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DYNAMICAL BIFURCATION OF THE BURGERS-FISHER EQUATION

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Abstract. In this paper, we study dynamical Bifurcation of the Burgers-Fisher equation. We show that the equation bifurcates an invariant set $\mathcal{A}_n(\beta)$ as the control parameter β crosses over n^2 with $n \in \mathbb{N}$. It turns out that $\mathcal{A}_n(\beta)$ is homeomorphic to S^1, the unit circle.

1. Introduction

The Burgers equation

$$u_t = u_{xx} + \alpha u u_x$$

is known as an important nonlinear diffusion equation describing the far field of wave propagation in the corresponding dissipative systems such as shallow water waves and gas dynamics [1,4]. Here, $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. On the other hand, the Fisher equation

$$u_t = u_{xx} + \beta u(1 - u)$$

is known to have close connection with biophysics such as diffusive population dynamics and nerve signal propagation [7]. Here, $\beta \in \mathbb{R}^+$. If

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we consider the nonlinear effect in both the Burgers equation and the Fisher equation, we are led to the Bugers-Fisher equation(BFE):

$$u_t = u_{xx} + \alpha u u_x + \beta u(1-u). \tag{1.1}$$

The Bugers-Fisher equation is regarded as a prototypical model for describing the interaction between the reaction mechanism, convection effect, and diffusion transport [7].

In this paper, we are interested in the dynamical bifurcation of the BFE as the control parameter β moves. It is not difficult to see by the linear stability analysis that the trivial solution $u = 0$ is unstable. We will prove that the BFE bifurcates from the trivial solution to an invariant sets as β passes over a sequence of nodal point n^2, $n = 1, 2, \cdots$. Such a bifurcation problem is quite interesting since it provides us long time dynamics of solutions near the trivial solution. For instance, consider the generalized Burgers equation

$$u_t = u_{xx} + \lambda u + \delta u_x + \alpha u u_x. \tag{1.2}$$

In [3], the dynamical bifurcation problem of (1.2) was studied for the case $\delta = 0$. Recently, this result was extended to the case $\delta \neq 0$ in [5]. In these results, the trivial solution bifurcates to an attractor which determines the final patterns of solutions. The main difference between [3] and [5] lies in the invariance of odd functions. Indeed, if the initial condition is an odd function, then the solution of (1.2) is also odd if $\delta = 0$. However, such an invariance is no longer true for the case $\delta \neq 0$. This means that the dimension of the center manifold the trivial solution may be doubled if $\delta \neq 0$. Consequently, the analysis is more complicated. See also [2] for the dynamical bifurcation of a fourth order differential equation in a similar spirit.

To set up our problem, we consider the BFE (1.1) under the periodic boundary condition on $\Omega = [-\pi, \pi]$. For the functional setting of the periodic BFE, let

$$\begin{aligned} H &= \left\{ u \in L^2(\Omega;\mathbb{R}) : u(-\pi) = u(\pi) \right\}, \\ H^2_{per}(\Omega;\mathbb{R}) &= \left\{ u \in H^2(\Omega;\mathbb{R}) : \frac{\partial^j u}{\partial x^j}(-\pi) = \frac{\partial^j u}{\partial x^j}(\pi) \text{ for } j = 0, 1 \right\}, \\ H_1 &= H^2_{per}(\Omega;\mathbb{R}) \cap H. \end{aligned}$$

Then, we can rewrite (1.1) into an abstract equation

$$\begin{cases} \dfrac{du}{dt} &= \mathcal{L}_\beta u + G(u,\alpha,\beta), \\ u(0) &= u_0, \end{cases} \tag{1.3}$$

where

$$\mathcal{L}_\beta u = \Big(\frac{\partial^2}{\partial x^2} + \beta\Big)u, \quad G(u,\alpha,\beta) = \alpha u u_x - \beta u^2.$$

It is easy to see that $\mathcal{L}_\beta,\ G(\cdot,\alpha,\beta) : H_1 \to H$ are well-defined.

To find the eigenvalues of $\mathcal{L}_\beta$, given $u \in H$ we set

$$u(x) = a_0 + \sum_{n=1}^{\infty}\big(a_n \cos nx + b_n \sin nx\big).$$

If λ is an eigenvalue of $\mathcal{L}_\beta$, then

$$\begin{aligned} \mathcal{L}_\beta u &= \beta a_0 + \sum_{n=1}^{\infty}\Big(a_n(\beta - n^2)\cos nx + b_n(\beta - n^2)\sin nx\Big) \\ &= \lambda a_0 + \lambda \sum_{n=1}^{\infty}\big(a_n \cos nx + b_n \sin nx\big) = \lambda u. \end{aligned}$$

Thus the eigenvalues of $\mathcal{L}_\beta$ are

$$\lambda_n(\beta) = \beta - n^2, \quad n = 0, 1, 2, \cdots$$

with the corresponding eigenvectors $\phi_0 = 1$,

$$\phi_n(x) = \cos nx, \quad \psi_n(x) = \sin nx, \quad n = 1, 2, 3, \cdots .$$

We note that

$$\|\phi_0\| = \sqrt{2\pi}, \quad \|\phi_n\| = \|\psi_n\| = \sqrt{\pi}, \quad n = 1, 2, 3, \cdots .$$

Now we are ready to state the main result of this paper as follows.

THEOREM 1.1. *As β passes through n^2 for $n = 1, 2, \cdots$, BFE* (1.1) *defined in H bifurcates to an invariant set $\mathcal{A}_n(\beta)$ which is homeomorphic to S^1.*

We prove Theorem 1.1 in subsequent section. We follow the method in [2] where the center manifold reduction was made by using of Theorem 3.8 in [6]. As in the equation (1.2) with $\delta \neq 0$, the equation (1.1) is not invariant under odd or even function spaces. As a consequence, the center manifolds is not represented as a single variable.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let $n \in \mathbb{N}$ be fixed and assume that β is slightly bigger than n^2. We note that

$$\lambda_n(\beta) = \beta - n^2 \begin{cases} < 0 & \text{for} \quad \beta < n^2, \\ = 0 & \text{for} \quad \beta = n^2, \\ > 0 & \text{for} \quad \beta > n^2. \end{cases}$$

Hence, by Theorem 5.2 of [6], the BFE bifurcates to an attractor $\mathcal{A}_n(\beta)$. In the following, we study the structure of $\mathcal{A}_n(\beta)$ by the center manifold analysis. The main issue is to find the reduced equation of (1.1) on the center manifold. To find a form of the center manifold function, let $E_1 = \text{span}\{\phi_n, \psi_n\}$ and $E_2 = E_1^{\perp}$ in H. Let $P_j : H \to E_j$ be the canonical projection and $\mathcal{L}_j = \mathcal{L}_\beta|_{E_j}$, for $j = 1, 2$. For $u \in H$, we expand it into

$$u = y_0\phi_0 + \sum_{k=1}^{\infty}(y_k\phi_k + z_k\psi_k).$$

If $\Phi : E_1 \to E_2$ is a center manifold function and $v = P_1 u = y_n\phi_n + z_n\psi_n$, then the reduced equation of (1.3) on the center manifold is

$$\frac{dv}{dt} = \mathcal{L}_1 v + P_1 G\big(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)\big). \tag{2.1}$$

Here, we used the notation $G(u) = G(u, \alpha, \beta)$ for simplicity. By taking the inner product of (2.1) with ϕ_n and ψ_n, we are led to

$$\begin{cases} \dfrac{dy_n}{dt} = \lambda_n y_n + F_1(y_n, z_n), \\ \dfrac{dz_n}{dt} = \lambda_n z_n + F_2(y_n, z_n), \end{cases} \tag{2.2}$$

where

$$F_1(y_n, z_n) = \frac{1}{\pi}\left\langle G(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)), \phi_n \right\rangle,$$
$$F_2(y_n, z_n) = \frac{1}{\pi}\left\langle G(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)), \psi_n \right\rangle.$$

Since β is slightly bigger than n^2, we have the following

$$\lambda_n(\beta) = \beta - n^2 = o(1) \tag{2.3}$$

as $\beta \to n^2$. By means of Theorem 3.8 in [6], the center manifold function Φ can be expressed as

$$\begin{aligned}\Phi(y_n\phi_n + z_n\psi_n) &= (-\mathcal{L}_2)^{-1}P_2G(y_n\phi_n + z_n\psi_n)\\ &\qquad + O\big(|\lambda_n| \cdot \pi(y_n^2 + z_n^2)\big) + o(\pi(y_n^2 + z_n^2))\\ &= (-\mathcal{L}_2)^{-1}P_2G(y_n\phi_n + z_n\psi_n) + o(y_n^2 + z_n^2)\end{aligned}$$

where the last equality comes from (2.3). By direct computation, we have

$$\begin{aligned}&G(y_n\phi_n + z_n\psi_n)\\ =\ &\alpha(y_n\phi_n + z_n\psi_n)(-y_n\psi_n + z_n\phi_n) - \beta(y_n^2\phi_n^2 + 2y_nz_n\phi_n\psi_n + z_n^2\psi_n^2)\\ =\ &-\alpha(y_n^2 - z_n^2)\phi_n\psi_n + \alpha y_nz_n(\phi_n^2 - \psi_n^2) - \beta(y_n^2\phi_n^2 + 2y_nz_n\phi_n\psi_n + z_n^2\psi_n^2)\\ =\ &-\frac{\beta}{2}(y_n^2 + z_n^2)\phi_0 + \Big(\alpha y_nz_n - \frac{\beta}{2}(y_n^2 - z_n^2)\Big)\phi_{2n} - \Big(\alpha\frac{y_n^2 - z_n^2}{2} + \beta y_nz_n\Big)\psi_{2n},\end{aligned}$$

where we used the trigonometric identities:

$$\phi_n\psi_n = \frac{1}{2}\psi_{2n}, \quad \phi_n^2 = \frac{\phi_0 + \phi_{2n}}{2}, \quad \psi_n^2 = \frac{\phi_0 - \phi_{2n}}{2}. \tag{2.4}$$

Let

$$\Phi(y_n\phi_n + z_n\psi_n) = a_0\phi_0 + \sum_{k\neq n, k\geq 1}(a_k\phi_k + b_k\psi_k).$$

From the relation

$$-\mathcal{L}_2\Phi(y_n\phi_n + z_n\psi_n) = P_2G(y_n\phi_n + z_n\psi_n) + o(y_n^2 + z_n^2),$$

we derive that

$$\begin{aligned}&-\lambda_0a_0\phi_0 - \sum_{k\neq n, k\geq 1}\lambda_k(a_k\phi_k + b_k\psi_k)\\ =\ &-\frac{\beta}{2}(y_n^2 + z_n^2)\phi_0 + \Big(\alpha y_nz_n - \frac{\beta}{2}(y_n^2 - z_n^2)\Big)\phi_{2n} - \Big(\alpha\frac{y_n^2 - z_n^2}{2} + \beta y_nz_n\Big)\psi_{2n}\\ &+ o(y_n^2 + z_n^2).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\Phi(y_n\phi_n + z_n\psi_n) = \frac{y_n^2 + z_n^2}{2}\phi_0 - \Big(\alpha y_nz_n - \frac{\beta}{2}(y_n^2 - z_n^2)\Big)\frac{\phi_{2n}}{\lambda_{2n}}\\ + \Big(\alpha\frac{y_n^2 - z_n^2}{2} + \beta y_nz_n\Big)\frac{\psi_{2n}}{\lambda_{2n}} + o(y_n^2 + z_n^2).\end{aligned} \tag{2.5}$$

Then, by tedious computation, we get

(2.6)

$$
\begin{aligned}
&G(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)) \\
&= \big(\alpha y_n z_n - \frac{\beta}{2}(y_n^2 + z_n^2)\big)\phi_0 \\
&+ \Big[-(\beta^2 + 2\lambda_{2n}\beta)y_n^3 + (\alpha\beta + \lambda_{2n}\alpha)y_n^2 z_n - (\beta^2 + 2\lambda_{2n}\beta)y_n z_n^2 + (\alpha\beta + \lambda_{2n}\alpha)z_n^3\Big]\frac{\phi_n}{2\lambda_{2n}} \\
&+ \Big[-(\alpha\beta + \lambda_{2n}\alpha)y_n^3 - (\beta^2 + 2\lambda_{2n}\beta)y_n^2 z_n - (\alpha\beta + \lambda_{2n}\alpha)y_n z_n^2 - (\beta^2 + 2\lambda_{2n}\beta)z_n^3\Big]\frac{\psi_n}{2\lambda_{2n}} \\
&- \frac{\beta}{2}(y_n^2 - z_n^2)\phi_{2n} - (\alpha y_n^2 + 2\beta y_n z_n - \alpha z_n^2)\frac{\psi_{2n}}{2} \\
&+ \Big[\big(\alpha^2 - \beta^2\big)y_n^3 + 6\alpha\beta y_n^2 z_n - 3\big(\alpha^2 - \beta^2\big)y_n z_n^2 - 2\alpha\beta z_n^3\Big]\frac{\phi_3}{2\lambda_{2n}} \\
&+ \Big[-2\alpha\beta y_n^3 + 3(\alpha^2 - \beta^2)y_n^2 z_n + 6\alpha\beta y_n z_n^2 - \big(\alpha^2 - \beta^2\big)z_n^3\Big]\frac{\psi_3}{2\lambda_{2n}} + o(y_n^3 + z_n^3).
\end{aligned}
$$

We postpone the derivation of (2.6) in the Appendix. As a consequence, we are led to

$$
\begin{aligned}
&\frac{1}{\pi}\left\langle G_2(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)), \phi_n\right\rangle \\
&= \frac{-(\beta^2 + 2\lambda_{2n}\beta)y_n^3 + (\alpha\beta + \lambda_{2n}\alpha)y_n^2 z_n - (\beta^2 + 2\lambda_{2n}\beta)y_n z_n^2 + (\alpha\beta + \lambda_{2n}\alpha)z_n^3}{2\lambda_{2n}} \\
&\qquad + o(|y_n|^3 + |z_n|^3), \\
&\frac{1}{\pi}\left\langle G_2(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)), \psi_n\right\rangle \\
&= \frac{-(\alpha\beta + \lambda_{2n}\alpha)y_n^3 - (\beta^2 + 2\lambda_{2n}\beta)y_n^2 z_n - (\alpha\beta + \lambda_{2n}\alpha)y_n z_n^2 - (\beta^2 + 2\lambda_{2n}\beta)z_n^3}{2\lambda_{2n}} \\
&\qquad + o(|y_n|^3 + |z_n|^3).
\end{aligned}
$$

In the sequel, (2.2) becomes

$$\frac{d\mathbf{y}}{dt} = \lambda_n\mathbf{y} - \mathbf{F}(\mathbf{y}) + o(|\mathbf{y}|^3), \tag{2.7}$$

where $\mathbf{y} = (y_n, z_n)$ and

$$
\begin{aligned}
\mathbf{F}(\mathbf{y}) = &\frac{1}{2\lambda_{2n}}\Big((\beta^2 + 2\lambda_{2n}\beta)(y_n^3 + y_n z_n^2) - (\alpha\beta + \lambda_{2n}\alpha)(y_n^2 z_n + z_n^3), \\
&\qquad (\alpha\beta + \lambda_{2n}\alpha)(y_n^3 + y_n z_n^2) + (\beta^2 + 2\lambda_{2n}\beta)(y_n^2 z_n + z_n^3)\Big).
\end{aligned}
$$

We notice that

$$\langle \mathbf{F}(\mathbf{y}), \mathbf{y} \rangle = \frac{\beta^2 + 2\lambda_{2n}\beta}{2\lambda_{2n}}(y_n^2 + z_n^2)^2 = \frac{\beta^2 + 2\lambda_{2n}\beta}{2\lambda_{2n}}|\mathbf{y}|^4.$$

If $0 < \beta < -2\lambda_{2n}$, or equivalently if $0 < \beta < 8n^2/3$, then $d = (\beta^2 + 2\lambda_{2n}\beta)/(2\lambda_{2n}) > 0$. Thus we obtain

$$d|\mathbf{y}|^4 \leq \langle \mathbf{F}(\mathbf{y}), \mathbf{y} \rangle \leq 2d|\mathbf{y}|^4.$$

This implies by Theorem 5.10 of [6] that (2.7) bifurcates from the trivial solution to an invariant set $\mathcal{A}_n(\lambda, \alpha)$ as β passes through n^2 which is homeomorphic to S^1. This finishes the proof. □

3. Appendix

In this appendix section, we verify the identity (2.6). By definition, we have

$$\begin{aligned}
&G(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)) \\
&= \alpha\big(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)\big)\big(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)\big)_x \\
&\quad - \beta\big(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)\big)^2.
\end{aligned}$$

Then, by (2.5),

$$\begin{aligned}
&G(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)) \\
&= \alpha\Big[y_n\phi_n + z_n\psi_n + \frac{y_n^2 + z_n^2}{2}\phi_0 - \Big(\alpha y_n z_n - \frac{\beta}{2}(y_n^2 - z_n^2)\Big)\frac{\phi_{2n}}{\lambda_{2n}} \\
&\qquad + \Big(\alpha\frac{y_n^2 - z_n^2}{2} + \beta y_n z_n\Big)\frac{\psi_{2n}}{\lambda_{2n}}\Big] \\
&\quad \times \Big[- y_n\psi_n + z_n\phi_n + \Big(\alpha y_n z_n - \frac{\beta}{2}(y_n^2 - z_n^2)\Big)\frac{\psi_{2n}}{\lambda_{2n}} \\
&\qquad + \Big(\alpha\frac{y_n^2 - z_n^2}{2} + \beta y_n z_n\Big)\frac{\phi_{2n}}{\lambda_{2n}}\Big] \\
&\quad - \beta\Big[y_n\phi_n + z_n\psi_n + \frac{y_n^2 + z_n^2}{2}\phi_0 - \Big(\alpha y_n z_n - \frac{\beta}{2}(y_n^2 - z_n^2)\Big)\frac{\phi_{2n}}{\lambda_{2n}} \\
&\qquad + \Big(\alpha\frac{y_n^2 - z_n^2}{2} + \beta y_n z_n\Big)\frac{\psi_{2n}}{\lambda_{2n}}\Big]^2. \\
&\quad + o(y_n^2 + z_n^2).
\end{aligned}$$

Expanding these terms, we get

$$
\begin{aligned}
& G(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)) \\
= & -\frac{2\beta(y_n^3 + y_nz_n^2) - \alpha(y_n^2z_n + z_n^3)}{2}\phi_n - \frac{\alpha(y_n^3 + y_nz_n^2) + 2\beta(y_n^2z_n + z_n^3)}{2}\psi_n \\
& (\alpha y_nz_n - \beta y_n^2)\phi_n^2 - (\alpha y_n^2 + 2\beta y_nz_n - \alpha z_n^2)\phi_n\psi_n + (\alpha y_nz_n - \beta z_n^2)\psi_n^2 \\
& + \Big[\big(\frac{\alpha^2}{2} - \beta^2\big)y_n^3 + \frac{7\alpha\beta}{2}y_n^2z_n - \big(\frac{3\alpha^2}{2} - \beta^2\big)y_nz_n^2 - \frac{\alpha\beta}{2}z_n^3\Big]\frac{\phi_n\phi_{2n}}{\lambda_2} \\
& + \Big[-\frac{3\alpha\beta}{2}y_n^3 + \big(\frac{3\alpha^2}{2} - 2\beta^2\big)y_n^2z_n + \frac{5\alpha\beta}{2}y_nz_n^2 - \frac{\alpha^2}{2}z_n^3\Big]\frac{\phi_n\psi_{2n}}{\lambda_{2n}} \\
& + \Big[-\frac{\alpha\beta}{2}y_n^3 + \big(\frac{3\alpha^2}{2} - \beta^2\big)y_n^2z_n + \frac{7\alpha\beta}{2}y_nz_n^2 - \big(\frac{\alpha^2}{2} - \beta^2\big)z_n^3\Big]\frac{\psi_n\phi_{2n}}{\lambda_{2n}} \\
& + \Big[-\frac{\alpha^2}{2}y_n^3 - \frac{5\alpha\beta}{2}y_n^2z_n + \big(\frac{3\alpha^2}{2} - 2\beta^2\big)y_nz_n^2 + \frac{3\alpha\beta}{2}z_n^3\Big]\frac{\psi_n\psi_{2n}}{\lambda_{2n}} \\
& + o(y_n^3 + z_n^3).
\end{aligned}
$$

Then, we obtain by (2.4) that

$$
\begin{aligned}
& G(y_n\phi_n + z_n\psi_n + \Phi(y_n\phi_n + z_n\psi_n)) \\
= & -\frac{2\beta(y_n^3 + y_nz_n^2) - \alpha(y_n^2z_n + z_n^3)}{2}\phi_n - \frac{\alpha(y_n^3 + y_nz_n^2) + 2\beta(y_n^2z_n + z_n^3)}{2}\psi_n \\
& + (\alpha y_nz_n - \beta y_n^2)\frac{\phi_0 + \phi_{2n}}{2} - (\alpha y_n^2 + 2\beta y_nz_n - \alpha z_n^2)\frac{\psi_{2n}}{2} \\
& + (\alpha y_nz_n - \beta z_n^2)\frac{\phi_0 - \phi_{2n}}{2} \\
& + \Big[\big(\frac{\alpha^2}{2} - \beta^2\big)y_n^3 + \frac{7\alpha\beta}{2}y_n^2z_n - \big(\frac{3\alpha^2}{2} - \beta^2\big)y_nz_n^2 - \frac{\alpha\beta}{2}z_n^3\Big]\frac{\phi_n + \phi_3}{2\lambda_{2n}} \\
& + \Big[-\frac{3\alpha\beta}{2}y_n^3 + \big(\frac{3\alpha^2}{2} - 2\beta^2\big)y_n^2z_n + \frac{5\alpha\beta}{2}y_nz_n^2 - \frac{\alpha^2}{2}z_n^3\Big]\frac{\psi_n + \psi_3}{2\lambda_{2n}} \\
& + \Big[-\frac{\alpha\beta}{2}y_n^3 + \big(\frac{3\alpha^2}{2} - \beta^2\big)y_n^2z_n + \frac{7\alpha\beta}{2}y_nz_n^2 - \big(\frac{\alpha^2}{2} - \beta^2\big)z_n^3\Big]\frac{-\psi_n + \psi_3}{2\lambda_{2n}} \\
& + \Big[-\frac{\alpha^2}{2}y_n^3 - \frac{5\alpha\beta}{2}y_n^2z_n + \big(\frac{3\alpha^2}{2} - 2\beta^2\big)y_nz_n^2 + \frac{3\alpha\beta}{2}z_n^3\Big]\frac{\phi_n - \phi_3}{2\lambda_{2n}} \\
& + o(y_n^3 + z_n^3)
\end{aligned}
$$

$$
\begin{aligned}
&= \big(\alpha y_n z_n - \frac{\beta}{2}(y_n^2 + z_n^2)\big)\phi_0 \\
&\quad + \Big[- (\beta^2 + 2\lambda_{2n}\beta)y_n^3 + (\alpha\beta + \lambda_{2n}\alpha)y_n^2 z_n - (\beta^2 + 2\lambda_{2n}\beta)y_n z_n^2 \\
&\qquad + (\alpha\beta + \lambda_{2n}\alpha)z_n^3\Big]\frac{\phi_n}{2\lambda_{2n}} \\
&\quad + \Big[- (\alpha\beta + \lambda_{2n}\alpha)y_n^3 - (\beta^2 + 2\lambda_{2n}\beta)y_n^2 z_n - (\alpha\beta + \lambda_{2n}\alpha)y_n z_n^2 \\
&\qquad - (\beta^2 + 2\lambda_{2n}\beta)z_n^3\Big]\frac{\psi_n}{2\lambda_{2n}} \\
&\quad - \frac{\beta}{2}(y_n^2 - z_n^2)\phi_{2n} - (\alpha y_n^2 + 2\beta y_n z_n - \alpha z_n^2)\frac{\psi_{2n}}{2} \\
&\quad + \Big[\big(\alpha^2 - \beta^2\big)y_n^3 + 6\alpha\beta y_n^2 z_n - 3\big(\alpha^2 - \beta^2\big)y_n z_n^2 - 2\alpha\beta z_n^3\Big]\frac{\phi_3}{2\lambda_{2n}} \\
&\quad + \Big[- 2\alpha\beta y_n^3 + 3(\alpha^2 - \beta^2)y_n^2 z_n + 6\alpha\beta y_n z_n^2 - \big(\alpha^2 - \beta^2\big)z_n^3\Big]\frac{\psi_3}{2\lambda_{2n}} \\
&\quad + o(y_n^3 + z_n^3).
\end{aligned}
$$

This gives (2.6).

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