Journal of the Korean Society of Marine Engineering, Vol. 40, No. 2 pp. 131~137, 2016
J. Korean Soc. of Marime Engineering (JKOSME)
http://dx.doi.org/10.5916/jkosme.2016.40.2.131

ISSN 2234-7925 (Print)
ISSN 2234-8352 (Online)
Original Paper

Analysis of generalized progressive hybrid censored competing risks data

Kyeong-Jun Lee[1]· Jae-Ik Lee[2]· Chan-Keun Park†

(Received January 21, 2016 ; Revised February 17, 2016 ; Accepted February 17, 2016)

Abstract: In reliability analysis, it is quite common for the failure of any individual or item to be attributable to more than one cause. Moreover, observed data are often censored. Recently, progressive hybrid censoring schemes have become quite popular in life-testing problems and reliability analysis. However, a limitation of the progressive hybrid censoring scheme is that it cannot be applied when few failures occur before time T. Therefore, generalized progressive hybrid censoring schemes have been introduced. In this article, we derive the likelihood inference of the unknown parameters under the assumptions that the lifetime distributions of different causes are independent and exponentially distributed. We obtain the maximum likelihood estimators of the unknown parameters in exact forms. Asymptotic confidence intervals are also proposed. Bayes estimates and credible intervals of the unknown parameters are obtained under the assumption of gamma priors on the unknown parameters. Different methods are compared using Monte Carlo simulations. One real data set is analyzed for illustrative purposes.

Keywords: Bayes estimate, Competing risk, Exponential distribution, Generalized progressive hybrid censoring, Maximum likelihood estimator

1. Introduction

In medical studies or reliability analysis, it is quite common that more than one cause or risk factor may be present at the same time. That is, a failure of test unit is often resulted by one of the several risk factors. This is what we call competing risks model, proposed by Cox **[1]**. In analyzing the competing risks model, it is assumed that data consists of a failure time and an indicator denoting the cause of failure. Recently, researchers are interested with one specific factor in the presence of other risk factors. It is also typically supposed that the different risk factors are independent so as to avoid the problem of model identifiability **[2]**.

Based on competing risks model, a simple step-stress accelerated life testing problem under different censoring scheme was discussed in Balakrishnan and Han **[3]** and Han and Balakrishnan **[4]**. They constructed exact confidence intervals (CIs) and approximate CIs by exact distributions, asymptotic distributions, the parametric bootstrap method and the Bayesian posterior distribution, respectively. For the Lomax distribution, Cramer and Schmiedt **[5]** developed a competing risks model under progressively type II censoring scheme, and addressed the problem of optimal censoring schemes based on the Fisher information matrix. For the Weibull distribution, Bhattacharya *et al.* **[6]** developed a competing risks model under hybrid censoring scheme. Mao *et al.* **[7]** developed competing risks model under generalized type I hybrid censoring scheme, and constructed exact CIs and approximate CIs by exact distributions, asymptotic distributions, the parametric bootstrap method and the Bayesian posterior distribution, respectively.

If an experimenter desires to remove live experimental units at points other than the final termination point of the experiment, the Type I and Type II censoring schemes will not be of use. The conventional censoring schemes do not allow for units to be removed from the test at points other than the final termination point. Intermediate removal may be desirable when a compromise between reduced time of experimentation and the observation of at least some extreme lifetimes is sought, or when some of the surviving units in the experiment that are removed early on can be used for some other tests. Therefore, the loss of units at points other than the final termination point may be unavoidable, as in the case of accidental breakage of experimental units or loss of contact with individuals under experiment. These reasons and motivations lead reliability practitioners and theoreticians directly into the area of progressive censoring.

Progressive censoring scheme can be described as follows. Immediately following the first observed failure, R_1 surviving units are removed from the test at random. Similarly, following

† Corresponding Author (ORCID: http://orcid.org/0000-0003-3388-1505): Department of Data Information, Korea Maritime and Ocean University, 727, Taejong-ro, Yeongdo-gu, Busan 49112, Korea, E-mail: chapark@kmou.ac.kr, Tel: 051-410-4372
1 Department of Data Information, Korea Maritime and Ocean University, E-mail: indra_74@naver.com, Tel: 051-410-4370
2 Department of Data Information, Korea Maritime and Ocean University, E-mail: ljaei90@naver.com, Tel: 051-410-4370

the second observed failure, R_2 surviving units are removed from the test at random. This process continues until, immediately following the mth observed failure, all the remaining $R_m = n - R_1 - \cdots - R_{m-1} - m$ units are removed from the experiment. In this experiment, the progressive censoring scheme $R = (R_1, R_2, \cdots, R_m)$ is pre-fixed. The resulting m ordered failure times, which we denote by $X_{1:m:n}$, $X_{2:m:n}$, $\cdots$, $X_{m:m:n}$, are referred to as progressive Type II censoring scheme.

The disadvantages of the progressive Type II censoring scheme is that the time of the experiment can be very long if the units are highly reliable. Because of that, Kundu and Joarder **[8]** and Childs *et al.* **[9]** proposed a progressive hybrid censoring scheme in the context of life-testing experiment in which n identical units are placed on experiment with progressive Type II censoring scheme $(R_1, R_2, \cdots, R_m)$, and the experiment is terminated at time $\min\{X_{m:m:n}, T\}$, where $T \in (0, \infty)$ and $1 \le m \le n$ are fixed in advance, and $X_{1:m:n} \le X_{2:m:n} \le \cdots \le X_{m:m:n}$ are the ordered failure times resulting from the experiment. Under progressive hybrid censoring scheme the time on experiment will be no more than T.

The disadvantages of the progressive hybrid censoring scheme is that there is a possibility that very few failures may occur before time T. In order to provide a guarantee in terms of the number of failures observed as well as time to complete the test, Cho *et al.* **[10]** propose generalized progressive hybrid censoring scheme. This is designed to fix the disadvantages inherent in the progressive hybrid censoring scheme. Cho *et al.* **[11]** discussed the entropy in the Weibull distribution for generalized progressive hybrid censoring. The detail description and its adventages will be described in the next section.

In this paper, we consider independent identically distributed (iid) exponential competing risks model under generalized progressive hybrid censoring. The detail description of the model and maximum likelihood estimates (MLEs) for parameters are presented in Section 2. We obtain an estimate of the asymptotic confidence intervals (CIs) in Section 3. Bayes estimate and credible intervals are also obtained under the assumption of the gamma prior on the unknown parameters in Section 4. A real data set has been analyzed in Section 5. Furthermore, A Monte Carlo simulation of inferential procedures is carried out in Section 6 and finally we conclude the paper in Section 7.

2. Model, likelihood and MLEs

Suppose that n identical items are simultaneously put on a generalized progressive hybrid censoring lifetime test with competing risks. There are two risk factors for the failure of items when the distributions of different factors are independently exponential. We also suppose that the lifetimes $X_1, X_2, \cdots, X_n$ are statistically independent, here $X_i = \min\{X_{1i}, X_{2i}\}$, X_{ji} denotes the lifetime of the \$i\$th item under the jth failure risk factor with cumulative density function (CDF) and probability density function (PDF) such as $G_{i(x)} = 1 - \exp(\lambda_i x)$ and $g_{i(x)} = \lambda_i \exp(\lambda_i x)$, $i = 1, 2$. It is also easy to obtain the CDF and PDF of lifetime as

$$F(x) = 1 - \prod_{i=1}^{2}(1 - G_i) = 1 - \exp\left[-\left(\sum_{i=1}^{2}\lambda_i\right)x\right],$$

$$f(x) = \left(\sum_{i=1}^{2}\lambda_i\right)\exp\left[-\left(\sum_{i=1}^{2}\lambda_i\right)x\right].$$

It is well known that each failure observation is composed of failure lifetime and the cause of failure under the competing risks model. For convenience, let $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ describe sorted progressive Type II censored lifetime of m items, and $Z = (\delta_1, \delta_2, \cdots, \delta_m)$ describe the indicator of risk factor corresponding to the sequential failure times above. Here $\delta_j = 1$, $j = 1, 2, \cdots, m$, denotes the failure of the jth item caused by risk factor 1. Obviously, $\delta_j = 0$ means that factor 2 is responsible for the jth failure. Based on our assumption, the joint PDF of lifetime and corresponding factor (X, Z) is given by

$$f_{X,Z}(x, i) = \lambda_i \exp\left[-\left(\sum_{j=1}^{2}\lambda_j\right)x\right],\ i = 1, 2.$$

Furthermore, generalized progressive hybrid censoring scheme is described as follows. The integer $k, m \in \{1, 2, \cdots, n\}$ is pre-fixed such that $k < m$ and also $R_1, R_2, \cdots, R_m$ are pre-fixed integers satisfying $\sum_{i=1}^{m} R_i + m = n$. $T \in (0, \infty)$ is a pre-fixed time point. At the time of first failure R_1 of the remaining units are randomly removed. Similarly at the time of the second failure R_2 of the remaining units are removed and so on. This process continues until, immediately following the terminated time $T^* = \max\{X_{k:m:n}, \min\{X_{m:m:n}, T\}\}$, all the remaining units are removed from the experiment. This generalized progressive hybrid censoring scheme modifies the progressive hybrid censoring scheme by allowing the experiment to continue beyond time T if very few failures had been observed up to time T. Under this scheme, the experimenter would ideally like to observe m failures, but is willing to accept a bare minimum of k failures. Let D denote the number of observed failures up to time T. In this scheme, we have one of the following types of observations;

Case I: $(X_{1:m:n}, \delta_1)$, $\cdots$, $(X_{k:m:n}, \delta_k)$, if $T < X_{k:m:n}$, Case II : $(X_{1:m:n},\delta_1),\cdots,(X_{k:m:n},\delta_k),\cdots,(X_{D:m:n},\delta_D)$, if $X_{k:m:n} < T$,

Case III : $(X_{1:m:n},\delta_1),\cdots,(X_{k:m:n},\delta_k),\cdots,(X_{m:m:n},\delta_m)$, if $X_{m:m:n} < T$.

Note that for case II, $X_{D:m:n} < T < X_{D+1:m:n}$ and $X_{D+1:m:n},\cdots,X_{m:m:n}$ are not observed. For case III, $T < X_{k:m:n} < X_{m:m:n}$ and $X_{k+1:m:n},\cdots,X_{m:m:n}$ are not observed. Based on the observable data, the likelihood function can be written as;

$$L(\lambda_1,\lambda_2) = C\,\lambda_1^{J_1}\lambda_2^{J_2}e^{-(\lambda_1+\lambda_2)W},$$

where $C = C_k$, $J_1 = \sum_{i=1}^{k}\delta_i$, $J_2 = k - J_1$, $R_k = n - \sum_{i=1}^{k-1}R_i - k$, $W = \sum_{i=1}^{k}(1+R_i)x_{i:m:n}$ for Case I, $C = C_d$, $J_1 = \sum_{i=1}^{D}\delta_i$, $J_2 = D - J_1$, $W = \sum_{i=1}^{D}(1+R_i)x_{i:m:n} + TR_{D+1}^{*}$ for Case II, and $C = C_m$, $J_1 = \sum_{i=1}^{m}\delta_i$, $J_2 = m - J_1$, $W = \sum_{i=1}^{m}(1+R_i)x_{i:m:n}$ for Case III.

From likelihood function, we have

$$\frac{\partial \log L(\lambda_1,\lambda_2)}{\partial \lambda_1} = \frac{J_1}{\lambda_1} - W \text{ and } \frac{\partial \log L(\lambda_1,\lambda_2)}{\partial \lambda_2} = \frac{J_2}{\lambda_2} - W.$$

Therefore, we obtain the MLEs of λ_1 and λ_2 as

$$\hat{\lambda}_1 = \frac{J_1}{W} \text{ and } \hat{\lambda}_2 = \frac{J_2}{W}.$$

3. Confidence intervals

In this section, we propose different CIs of the unknown parameters of λ_1 and λ_2 for $J_1 > 0$ and $J_2 > 0$. It is very difficult to obtain the CIs of λ_1 and λ_2 for $J_1 = 0$ and $J_2 = 0$, and it is not pursued here.

The $100(1-\alpha)\%$ CIs for λ_1 and λ_2 can be obtained from the usual asymptotic normality of the MLEs with $Var(\hat{\lambda}_1)$ and $Var(\hat{\lambda}_2)$ estimated from the inverse of the observed Fisher information matrix.

From the log-likelihood function, the second derivatives of log-likelihood function with respect to λ_1 and λ_2 are given by

$$\frac{\partial^2 \log L(\lambda_1,\lambda_2)}{\partial \lambda_1^2} = -\frac{J_1}{\lambda_1^2},\quad \frac{\partial^2 \log L(\lambda_1,\lambda_2)}{\partial \lambda_2^2} = -\frac{J_2}{\lambda_2^2},$$

$$\text{and } \frac{\partial^2 \log L(\lambda_1,\lambda_2)}{\partial \lambda_1\lambda_2} = 0. \tag{1}$$

Let $I(\lambda_1,\lambda_2)$ denote the Fisher information matrix of the parameters λ_1 and λ_2. The Fisher information matrix is then obtained by taking expectations of minus **Equation (1)**.

$$I(\lambda_1,\lambda_2) = I_{ij}(\lambda_1,\lambda_2) = -E\left[\frac{\partial^2 \log L}{\partial \lambda_i \lambda_j}\right].$$

It follows that

$$I_{11}(\lambda_1,\lambda_2) = -\frac{E(J_1)}{\lambda_1^2},\ I_{22}(\lambda_1,\lambda_2) = -\frac{E(J_2)}{\lambda_2^2},$$

and $I_{12}(\lambda_1,\lambda_2) = I_{21}(\lambda_1,\lambda_2) = 0$.

Observe that $E(J_1) = \sum_{i=1}^{J_1}P(X_{i:m:n} < T)$ and $E(J_2) = \sum_{i=1}^{J_2}P(X_{i:m:n} < T)$.

Under some mild regularity conditions, $(\hat{\lambda}_1,\hat{\lambda}_2)$ is approximately bivariately normal with mean (λ_1,λ_2) and covariance matrix $I^{-1}(\lambda_1,\lambda_2)$.

However it is not easy to compute $P(X_{i:m:n} < T)$ for general i, because $X_{i:m:n}$ is a sum of i independent but not identically distributed exponential random variables. Therefore, for $J_1 > 0$ and $J_2 > 0$, we estimate $I^{-1}(\lambda_1,\lambda_2)$ by $I^{-1}(\hat{\lambda}_1,\hat{\lambda}_2)$.

A simpler and equally valid procedure is to use the approximation

$$(\hat{\lambda}_1,\hat{\lambda}_2) \sim N\left[(\lambda_1,\lambda_2), I^{-1}(\hat{\lambda}_1,\hat{\lambda}_2)\right],$$

where $I^{-1}(\hat{\lambda}_1,\hat{\lambda}_2) = \begin{pmatrix} \hat{\lambda}_1^2/J_1 & 0 \\ 0 & \hat{\lambda}_2^2/J_2 \end{pmatrix}$.

Therefore, the $100(1-\alpha)\%$ normal approximate CIs for λ_1 and λ_2 are

$$\left(\hat{\lambda}_1 - Z_{\alpha/2}\frac{\hat{\lambda}_1}{\sqrt{J_1}}, \hat{\lambda}_1 + Z_{\alpha/2}\frac{\hat{\lambda}_1}{\sqrt{J_1}}\right)$$

and $\left(\hat{\lambda}_2 - Z_{\alpha/2}\frac{\hat{\lambda}_2}{\sqrt{J_2}}, \hat{\lambda}_2 + Z_{\alpha/2}\frac{\hat{\lambda}_2}{\sqrt{J_2}}\right)$,

where $Z_{\alpha/2}$ is the percentile of the standard normal distribution with right-tail probability $\alpha/2$.

4. Bayes inference

In this section we approach the problem from the Bayesian point of view. It is assumed that the parameters λ_1 and λ_2 are independent and follow the gamma(a_1,b_1) and gamma(a_2,b_2) prior distributions with $a_1 > 0$, $b_1 > 0$, $a_2 > 0$ and $b_2 > 0$. Therefore, the joint prior distribution of λ_1 and λ_2 is of the form:

$$\pi(\lambda_1,\lambda_2) \propto \lambda_1^{\alpha_1 - 1}\lambda_2^{\alpha_2 - 1}e^{-\lambda_1 b_1}e^{-\lambda_2 b_2}.$$

Based on the above joint prior distribution, the joint density of the λ_1, λ_2 and X can be written as follows.

$$\pi(\lambda_1,\lambda_2,X) \propto \lambda_1^{J_1+\alpha_1-1}\lambda_2^{J_2+\alpha_2-1}e^{-\lambda_1(W+b_1)}e^{-\lambda_2(W+b_2)}.$$

Then, the posterior distribution of λ_1 and λ_2, given X, is obtained as:

$$\pi(\lambda_1,\lambda_2|X) \propto \frac{\pi(\lambda_1,\lambda_2,X)}{\int_0^\infty \int_0^\infty \pi(\lambda_1,\lambda_2,X)d\lambda_1 d\lambda_2}.$$

Now, we obtain Bayes estimates of λ_1 and λ_2 against the squared error loss (SEL) and linex loss (LL) functions when the prior distribution is taken to be $\pi(\lambda_1,\lambda_2)$. The Bayes estimate of λ_1 against the SEL function is respectively obtained as,

$$\lambda_{S1} = E[\lambda_1|X] = \frac{(W+b_1)^{J_1+a_1}}{\Gamma(J_1+a_1)}\int_0^\infty \lambda_1^{(J_1+a_1+1)-1} e^{-\lambda_1(W+b_1)} d\lambda_1 = \frac{J_1+a_1}{W+b_1}.$$

Similary, we can obtain the Bayes estimate of λ_2 against the SEL function. Interestingly, when the non-informative priors $a_1=b_1=a_2=b_2=0$, the Bayes estimators under SEL function coincide with the corresponding MLEs.

Next, the Bayes estimate of λ_1 against the LL function is respectively obtained as,

$$\lambda_{L1} = -\frac{1}{h}\log\{E[e^{-h\lambda_1}|X]\},$$

where

$$E[e^{-h\lambda_1}|X] = \frac{(W+b_1)^{J_1+a_1}}{\Gamma(J_1+a_1)}\int_0^\infty \lambda_1^{(J_1+a_1)-1} e^{-\lambda_1(W+b_1+h)} d\lambda_1 = \left[\frac{W+b_1}{W+b_1+h}\right]^{J_1+a_1}.$$

Similary, we can the Bayes estimate of λ_2 against the LL function.

Also, the credible intervals for λ_1 and λ_2 can be obtained using the posterior distributions of λ_1 and λ_2. Note that $Z_1 = 2\lambda(W+b_1)$ and $Z_2 = 2\lambda(W+b_2)$ follows χ^2 distribution with $2(J_1+a_1)$ and $2(J_2+a_2)$ degrees of freedom provided $2(J_1+a_1)$ and $2(J_2+a_2)$ are positive integer. Therefore, $100(1-\alpha)\%$ credible interval (BA) for λ_1 and λ_2 can be obtained as

$$\left[\frac{\chi^{2_{1-\alpha/2}}(2(J_1+a_1))}{2(W+b_1)}, \frac{\chi^{2_{\alpha/2}}(2(J_1+a_1))}{2(W+b_1)}\right] \text{ and } \left[\frac{\chi^{2_{1-\alpha/2}}(2(J_2+a_2))}{2(W+b_2)}, \frac{\chi^{2_{\alpha/2}}(2(J_2+a_2))}{2(W+b_2)}\right],$$ respectively for $J_{1+}a_1 > 0$ and $J_2+a_2>0$. Here, $\chi^2_{1-\alpha/2}(df)$ and $\chi^2_{\alpha/2}(df)$ denote the upper and lower $\alpha/2$th percentile points of a χ^2 distribution. Note that if J_1+a_1 and J_2+a_2 are not integer values then gamma distribution can be used to construct the credible intervals for λ_1 and λ_2 . If no prior information is available, then non-informative priors can be used to compute the credible intervals for λ_1 and λ_2 .

5. Illustrative example

For illustrative purposes, we present here a real data analysis using the proposed methods. The following data set are some small vessel electronic appliances exposed to the automatic test machine (Lawless, **[12]**). This data set was analyzed by Mao et al. [7]. There were 18 risk factors for the failure of the appliances. Among the 18 failure modes, only failure mode 9 appeared the most times, to be accurate 17 times. Obviously, it was desirable to consider inference of failure mode 9 in the presence of other modes including 17 failure risks modes and censoring mode. Considering this, let us express the ith failure appliance due to failure mode 9 with $\delta_i=1$, and then $\delta_i=0$ denotes failure caused by other failure modes. And the ordered failure lifetimes and corresponding failure factors were presented at the following: (11,0), (35,0), (49,0), (170,0), (329,0), (381,0), (708,0), (958,0), (1062,0), (1167,1), (1594,0), (1925,1), (1990,1), (2223,1), (2327,0), (2400,1), (2451,0), (2471,1), (2551,1), (2565,0), (2568,1), (2694,1), (2702,0), (2761,0), (2831,0), (3034,1), (3059,0), (3112,1), (3214,1), (3478,1), (3504,1), (4329,1), (6367,0), (6976,1), (7846,1), (13403,0).

From the above sample, we created an artificial data by progressive Type II censored sample. We have $n=36$ and we took $m=28$, $R_1=R_{28}=4$ and $R_i=0$ for $i=2,\cdots,27$. Thus, the progressive Type II censored sample is (11,0), (170,0), (329,0), (708,0), (1062,0), (1167,1), (1594,0), (1925,1), (2223,1), (2327,0), (2400,1), (2451,0), (2471,1), (2551,1), (2565,0), (2568,1), (2694,1), (2702,0), (2761,0), (2831,0), (3034,1), (3059,0), (3112,1), (3214,1), (3478,1), (3504,1), (4329,1), (6976,1). In this example, we take case I ($T=2000$ and $k=10$), case II ($T=2000$ and $k=6$), and case III ($T=7000$ and $k=6$).

Table 1 presents inferences of λ_1 and λ_2, and the 95% CIs and credible intervals for λ_1 and λ_2 for values of case I, II, and III of generalized progressive hybrid censoring schemes.

Table 1: The MLEs, Bayes estimates and confidence/credible intervals of λ_1 and λ_2 for example

Sch.	J_1	λ_1			J_2	λ_2		
		MLE	SEL	LL		MLE	SEL	LL
I	3	4.78×10^{-5}	6.37×10^{-5}	6.37×10^{-5}	7	1.12×10^{-4}	1.27×10^{-4}	1.27×10^{-4}
II	2	3.64×10^{-5}	5.45×10^{-5}	5.45×10^{-5}	6	1.09×10^{-4}	1.27×10^{-4}	1.27×10^{-4}
III	15	1.53×10^{-4}	1.63×10^{-4}	1.63×10^{-4}	13	1.32×10^{-4}	1.43×10^{-4}	1.43×10^{-4}
Sch.	J_1	NA	BA		J_2	NA	BA	
I	3	$(0.00\times10^{-5},1.02\times10^{-4})$	$(0.00\times10^{-5},8.87\times10^{-5})$		7	$(2.89\times10^{-5},1.94\times10^{-4})$	$(1.42\times10^{-6},1.77\times10^{-4})$	
II	2	$(0.00\times10^{-5},8.67\times10^{-5})$	$(0.00\times10^{-5},7.17\times10^{-5})$		6	$(2.18\times10^{-5},1.96\times10^{-4})$	$(0.00\times10^{-5},1.74\times10^{-4})$	
III	15	$(7.54\times10^{-5},2.30\times10^{-4})$	$(6.13\times10^{-5},2.18\times10^{-4})$		13	$(6.04\times10^{-5},2.04\times10^{-4})$	$(3.75\times10^{-5},1.92\times10^{-4})$	

6. Simulation results

In this section, a Monte Carlo simulation study is conducted to compare the performance of different estimators. We consider different n, m, k, and T. We have used three different progressive Type II censored sampling schemes, namely; Scheme I : $R_m = n-m$ and $R_i = 0$ for $i \neq m$. Scheme II : $R_{m/2} = n-m$ and $R_i = 0$ for $i \neq m/2$. Scheme III : $R_m = n-m$ and $R_i = 0$ for $i \neq m$.

All Bayes estimates are computed with respect to the gamma prior distribution. This corresponds to the case when hyper-parameters take values of $a_1 = b_1 = a_2 = b_2 = 0$. Bayes estimates of parameters are derived with respect to three different loss functions, SEL and LL function. Under LL associated estimates are obtained for $h=1$. Finally, different schemes have been taken into consideration to compute MSE values of all estimates, and these values are tabulated in **Table 2**. We also compute the average confidence/credible lengths and the corresponding coverage percentages. The results are presented in **Table 3**. The corresponding coverage percentages are reported within brackets.

From **Table 2**, the following general observations can be made. The MSEs of both λ_1 and λ_2 decrease as sample size n increases. For fixed sample size, the MSEs of both λ_1 and λ_2 decrease generally as the number of progressive censored samples R_i decreases. For Fixed sample and progressive censoring data size, the MSEs of both λ_1 and λ_2 decrease generally as the time T increases. For fixed time T, sample and progressive censoring data size, the MSEs of both λ_1 and λ_2 decrease generally as the number of guarantee sample size k increase. It is also observed that the MLEs for schemes 1 and 2 behave quite similarly in terms of MSEs of both λ_1 and λ_2. The MLEs for scheme 3 have smaller MSEs of both λ_1 and λ_2 than the corresponding MLEs for the other two schemes.

From **Table 3**, the following general observations can be made. The confidence/credible lengths decrease as sample size n increase. For fixed sample size, the confidence/credible lengths of both λ_1 and λ_2 decrease generally as the number of progressive censored samples R_i decrease. For fixed sample and progressive censoring data size, the confidence/credible lengths of both λ_1 and λ_2 decrease generally as the time T increases. For fixed time T, sample and progressive censoring data size, the confidence/credible lengths of both λ_1 and λ_2 decrease generally as the number of guarantee sample size k increase. For most of the methods, scheme 2 and scheme 3 behave very similarly although the confidence/credible intervals of both λ_1 and λ_2 for scheme 3 are slightly shorter than scheme 2. The confidence/credible intervals for scheme 1 have longer than the other two schemes.

It is observed that the Bayes credible intervals (BA) with respect to the gamma prior work quite well for all sample sizes and for all the schemes. In most of the cases, the coverage percentages are quite close to the nominal level. The asymptotic CIs (NA) do not work well. It can not maintain the nominal level even when n, m and T is large.

7. Conclusions

In this paper, we consider iid exponential competing risks model under generalized progressive hybrid censoring. We obtain the exact inference for the parameters. We also obtain an estimate of the asymptotic CIs. Moreover, Bayes estimate and credible intervals are also obtained under the assumption of the gamma prior on the unknown parameters under two types loss functions. A real data set and a numerical simulation have been conducted to evaluate the performances of estimators in this article. The results show that Bayes method outperforms generally approximate method. Although we have assumed that the lifetime distributions are exponential but most of the methods can be extended for other distributions also, like Weibull, log-normal and rayleigh distribution.

Table 2: The average confidence/credible length and the corresponding coverage percentage of λ_1 and λ_2

n	m	k	Sch.	λ_1						λ_2					
				$T=0.5$			$T=1.0$			T=0.5			T=1.0		
				MLE	SEL	LL	MLE	SEL	LL	MLE	SEL	LL	MLE	SEL	LL
20	18	3	I	0.095	0.077	0.064	0.060	0.053	0.046	0.052	0.049	0.039	0.042	0.041	0.035
			II	0.085	0.071	0.059	0.057	0.052	0.045	0.050	0.047	0.038	0.041	0.040	0.034
			III	0.081	0.067	0.056	0.053	0.048	0.042	0.049	0.046	0.038	0.039	0.038	0.033
		5	I	0.089	0.074	0.061	0.060	0.053	0.046	0.051	0.048	0.039	0.042	0.041	0.035
			II	0.084	0.070	0.058	0.057	0.052	0.045	0.048	0.045	0.037	0.041	0.040	0.034
			III	0.080	0.067	0.056	0.053	0.048	0.042	0.048	0.045	0.037	0.039	0.038	0.033
	14	3	I	0.113	0.085	0.068	0.084	0.071	0.059	0.069	0.067	0.050	0.049	0.048	0.039
			II	0.105	0.085	0.068	0.074	0.064	0.053	0.061	0.059	0.047	0.046	0.045	0.038
			III	0.087	0.072	0.060	0.064	0.056	0.048	0.053	0.050	0.041	0.040	0.039	0.033
		5	I	0.106	0.081	0.065	0.084	0.071	0.059	0.064	0.063	0.047	0.049	0.048	0.039
			II	0.102	0.082	0.066	0.074	0.064	0.053	0.060	0.058	0.046	0.046	0.045	0.038
			III	0.084	0.069	0.058	0.064	0.056	0.048	0.052	0.050	0.041	0.040	0.039	0.033
30	26	6	I	0.062	0.055	0.048	0.044	0.041	0.037	0.041	0.038	0.033	0.027	0.026	0.023
			II	0.059	0.053	0.046	0.041	0.038	0.035	0.039	0.036	0.032	0.025	0.025	0.022
			III	0.057	0.051	0.045	0.037	0.035	0.032	0.038	0.035	0.031	0.023	0.022	0.020
		8	I	0.060	0.053	0.046	0.044	0.041	0.037	0.039	0.036	0.031	0.027	0.026	0.023
			II	0.057	0.051	0.045	0.041	0.038	0.035	0.039	0.036	0.032	0.025	0.025	0.022
			III	0.055	0.049	0.043	0.037	0.035	0.032	0.038	0.035	0.031	0.023	0.022	0.020
	22	6	I	0.080	0.069	0.058	0.050	0.047	0.041	0.050	0.048	0.040	0.033	0.032	0.028
			II	0.067	0.060	0.051	0.046	0.043	0.038	0.042	0.040	0.034	0.031	0.031	0.027
			III	0.055	0.049	0.043	0.039	0.036	0.033	0.036	0.034	0.030	0.027	0.026	0.024
		8	I	0.078	0.067	0.057	0.050	0.047	0.041	0.048	0.046	0.038	0.033	0.032	0.028
			II	0.066	0.059	0.051	0.046	0.043	0.038	0.041	0.040	0.034	0.031	0.031	0.027
			III	0.054	0.048	0.042	0.039	0.036	0.033	0.035	0.034	0.029	0.027	0.026	0.024

Table 3: The MSEs and average biases of all estimators of λ_1 and λ_2.

n	m	k	Sch.	λ_1								λ_2							
				$T=0.5$				$T=1.0$				$T=0.5$				$T=1.0$			
				AL		BL		AL		BL		AL		BL		AL		BL	
				length	CP	length	CP	length	CP	length	CP	length	CP	length	CP	length	CP	length	CP
20	18	3	I	1.13	89.6	1.08	96.0	.91	92.0	.89	96.2	.92	91.3	.93	98.0	.75	91.8	.75	95.2
			II	1.09	88.9	1.05	95.2	.90	93.3	.87	96.2	.88	87.2	.88	97.7	.73	91.9	.73	94.9
			III	1.08	88.9	1.04	95.0	.87	92.7	.85	96.0	.87	87.0	.88	97.5	.71	91.3	.71	94.6
		5	I	1.14	91.8	1.09	96.4	.91	92.0	.89	96.2	.92	91.8	.92	98.1	.75	91.8	.75	95.2
			II	1.09	90.5	1.05	95.9	.90	93.3	.87	96.2	.88	88.9	.88	97.7	.73	91.9	.73	94.9
			III	1.08	90.5	1.04	95.5	.87	92.7	.85	96.0	.87	88.7	.88	97.4	.71	91.3	.71	94.6
	14	3	I	1.26	88.7	1.20	97.7	1.04	92.3	1.00	95.7	1.08	98.1	1.07	97.2	.85	91.2	.85	97.3
			II	1.15	89.0	1.10	95.4	.99	93.3	.95	95.9	.96	91.3	.95	97.2	.81	92.6	.81	96.9
			III	1.07	88.4	1.03	95.1	.88	92.4	.85	95.4	.90	89.1	.90	97.4	.72	92.0	.72	94.9
		5	I	1.26	90.6	1.19	97.6	1.04	92.3	1.00	95.7	1.05	95.2	1.04	97.4	.85	91.2	.85	97.3
			II	1.15	89.9	1.10	96.0	.99	93.3	.95	95.9	.96	91.8	.95	97.1	.81	92.6	.81	96.9
			III	.08	89.4	1.04	95.7	.88	92.4	.85	95.4	.90	89.8	.90	97.5	.72	92.0	.72	94.9
30	26	6	I	.95	92.0	.92	96.0	.76	92.4	.74	94.6	.76	91.9	.76	97.2	.61	91.8	.61	95.8
			II	.90	93.3	.88	95.4	.74	93.5	.73	95.1	.72	88.2	.72	95.6	.59	91.9	.59	94.5
			III	.89	93.0	.87	95.3	.71	93.3	.70	95.1	.71	88.0	.71	95.6	.57	91.4	.57	94.9
		8	I	.95	93.1	.92	96.3	.76	92.4	.74	94.6	.76	92.1	.76	96.3	.61	91.8	.61	95.8
			II	.90	93.5	.88	95.6	.74	93.5	.73	95.1	.71	88.4	.72	95.7	.59	91.9	.59	94.5
			III	.89	93.3	.87	95.6	.71	93.3	.70	95.1	.71	88.2	.71	95.6	.57	91.4	.57	94.9
	22	6	I	1.03	92.5	1.00	95.9	.84	94.4	.82	95.0	.84	91.7	.84	97.0	.67	91.0	.67	94.8
			II	.93	93.0	.90	94.6	.80	94.3	.78	94.8	.75	90.2	.75	95.8	.64	92.2	.64	95.1
			III	.89	93.6	.87	95.4	.72	93.3	.71	95.1	.72	89.7	.72	96.4	.58	92.0	.58	94.3
		8	I	1.04	93.0	1.00	95.4	.84	94.4	.82	95.0	.84	90.9	.83	95.6	.67	91.0	.67	94.8
			II	.94	93.9	.91	94.9	.80	94.3	.78	94.8	.75	90.1	.75	95.5	.64	92.2	.64	95.1
			III	.89	94.5	.87	95.5	.72	93.3	.71	95.1	.72	89.6	.72	96.0	.58	92.0	.58	94.3

References

[1] D. R. Cox, “The analysis of exponentially distributed lifetimes with two types of failure,” Journal of the Royal Statistical Society. Series B (Methodological), vol. 21, no. 2, pp. 411-421, 1959.

[2] M. Crowder, Classical Competing Risks, Boca Raton (FL): Chapman & Hall, 2010.

[3] N. Balakrishnan and D. Han, “Exact inference for a simple step stress model with competing risks for failure from exponential distribution under type II censoring,” Journal of Statistical Planning and Inference, vol. 138, no. 12, pp. 4172-4186, 2008.

[4] D. Han and N. Balakrishnan, “Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint,” Computational Statistics & Data Analysis, vol. 54, no. 9, pp. 2066-2081, 2010.

[5] E. Cramer and A. Schmiedt, “Progressively type II censored competing risks data from lomax distributions,” Computational Statistics & Data Analysis, vol. 55, no. 3, pp. 1285-1303, 2011.

[6] S. Bhattacharya, B. Pradhan, and D. Kundu, “Analysis of hybrid censored competing risks data,” Statistics, vol. 48, no. 5, pp. 1138-1154, 2014.

[7] S. Mao, Y. M. Shi, and Y. D. Sun, “Exact inference for competing risks model with generalized type I hybrid censored exponential data,” Journal of Statistical Computation and Simulation, vol. 84, no. 11, pp. 2506-2521, 2014.

[8] D. Kundu, and A. Joarder, “Analysis of type II progressively hybrid censored data,” Computational Statistics and Data Analysis, vol. 50, no. 10, pp. 2509-2528, 2006.

[9] A. Childs, B. Chandrasekar, and N. Balakrishnan, “Exact likelihood inference for an exponential parameter under progressive hybrid schemes,” Statistical Models and Methods for Biomedical and Technical Systems, Birkhauser, Boston, pp. 319-330, 2007.

[10] Y. Cho, H. Sun, and K. Lee, “Exact likelihood inference for an exponential parameter under generalized progressive hybrid censoring scheme,” Statistical Methodology, vol. 23, pp. 18-34, 2015.

[11] Y. S. Cho, H. K. Sun, and K. J. Lee, “Estimating the entropy of a weibull distribution under generalized progressive hybrid censoring,” Entropy, vol. 17, no. 1, pp. 102-122, 2015.

[12] J. Lawless, Statistical Models and Methods for Lifetime Data, Hoboken (NJ): Wiley, 2011.