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M/G/1/1 대기체계의 고객 손실간격 분포에 대한 소고

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A Note on the Inter-Loss Time Distribution of an M/G/1/1 Queuing System

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■ Abstract ■

This note discusses the inter-loss time ofan M/G/1/1 queuing system. The inter-loss time is defined as the time duration between two consecutive losses of arriving customers. In this study, we present the explicit Laplace transform of the inter-loss time distribution of an M/G/1/1 queuing system.

Keywords : Inter-Loss Time, M/G/1/1 Queue, Laplace Transform, Loss Probability

1. Introduction

Ferrante [1] has recently investigated the inter-loss time for an M/M/1/1 Erlang loss model to solve the location problem of the ambulance in the Emergency Medical Systems (EMS). In EMS, each ambulance behaves as an M/M/1/1 queueing system because its client cannot wait for an emergency service. The information on the inter-loss not only has an influence on the performance measures of the system but also improves the operation efficiency and the quality of the emergency service.

Erlang loss model eliminates all delays by set-

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ting the number of the buffers to the number of servers. In this case, the so-called Erlang-B formula characterizes the loss (blocking) probability for the associated system. The serendipity is the well-known *invariance property* of the loss probability with respect to service time distribution. This accommodates general, rather than exponential, service time distributions. The objective of this note is to extend the work of Ferrante [1] in the general service time setting.

We consider an M/G/1/1 queueing system having following features. Customers arrive at the system, according to Poisson process. They are served by a single server. It is assumed that there is no waiting space in the system. Therefore, if a customer arrives in the system while a server is busy, he/she is blocked and lost. The inter-loss time is defined as the length of the time duration between two consecutive customer losses. In this work, we present that the Laplace transform (LT) of the inter-loss time in the M/G/1/1 queue. Throughout this note, $X^*(\theta)$, the LT of any continuous random variable (RV) X, is defined as $X^*(\theta)=\int_0^\infty e^{-\theta x}\Pr\{x<X<x+dx\}$.

2. Preliminaries

In this section, we carry out the preliminary analysis for the derivation of the main result. Let A and S denote the inter-arrival time and the service time, respectively. A has the exponential distribution having a mean λ^{-1}. S is the generally distributed RV. We assume that A and S are mutually independent. Let us define $G(x)$ and $S^*(\theta)$ as the cumulative density function (CDF) and its LT, respectively. Let $T_0(x)$ denote length of time from the point a busy period begins to the point the next loss occurs given that the length of the busy period is x. Let T_0 denote the unconditional length of time from the point a busy period begins to the point the next loss occurs. Note that, since only one customer is served per busy period, S is equal to the length of the busy period. Thus, we have

$$\Pr\{t<T_0<t+dt\}=\int_0^\infty \Pr\{t<T_0(x)<t+dt\}\,dG(x) \quad (1)$$

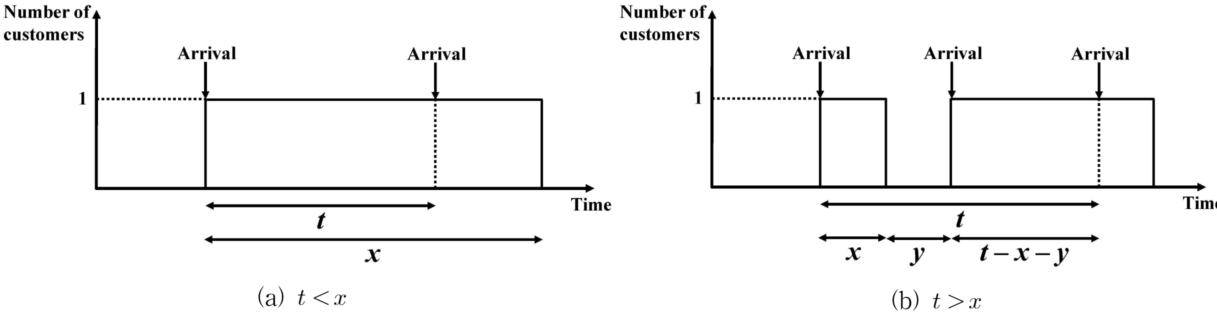
We must state here that, according to the range of t, $\Pr\{t<T_0(x)<t+dt\}$ in (1) is expressed as two different forms. If $t>x$, $\Pr\{t<T_0(x)<t+dt\}$ is expressed as

$$\Pr\{t<T_0(x)<t+dt\}=\Pr\{t<A<t+dt\}=\lambda e^{-\lambda t}dt,\quad x>t \quad (2)$$

In other words, $T_0(x)$ is equal to A by the memoryless property of the exponential inter-arrival time as long as $t<x$ (see [Figure 1(a)]). On the other hand, if $t<x$, the next loss does not occur during the currently ongoing busy period. At the point of the next customer arrival, $T_0(x)$ is equal to T_0 (see [Figure 1(b)]). Conditioning on the inter-arrival time of the next customer, $\Pr\{t<T_0(x)<t+dt\}$ is represented as

$$\begin{aligned}\Pr\{t<T_0(x)<t+dt\} &= \int_{y=0}^{t-x}\Pr\{t-x-y<T_0<t-x-y+dt\}\\ &\quad \Pr\{x+y<A<x+y+dy\} \quad (3)\\ &= e^{-\lambda x}\int_{y=0}^{t-x}\Pr\{t-x-y<T_0<t-x-y+dt\}\\ &\quad \lambda e^{-\lambda y}dy,\ x<t\end{aligned}$$

Let $T_0^*(\theta,x)$ and $T_0^*(\theta)$ denote the LT of $T_0(x)$ and that of T_0, respectively. From the probability



[Figure 1] Sample Path Examples

expression for $T_0(x)$ in (2) and (3), $T_0^*(\theta, x)$ is obtained as follows :

$$\begin{aligned} T_0^*(\theta, x) &= \int_0^x e^{-\theta t} \Pr\{t < T_0(x) < t+dt\} \\ &\quad + \int_x^\infty e^{-\theta t} \Pr\{t < T_0(x) < t+dt\} \\ &= \frac{\lambda}{\lambda+\theta}\left[1-e^{-(\lambda+\theta)x}\left(1-T_0^*(\theta)\right)\right] \end{aligned} \tag{4}$$

Remark 1: The detailed derivation of (4) is in the <Appendix A>.

Furthermore, taking the LT of (1) results in

$$\begin{aligned} T_0^*(\theta) &= \int_{t=0}^\infty e^{-\theta t} \int_{x=0}^\infty \\ &\quad \Pr\{t < T_0(x) < t+dt\} dG(x) \\ &= \int_0^\infty T_0^*(\theta, x) dG(x) \end{aligned} \tag{5}$$

Substituting (4) into (5), we have

$$\begin{aligned} T_0^*(\theta) &= \int_0^\infty \frac{\lambda}{\lambda+\theta}\left[1-e^{-(\lambda+\theta)x}\left(1-T_0^*(\theta)\right)\right]dG(x) \\ &= \frac{\lambda}{\lambda+\theta}\left[1-S^*(\theta+\lambda)\left(1-T_0^*(\theta)\right)\right] \end{aligned}$$

which leads to

$$T_0^*(\theta) = \frac{\lambda(1-S^*(\theta+\lambda))}{\theta+\lambda-\lambda S^*(\theta+\lambda)} \tag{6}$$

3. Main results

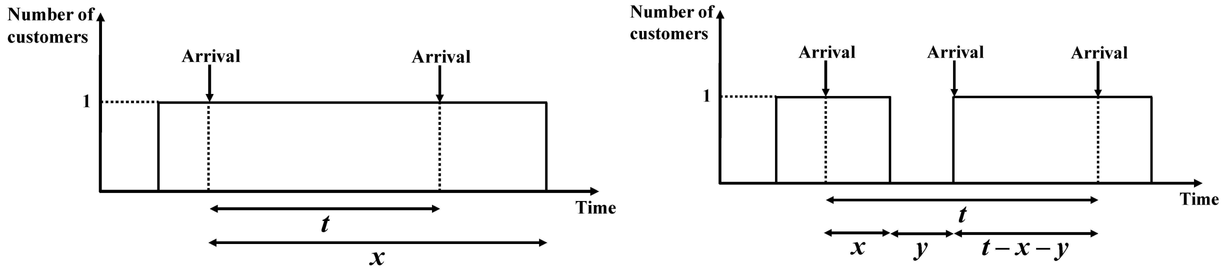
This section deals with the main result, the LT of the inter-loss time in the M/G/1/1 queue. Let S_R denote the remaining service time at the point the current loss occurs. A CDF and its LT is denoted by $F(x)$ and $S_R^*(\theta)$. By Green's theorem[2], we have

$$\begin{aligned} dF(x) &= \frac{\Pr\{S > x\}}{E[S]}dx \\ &\text{and} \\ S_R^*(\theta) &= \int_0^\infty e^{-\theta x} dF(x) = \frac{1-S^*(\theta)}{\theta E[S]} \end{aligned} \tag{7}$$

Remark 2: (7) shows that S_R is stochastically equivalent to the remaining time of the ongoing service at the arbitrary point. For a proof, see Appendix B.

Let $T(x)$ denote the conditional inter-loss time given that the current loss occurs when the remaining service time is x and T denote the inter-loss time in the M/G/1/1 queue. Then, $\Pr\{t < T < t+dt\}$ is given by

$$\begin{aligned} &\Pr\{t < T < t+dt\} \\ &\quad = \int_0^\infty \Pr\{t < T(x) < t+dt\} dF(x) \end{aligned} \tag{8}$$



(a) $t<x$ (b) $t>x$

[Figure 2] Sample Path Examples

Similar to (2) and (3), $\Pr\{t<T(x)<t+dt\}$ in (8) has two different expressions, according to the range of t. In case that $t<x$ (see [Figure 2(a)]), $\Pr\{t<T(x)<t+dt\}$ is expressed as

$$\begin{aligned}&\Pr\{t<T(x)<t+dt\}\\&\quad=\Pr\{t<A<t+dt\}=\lambda e^{-\lambda t}dt\end{aligned} \tag{9}$$

Meanwhile, for $t>x$ (see [Figure 2(b)]), $\Pr\{t<T(x)<t+dt\}$ is represented as

$$\begin{aligned}&\Pr\{t<T(x)<t+dt\}\\&=\int_{y=0}^{t-x}\Pr\{t-x-y<T_0<t-x-y+dt\}\\&\quad\Pr\{x+y<A<x+y+dy\}\\&=e^{-\lambda x}\int_{y=0}^{t-x}\\&\quad\Pr\{t-x-y<T_0<t-x-y+dt\}\lambda e^{-\lambda y}dy\end{aligned} \tag{10}$$

Let $T^*(\theta, x)$ and $T^*(\theta)$ denote the LT of $T(x)$ and the LT of T, respectively. By (9) and (10), $T^*(\theta, x)$ is obtained as follows :

$$\begin{aligned}T^*(\theta, x)&=\int_0^x e^{-\theta t}\Pr\{t<T(x)<t+dt\}\\&\quad+\int_x^\infty e^{-\theta t}\Pr\{t<T(x)<t+dt\}\\&=\frac{\lambda}{\lambda+\theta}\left[1-e^{-(\lambda+\theta)x}\left(1-T_0^*(\theta)\right)\right]\end{aligned} \tag{11}$$

Remark 3: The detailed derivation of (11) is in the <Appendix A>.

Taking the LT of (8), we have

$$\begin{aligned}T^*(\theta)&=\int_{t=0}^\infty e^{-\theta t}\int_{x=0}^\infty\\&\quad\Pr\{t<T(x)<t+dt\}dF(x)\\&=\int_0^\infty T^*(\theta, x)dF(x)\end{aligned} \tag{12}$$

Utilizing (6) and (11) in (12), we have

$$\begin{aligned}T^*(\theta)&=\int_0^\infty\frac{\lambda}{\lambda+\theta}\left[1-e^{-(\lambda+\theta)x}\left(1-T_0^*(\theta)\right)\right]dF(x)\\&=\frac{\lambda}{\lambda+\theta}\left[1-\left(1-T_0^*(\theta)\right)\int_0^\infty e^{-(\lambda+\theta)x}dF(x)\right]\\&=\frac{\lambda}{\lambda+\theta}\left[1-S_R^*(\theta+\lambda)\left\{1-\frac{\lambda\left(1-S^*(\theta+\lambda)\right)}{\theta+\lambda-\lambda S^*(\theta+\lambda)}\right\}\right]\\&=\frac{\lambda}{\lambda+\theta}\left[1-\frac{\theta S_R^*(\theta+\lambda)}{\theta+\lambda-\lambda S^*(\theta+\lambda)}\right]\end{aligned} \tag{13}$$

which leads to $E[T]=\lambda^{-1}(1+\rho)/\rho$, where $\rho=\lambda E[S]$. Note that $1/E[T]$ represents the mean number of lost customers during a unit time, i.e. the loss rate of the M/G/1/1 queueing system. Then the fraction of customers that arrive to find the server busy, i.e., the loss probability of the M/G/1/1 queueing system, is given by

$$\Pr\{\text{an arriving customer is lost}\} = \frac{1}{\lambda E[T]} \quad (14)$$

It is interesting to notice that this, in (14), the probability does not depend on the type of the service time distribution and it depends only on its mean; that is, the Erlang B-formula holds not only for the exponential service time distribution but also for the arbitrary service time distribution in case that the number of the buffer is set to 1.

References

[1] Ferrante, P., "Interloss time in M/M/1/1 loss system," *Journal of Applied Mathematics and Stochastic Analysis*, Vol.2009(2009), article ID 308025.

[2] Green, L., "A limit theorem on subintervals of interrenewal times," *Operations Research*, Vol.30, No.1(1982), pp.210-216.

<Appendix A> The Derivation of Both (4) and (11)

We first derive (4) in detail as follows:

$$\begin{aligned}
T_0^*(\theta, x) &= \int_{t=0}^{x} e^{-\theta t} \Pr\{t < T_0(x) < t+dt\} + \int_{t=x}^{\infty} e^{-\theta t} \Pr\{t < T_0(x) < t+dt\} \\
&= \int_{t=0}^{x} e^{-\theta t} \lambda e^{-\lambda t} dt + \int_{t=x}^{\infty} e^{-\theta t} e^{-\lambda x} \int_{y=0}^{t-x} \Pr\{t-x-y < T_0 < t-x-y+dt\} \lambda e^{-\lambda y}\, dy \\
&= \frac{\lambda}{\lambda+\theta}[1-e^{-(\lambda+\theta)x}] + e^{-\lambda x} \int_{y=0}^{\infty} \lambda e^{-\lambda y} \int_{t=x+y}^{\infty} e^{-\theta t} \Pr\{t-x-y < T_0 < t-x-y+dt\} dy \\
&= \frac{\lambda}{\lambda+\theta}[1-e^{-(\lambda+\theta)x}] + e^{-\lambda x} \int_{y=0}^{\infty} \lambda e^{-\lambda y} \int_{u=0}^{\infty} e^{-\theta(u+x+y)} \Pr\{u < T_0 < u+du\} dy \\
&= \frac{\lambda}{\lambda+\theta}[1-e^{-(\lambda+\theta)x}] + e^{-\lambda x} \int_{y=0}^{\infty} \lambda e^{-\lambda y} e^{-\theta(x+y)} \int_{u=0}^{\infty} e^{-\theta u} \Pr\{u < T_0 < u+du\} dy \\
&= \frac{\lambda}{\lambda+\theta}[1-e^{-(\lambda+\theta)x}] + e^{-(\lambda+\theta)x} \int_{y=0}^{\infty} \lambda e^{-(\lambda+\theta)y}\, T_0^*(\theta) dy \\
&= \frac{\lambda}{\lambda+\theta}[1-e^{-(\lambda+\theta)x}] + \frac{\lambda e^{-(\lambda+\theta)x}}{\lambda+\theta} T_0^*(\theta) \\
&= \frac{\lambda}{\lambda+\theta}\left[1-e^{-(\lambda+\theta)x}\left(1-T_0^*(\theta)\right)\right]
\end{aligned} \tag{A.1}$$

In (A.1), replacing $T_0^*(\theta, x)$ with $T^*(\theta, x)$ and $T_0(x)$ with $T(x)$, we can derive (11).

<Appendix B> The proof of Remark 2

Let $U(t)$ denote the remaining service time at time t. Let us define the following probabilities: $p(0, t) = \Pr\{U(t) = 0\}$ and $p(x, t)dx = \Pr\{x < U(t) < x+dx\}$.We define $p'(x, t)dx$ and $q(x, t)dx$ as the conditional probability of the form: $p'(x, t)dx = \Pr\{x < U(t) < x+dx \mid U(t) > 0\}$ and $q(x, t)dx = \Pr\{x < U(t) < x+dx \mid L(t, t+dt) = 1\}$, where $L(t, t+dt)$ is the number of the losses which occur during the interval $(t, t+dt)$. If it is satisfied that $q(x, t) = p'(x, t)$, we prove Remark 2. Let $A(t, t+dt)$ be the number of arrivals during the interval $(t, t+dt)$. We have

$$\begin{aligned}
q(x, t)dx &= \Pr\{x < U(t) < x+dx \mid L(t, t+dt) = 1\} \\
&= \Pr\{x < U(t) < x+dx \mid U(t) > 0,\ A(t, t+dt) = 1\} \\
&= \frac{\Pr\{x < U(t) < x+dx,\ U(t) > 0,\ A(t, t+dt) = 1\}}{\Pr\{U(t) > 0,\ A(t, t+dt) = 1\}} \\
&= \frac{\Pr\{U(t) > 0\}\Pr\{x < U(t) < x+dx,\ A(t, t+dt) = 1 \mid U(t) > 0\}}{\Pr\{U(t) > 0\}\Pr\{A(t,t+dt) = 1 \mid U(t) > 0\}} \\
&= \frac{\Pr\{x < U(t) < x+dx,\ A(t,t+dt) = 1 \mid U(t) > 0\}}{\Pr\{A(t, t+dt) = 1 \mid U(t) > 0\}} \\
&= \frac{\Pr\{x < U(t) < x+dx \mid U(t) > 0\}\Pr\{A(t, t+dt) = 1 \mid x < U(t) < x+dx\}}{\Pr\{A(t, t+dt) = 1 \mid U(t) > 0\}}
\end{aligned} \tag{B.1}$$

Since $U(t)$ is determined by both the past arrivals and services which occur during the interval $(0, t)$, $A(t, t+dt)$ and $U(t)$ are mutually independent. In other words, the following relation holds:

$$\Pr\{A(t, t+dt)=1 | x < U(t) < x+dx\} = \Pr\{A(t, t+dt)=1 | U(t)>0\} = \Pr\{A(t, t+dt)=1\}.$$

Therefore, (B.1) is rewritten as $q(x, t)dx = \Pr\{x < U(t) < x+dx | U(t)>0\} = p'(x, t)dx$.