}} f(0.r_{k+1}r_{k+2}\ldots)}{(0.1)^k 0.r_{k+1}r_{k+2}\ldots}\\ =&(0.1)^{n_{k0}}(1.1)^{n_{k1}} \frac{f(0.r_{k+1}r_{k+2}\ldots)}{0.r_{k+1}r_{k+2}\ldots}\\ =&a_k(r)\frac{f(0.r_{k+1}r_{k+2}\ldots)}{0.r_{k+1}r_{k+2}\ldots} \end{aligned}$$

Since $f(z) \le z$ for all real number $z \in [0,1]$, we get an inequality,

$$0 \le \frac{f(r) - f(r(k))}{r - r(k)} \le a_k(r),$$

and this implies that if f is differentiable at r then $0 \le f'(r) \le \lim_{k\to\infty} a_k(r)$. Therefore $f'(r) = 0$ for all $r \in E_1$ and thus f is a singular function, which completes the proof.

□

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