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A BOUND FOR THE MILNOR SUM OF PROJECTIVE PLANE CURVES IN TERMS OF GIT

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Abstract. Let C be a projective plane curve of degree d whose singularities are all isolated. Suppose C is not concurrent lines. Płoski proved that the Milnor number of an isolated singlar point of C is less than or equal to $(d-1)^2 - \lfloor \frac{d}{2} \rfloor$. In this paper, we prove that the Milnor sum of C is also less than or equal to $(d-1)^2 - \lfloor \frac{d}{2} \rfloor$ and the equality holds if and only if C is a Płoski curve. Furthermore, we find a bound for the Milnor sum of projective plane curves in terms of GIT.

1. Introduction

Let $C = V(f)$ be a projective plane curve of degree d. In this paper, a plane curve C means a projective plane curve that has at most isolated singularities. Moreover, we assume that C is not concurrent lines. We assume that the base field k is algebraically closed and char(k)=0. Let $f = 0$ at [0,0,1]. Then, we define its Milnor number at 0 (in the sense of affine chart) by

$$\mu_0(f) = \dim_k(O_0/J_f),$$

where O_0 is a function germ of f at the origin and $J_f = (\partial f/\partial x, \partial f/\partial y)$ is the Jacobian ideal of f. Since $\mu_0(f)$ is finite if and only if the origin is an isolated singular point, the Milnor number is closely related to the local properties of isolated singular points. In fact, the Milnor number has an important topological meaning.

Proposition 1.1 ([4, Lemma 2.10]). *The Milnor number is a topological invariant for IHS* (*isolated hypersurface singularities*)*.*

By the importance of the Milnor number for IHS, there are some critical results. One of them was proven by Płoski which says that for a projective plane curve C of degree d whose singularities are all isolated, not concurrent lines, the Milnor number of an isolated singlar point of C is less than or equal to $(d-1)^2 - \lfloor \frac{d}{2} \rfloor$ with equality holds if and only if C is a Płoski curve (see [1,

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Definition 1.9, 1.10]). By this result, for any given point of a projective plane curve which is not concurrent lines, we get an upper bound for the Milnor number which is useful for computing the Milnor number of a given point. Also, one of the others was that of Huh which gives an upper bound for the Milnor sum of projective hypersurfaces which are not the cone over a smooth hypersurface (see [5, Theorem 1.1]). However, since the result of Huh applies to general cases, we can expect that a bound for the Milnor sum of projective plane curves can be reduced. So the purpose of this paper is to find an upper bound for the Milnor sum of a projective plane curve and to see how such a bound can be reduced by GIT conditions. In fact, without GIT conditions, we can get the following theorem which is one of our main results:

Theorem 1.2. *Let C be a plane curve whose singularities are all isolated and $\deg C = d \geq 5$. Then $\mathrm{pd}(C) = \lfloor \frac{d}{2} \rfloor$ if and only if C is a Płoski curve.*

Recall that the gradient map of $C = V(h)$, $grad(h) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, $[x, y, z] \mapsto [\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}]$, is a map obtained from the partial derivatives of h. Define the polar degree of a plane curve $C = V(h)$, denoted by $\mathrm{pd}(C)$, is the degree of a gradient map of h. There is a lemma that connects $\mathrm{pd}(C)$ with Milnor sum, which we call it Milnor formula. Before stating it, we give a simple example of polar degree.

Example 1.1. Let $C = V(h)$ be a cuspidal cubic, where $h = y^3 - x^2 z$. Then, $grad(h) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $[x, y, z] \mapsto [-2xz, 3y^2, -x^2]$ is the gradient map of h. Since $\mathbb{A}^2$ is birational to $\mathbb{P}^2$, by considering it in affine chart, we get that the field extension corresponding to $grad(h)$ is $k(x, y^2) \hookrightarrow k(x, y)$, which is of degree 2. So $\mathrm{pd}(C) = 2$.

Lemma 1.3 ([3, Proposition 2.3], Milnor formula)**.** *Let $C = V(h) \subset \mathbb{P}^n$ be a hypersurface with isolated singularities with $\deg(C) = d$. Then*

$$\mathrm{pd}(C) = (d-1)^n - \sum \mu_p(h),$$

where $\mu_p(h)$ is the Milnor number of h at p.

By the above Lemma 1.3 and Proposition 3.3, the Milnor sum of a plane curve is bounded above by $(d-1)^2 - \lfloor \frac{d}{2} \rfloor$ unless it is concurrent lines. Therefore, as in the case of the Milnor number of a plane curve, the Milnor sum of a plane curve also has the same bound and the equality holds only when the curve is exactly the same case as in [6, Theorem 1.4].

Finally, by using Hilbert-Mumford criterion (Theorem 2.1), we prove that even Płoski curves are strictly semi-stable and odd Płoski curves are unstable (see Proposition 3.8). By the previous theorem, we expect that the polar degree can be reduced by GIT conditions. Since there are many irreducible, stable plane curves of degree d with polar degree $d-1$, a bound for the Milnor sum should be less than or equal to $(d-1)^2 - (d-1)$. However, the following theorem which is one of our main results says that for some cases, this bound is very close.

Theorem 1.4. *Let* $\deg C = d \geq 5$. *Then we have the followings*:

1) *Suppose* C *is a stable curve that has either a line or a conic as an irreducible component. Then* $\sum \mu_p \leq (d-1)^2 - (d-2)$.
2) *Let* d *be odd. Suppose* C *is a semi-stable curve that has either a line or a conic as an irreducible component. Then* $\sum \mu_p \leq (d-1)^2 - (d-2)$.
3) *Suppose all irreducible components of* C *are of* $\deg \geq 3$. *Then* $\sum \mu_p \leq (d-1)^2 - \lceil \frac{2d}{3} \rceil$.

In Section 2, we recall Hilbert-Mumford criterion (see Theorem 2.1) and its application to projective plane curves. Moreover, some definitions and well-known results are mentioned. Finally, in the last section, we will prove main theorems of this paper.

2. GIT criterion and polar degree of plane curves

The purpose of this section is to introduce some preliminaries that are useful to prove the main theorem. From now on, we denote the polar degree of a plane curve C by $\mathrm{pd}(C)$. First, recall that the definition of semi-stability and stability in [2, Chapter 8]. Let $T = G_m^r$ be a torus and let V be a vector space. Then, a linear representation of T splits V into the direct sum of eigenspaces $V = \oplus_{\chi \in \chi(T)} V_\chi$, where $\chi(T)$ is a set of rational characters of T and $V_\chi = \{v \in V : t \cdot v = \chi(t) \cdot v\}$. Since there is a natural identification between $\chi(T)$ and $\mathbb{Z}^r$ of abelian groups, by identifying them, we define the weight set of V by $wt(V) = \{\chi \in \chi(T) : V_\chi \neq \{0\}\} \subset \mathbb{Z}^r$. In particular, let $\overline{wt(V)}$ = convex hull of $wt(V)$ in $\chi(T) \otimes \mathbb{R} \cong \mathbb{R}^r$ (see [2, Chapter 9]).

Theorem 2.1 ([2, Theorem 9.2], Hilbert-Mumford criterion)**.** *Let* G *be a torus and let* L *be an ample* G*-linearlized line bundle on a projective* G*-variety* X. *Then*

1) x *is semi-stable if and only if* $0 \in \overline{wt(x)}$.
2) x *is stable if and only if* $0 \in interior(\overline{wt(x)})$.

Also, we can check immediately that a given projective plane curve of degree d is unstable by using the following proposition.

Proposition 2.2 ([2, Chapter 10])**.** *A projective plane curve of degree* d *is unstable if it has a singular point of multiplicity* $> \frac{2d}{3}$.

Now, we recall definitions of Płoski curve.

Definition 2.1 ([1, Definition 1.9])**.** The curve C is called an even Płoski curve if $\deg C = 2n$, it has n irreducible components that are smooth conics passing through P, and all irreducible components intersect each other pairwise at P with multiplicity 4.

Definition 2.2 ([1, Definition 1.10])**.** The curve C is called an odd Płoski curve if $\deg C = 2n+1$, it has n irreducible components that are smooth conics

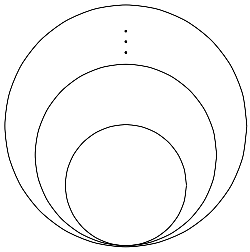

FIGURE 1. An even Płoski curve

passing through P and intersect each other pairwise at P with multiplicity 4, and the remaining irreducible component is a line that is tangent at P to all other irreducible components.

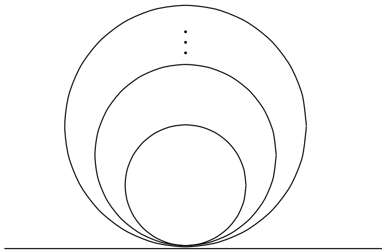

FIGURE 2. An odd Płoski curve

It is hard to compute the Milnor sum of a given projective plane curves directly. However, polar degree is a global one, so we can compute that more easily. So we will find a lower bound for the polar degree and use Lemma 1.3 in order to get an upper bound for the Milnor sum of plane curves. So the problem of computing the Milnor sum of plane curves can be reduced to that of computing the polar degree. However, we can easily get the polar degree of a plane curve by the following two lemmas.

Lemma 2.3 ([3, Theorem 3.1]). *Given an irreducible curve $C \subset \mathbb{P}^2$ of degree d, we have*

$$\mathrm{pd}(C) = d - 1 + 2p_g + \sum (r_p - 1),$$

where p_g is the geometric genus and r_p is the number of branches at p.

Lemma 2.4 ([3, Theorem 3.1]). *Given two reduced curves C, D in $\mathbb{P}^2$ with no common components, we have*

$$\mathrm{pd}(C \cup D) = \mathrm{pd}(C) + \mathrm{pd}(D) + \sharp(C \cap D) - 1.$$

The following lemma is the result of Płoski (see [6, Theorem 1.4]) that makes a Płoski curve important.

Lemma 2.5 ([6, Theorem 1.4]). *If $C = V(h)$ is a plane curve of degree $d \geq 5$, then $\mu_p(h) = (d-1)^2 - \lfloor \frac{d}{2} \rfloor$ if and only if C is a Płoski curve and p is a singular point.*

In order to check the semi-stability of a given plane curve, we need to consider the weight set of that one. The following remark gives a way to compute the weight set for plane curves.

Remark 2.6 ([2, Chapter 10], wt for plane curves). Let $Pol_d(E)$ be the space of degree d homogeneous polynomial on E, where E is a finite dimensional vector space. Let the standard torus G_m^2 act on $V = Pol_d(k^3)$ via its natural homomorphism $G_m^2 \to SL_3, (t_1, t_2) \mapsto (a_{ij})_{1 \leq i,j \leq 3}$, where $a_{11} = t_1$, $a_{22} = t_2$, $a_{33} = t_1^{-1} t_2^{-1}$, $a_{ij} = 0$ for all $i \neq j$, i.e., $(t_1, t_2) \cdot x^i y^j z^k = t_1^{i-k} t_2^{j-k} x^i y^j z^k$, $i + j + k = d$. Let $V(h)$ be a plane curve of degree d, i.e., $i + j + k = d$, i.e., $(i - k, j - k) = (2i + j - d, 2j + i - d)$. So $wt = \{(2i + j - d, 2j + i - d) \in \mathbb{Z}^2 : i, j \geq 0, i + j \leq d, a_{ijk} \neq 0\}$. Moreover, by considering $\mathbb{R} \otimes \mathbb{Z}^2$, define $\bar{wt}$ by the closure of wt in $\mathbb{R}^2$.

3. Main result

Now, we are ready to prove our main theorems of this paper. For notational convenience, let r_p be the number of branches at p as in Lemma 2.4.

Lemma 3.1. *Płoski curves are of polar degree $\lfloor \frac{d}{2} \rfloor$, where* $\deg = d$.

Proof. First, we consider an even Płoski curve, i.e., $d = 2n$. Let $C = C_1 \cdots C_n$ be an even Płoski curve, where C_i's are conics. Then $\mathrm{pd}(C) = \mathrm{pd}(C_1) + \cdots + \mathrm{pd}(C_n) + \sharp(C_1 \cap C_2) + \cdots + \sharp((C_1 \cdots C_{n-1} \cap C_n) - (n-1) = n$. Next, we consider an odd Płoski curve, i.e., $d = 2n + 1$. Let $C = lC_1 \cdots C_n$, where l is a tangent line, C_i's are conics. Then $\mathrm{pd}(C) = \mathrm{pd}(l) + \mathrm{pd}(C_1 \cdots C_n) + \sharp(l \cap C_1 \cdots C_n) - 1 = n$. □

Lemma 3.2. *Let $C = C_1 \cdots C_m C_{m+1} \cdots C_k$ be a plane curve of degree $2n$ (respectively, $2n + 1$) with $m \geq 1$, $k > n$ (respectively, $k > n + 1$), where $C_1, \ldots, C_m$ are irreducible, singular plane curves and $C_{m+1} \cdots C_k$ is concurrent lines. Then* $\mathrm{pd}(C) \geq n$.

Proof. Let $l_i = \deg C_i$. Clearly, $2n = \deg C = \deg(C_1 \cdots C_m) + \deg(C_{m+1} \cdots C_k) \geq 2m + k > 2m + n$, i.e., $m < \frac{n}{2}$. Then $\mathrm{pd}(C) = \mathrm{pd}(C_1) + \cdots + \mathrm{pd}(C_m) + \sharp(C_1 \cdots C_m \cap C_{m+1} \cdots C_k) - 1 \geq \sum_{i=1}^m (l_i - 1) + (\sum_p (r_p - 1) + \sharp(C_1 \cdots C_m \cap C_{m+1} \cdots C_k)) - 1 \geq ((2n - k + m) - m) + (k - m) - 1 = 2n - m - 1 > \frac{3n}{2} - 1 \geq n - 1$, i.e., $\mathrm{pd}(C) \geq n$. By the same argument, we can get the result when $\deg C = 2n + 1$ with $k > n + 1$. □

Proposition 3.3. *Let C be a plane curve of* $\deg C = d$. *Then* $\mathrm{pd}(C) \geq \lfloor \frac{d}{2} \rfloor$ *unless C is concurrent lines.*

Proof. First, we consider the case when $\deg C = 2n$. If C is irreducible, it is clear by Lemma 2.3. So let $C = C_1 \cdots C_k$, where C_i's are irreducible plane curves and $\deg C_i = l_i$. Then, $\mathrm{pd}(C) \geq \sum_{i=1}^{k}(l_i - 1) = 2n - k$. So if $k \leq n$, then $\mathrm{pd}(C) \geq n$. So let $k > n$. Then, there exists at least 2 components which are lines. So we use induction on n. For small n, we know that the result is true (see [3, Theorems 3.3 and 3.4]). So suppose it holds for $n-1$. Let $C = C_1 \cdots C_{k-2}C_{k-1}C_k$, where C_{k-1}, C_k are lines. Then, $\mathrm{pd}(C) = \mathrm{pd}(C_1 \cdots C_{k-2}) + \mathrm{pd}(C_{k-1}C_k) + \sharp(C_1 \cdots C_{k-2} \cap C_{k-1}C_k) - 1 \geq (n-1) + \sharp(C_1 \cdots C_{k-2} \cap C_{k-1}C_k) - 1 = n - 2 + \sharp(C_1 \cdots C_{k-2} \cap C_{k-1}C_k)$ by induction hypothesis. It is enough to consider the case when $\sharp(C_1 \cdots C_{k-2} \cap C_{k-1}C_k) = 1$. However, by Bézout's Theorem, it can happen only for the following two cases: first case is when all smooth components are lines that intersect at one point, and singular, irreducible components exist, and the second case is when C is concurrent lines. However, by Lemma 3.2, for case 1, $\mathrm{pd}(C) \geq n$. Therefore, $\mathrm{pd}(C) \geq n$ unless C is concurrent lines. For $d = 2n+1$, we can use the same argument to get the result. □

Corollary 3.4. *Let $C = V(h)$ be a plane curve of degree d in $\mathbb{P}^2$ whose singularities are all isolated. Then, $\sum_p \mu_p(h) \leq (d-1)^2 - \lfloor \frac{d}{2} \rfloor$ unless C is concurrent lines.*

Proof. By Proposition 3.3 and Lemma 1.3, $\sum_p \mu_p(h) = (d-1)^2 - \mathrm{pd}(C) \leq (d-1)^2 - \lfloor \frac{d}{2} \rfloor$. □

Since the Milnor number is nonnegative, we get the following corollary. (For another proof, see [6, Theorem 1.1].)

Corollary 3.5. *Let $C = V(h)$ be a plane curve of degree d in $\mathbb{P}^2$ whose singularities are all isolated. Then, for any singular points p, $\mu_p(h) \leq (d-1)^2 - \lfloor \frac{d}{2} \rfloor$ unless C is concurrent lines.*

Lemma 3.6. *Let C be a plane curve whose singularities are all isolated and* $\deg C = d \geq 5$. *Suppose that all irreducible components of C are smooth. Then C has only one isolated singular point if* $\mathrm{pd}(C) = \lfloor \frac{d}{2} \rfloor$.

Proof. Let $C = C_1 \cdots C_k$ be a given curve, where C_i's are irreducible components of C with $\deg C_i = l_i$. Suppose that C has at least two isolated singular points with $\mathrm{pd}(C) = \lfloor \frac{d}{2} \rfloor$. First, let $d = 2n$. In this case, $n = \mathrm{pd}(C) = \mathrm{pd}(C_1) + \cdots + \mathrm{pd}(C_k) + (\sharp(C_1 \cap C_2) + \cdots + \sharp(C_1 \cdots C_{k-1} \cap C_k)) - (k-1) \geq \sum_{i=1}^{k}(l_i - 1) - (k-1) + (*) = (2n - 2k + 1) + (*)$, where $(*) = \sharp(C_1 \cap C_2) + \cdots + \sharp(C_1 \cdots C_{k-1} \cap C_k)$, i.e., $n \geq (2n - 2k + 1) + (*)$. Since C has at least 2 isolated singularities and all $C_i's$ are smooth, some $\sharp$ in $(*)$ should be bigger than or equal to 2, i.e., $(*) \geq k$. So $n \geq (2n - 2k + 1) + (*) \geq 2n - k + 1$, i.e., $k \geq n + 1$. It means that C has at least two lines as its irreducible components. Let $C = C_1C_2C_3 \cdots C_k$, where C_1, C_2 are lines. Now, we consider $(*)$ again. Also, by reordering, if necessary, we can let m to be the maximal number such that $C_1, \ldots C_m$ are lines and intersect at one point. If $m = 2$, since $\sharp(C_1 \cap C_2) = 1$ and $\sharp(C_1C_2 \cap C_3) \geq 2, \cdots \sharp(C_1 \cdots C_{k-1} \cap C_k) \geq 2$, then $(*) \geq 2k - 3$. So $n \geq 2n - 2$, i.e., $n \leq 2$, which is a contradiction because $d \geq 5$. So $m > 2$. Then, $n = \mathrm{pd}(C_1 \cdots C_m) + \mathrm{pd}(C_{m+1} \cdots C_k) + \sharp(C_1 \cdots C_m \cap C_{m+1} \cdots C_k) - 1$, i.e., $\mathrm{pd}(C_{m+1} \cdots C_k) = (n + 1) - \sharp(C_1 \cdots C_m \cap C_{m+1} \cdots C_k)$. Since $\sharp(C_1 \cdots C_m \cap C_{m+1} \cdots C_k) \geq m$ (by using the fact that all C_i's are smooth and by Bézout's Theorem) and $\mathrm{pd}(C_{m+1} \cdots C_k) \geq \lfloor \frac{2n-m}{2} \rfloor$, we get $\lfloor \frac{2n-m}{2} \rfloor \leq \mathrm{pd}(C_{m+1} \cdots C_k) \leq n - m + 1$. If $m = 2s$, then $n - s \leq n - 2s + 1$, i.e., $s \leq 1$, which is a contradiction because $m > 2$. If $m = 2s + 1$, then $n - s - 1 \leq n - 2s$, i.e., $s \leq 1$. Since $m > 2$, we only need to check when $m = 3$. If $m = 3$, $n \geq (2n - 2k + 1) + (*) \geq 2n - 3$, i.e., $n \leq 3$. However, it does not happen when $d = 6$ by [3, Theorems 3.3, 3.4]. Therefore, we prove it for $d = 2n$. So we need to consider when $d = 2n + 1$. However, by the same argument, we can prove it. □

Theorem 3.7. *Let C be a plane curve whose singularities are all isolated and $\deg C = d \geq 5$. Then $\mathrm{pd}(C) = \lfloor \frac{d}{2} \rfloor$ if and only if C is a Płoski curve.*

Proof. We already proved the reverse direction, so we need to prove the remaining one. Let $C = C_1 \cdots C_k$ of degree d, where C_i's are irreducible plane curves of $\deg C_i = l_i$. Now, we consider the following 2 cases:

Case 1) First, suppose that all irreducible components of C are smooth, i.e., C_i's are all smooth. By Lemma 2.5, it suffices to show that if $\mathrm{pd}(C) = \lfloor \frac{d}{2} \rfloor$, then C has only one isolated singular point. However, by Lemma 3.6, we are done in the first case.

Case 2) Suppose that C has singular irreducible components. So let $C = C_1 \cdots C_m C_{m+1} \cdots C_k$, where $C_1, \ldots, C_m$ are singular and $C_{m+1}, \ldots, C_k$ are smooth of $\deg C_i = l_i$ and $m \geq 1$. First, let $d = 2n$. In this case, $n = \mathrm{pd}(C) \geq \mathrm{pd}(C_1) + \cdots + \mathrm{pd}(C_k) \geq \sum_{i=1}^{k}(l_i - 1) = 2n - k$, i.e., $k \geq n$. If $k > n$, then there exists at least 2 irreducible components of C which are lines. Since they are smooth, we assume that $C = (C_1 \cdots C_m)(C_{m+1}C_{m+2} \cdots C_k)$, where C_{m+1}, C_{m+2} are lines. Let $\deg(C_1 \cdots C_m) = l$, $\deg(C_{m+1} \cdots C_k) = 2n - l$. Since $C_{m+1} \cdots C_k$ is not a Płoski curve, by Case 1), $\mathrm{pd}(C_{m+1} \cdots C_k) > \lfloor \frac{2n-l}{2} \rfloor$. Then, if $l = 2s$, $n = \mathrm{pd}(C) \geq (\mathrm{pd}(C_1) + \cdots \mathrm{pd}(C_m)) + \mathrm{pd}(C_{m+1} \cdots C_k) > \sum_{i=1}^{m}(l_i - 1) + n - s = s + n - m$, i.e., $m > s$. However, $2n = \deg(C_1 \cdots C_m) + \deg(C_{m+1} \cdots C_k) \geq 3m + 2n - l > 2n + s$, which is a contradiction. So let $l = 2s + 1$. Also, $n = \mathrm{pd}(C) > l - m + n - s - 1$, i.e., $m > s$. Then $2n = \deg(C_1 \cdots C_m) + \deg(C_{m+1} \cdots C_k) \geq 3m + 2n - l > 3s + 2n - 2s - 1 = 2n + (s - 1) \geq 2n$, which is a contradiction. So when $k > n$, $\mathrm{pd}(C) \neq n$. Finally, it remains to prove when $k = n$. Let $k = n$. Then C has at least one line component. If there exists more than two line components in C, we can use the same argument so that we get a contradiction. So we only need to consider when C has only one line component. It is clear that C must be of the form $C = C_1C_2 \cdots C_n$, where C_1 is of degree 3, C_2 is a line, and all C_i, $i \geq 3$, are smooth conics. For convenience, let $F = C_2C_3 \cdots C_n$. Then

$n = \mathrm{pd}(C) = \mathrm{pd}(C_1) + \mathrm{pd}(F) + \sharp(C_1 \cap F) - 1$. Since C_i's, $i \geq 2$, are all smooth, we consider the following 2 cases:

Case 2-1) First, let F be a Płoski curve. Since irreducible singular plane curves of degree 3 are either cusps or nodal curves, we need to consider two cases. First, let C be a cusp. If $k = 3$, i.e., $\deg C = 6$, by [3, Theorems 3.3, 3.4], $\mathrm{pd}(C) > 3$. For $k \geq 4$, we can easily get that $\sharp(C_1 \cap F) \geq 2$. So $n = \mathrm{pd}(C_1) + \mathrm{pd}(F) + \sharp(C_1 \cap F) - 1 \geq 2 + (n-2) + 2 - 1 = n + 1$, which is a contradiction. So we need to consider when C is a nodal curve. Since $\mathrm{pd}(C_1) \geq 4$ [3, Theorem 3.4], $n = \mathrm{pd}(C_1) + \mathrm{pd}(F) + \sharp(C_1 \cap F) - 1 \geq 4 + (n-2) + 1 - 1$ $\geq n + 2$, which is a contradiction.

Case 2-2) Next, let F be not a Płoski curve. Then $n = \mathrm{pd}(C_1) + \mathrm{pd}(F) + \sharp(C_1 \cap F) - 1 > 2 + (n-2) + 1 - 1 = n$, which is a contradiction.

For $d = 2n+1$, we can use the same argument to get the result. Therefore, if C contains singular irreducible components, $\mathrm{pd}(C) \neq n$.

So by Case 1), 2), if $\mathrm{pd}(C) = n$ and $\deg C \geq 5$, then C is a Płoski curve. □

By Hilbert-Mumford criterion, we can check the semi-stability of Płoski curves (see [1, Example 1.18]).

Proposition 3.8. *An even Płoski curve is strictly semi-stable, and an odd Płoski curve is unstable.*

Proof. Let C be an even Płoski curve. By changing projective coordinate, if necessary, we may assume that $C = (x^2 - yz + z^2)(x^2 - yz + 2z^2) \cdots (x^2 - yz + nz^2)$. Then, any variable that has nonzero coefficient is of the form $x^{2a}(yz)^b z^{2(n-a-b)} = x^{2a} y^b z^{2n-2a-b}$, where $0 \leq a, b \leq n$, $a + b \leq n$. So $wt = \{(4a + b - 2n, 2b + 2a - 2n) \in \mathbb{Z}^2 : 0 \leq a, b \leq n, a + b \leq n\}$. Since $2b + 2a - 2n \leq 0$, $\bar{wt}$ lies in lower half-space of $\mathbb{R}^2$. Also, since $(2n, 0)$, $(-n, 0)$, $(-2n, -2n) \in wt$, $(0,0) \in \bar{wt}$, but $(0,0) \notin$ interior of $\bar{wt}$. Therefore, an even Płoski curve is strictly semi-stable.

Also, by changing projective coordinate, if necessary, we may assume that an odd Płoski curve is of the form $C = z(x^2 - yz + z^2)(x^2 - yz + 2z^2) \cdots (x^2 - yz + nz^2)$. So by the similar argument, we can get $(0,0) \notin \bar{wt}$. Therefore, an odd Płoski curve is unstable. □

So we can summarize what we get.

Theorem 3.9. *Let C be a plane curve of degree $d \geq 5$ in $\mathbb{P}^2$ whose singularities are all isolated. Suppose C is not concurrent lines. Then we have the followings*:

1) *When $d = 2n$, $\sum \mu_p \leq (d-1)^2 - \lfloor \frac{d}{2} \rfloor$ with equality if and only if C is an even Płoski curve.*
   For semi-stable curves, $\sum \mu_p \leq (d-1)^2 - \lfloor \frac{d}{2} \rfloor$ with equality if and only if C is an even Płoski curve.
   For stable curves, $\sum \mu_p \leq (d-1)^2 - \lfloor \frac{d}{2} \rfloor - 1$.
2) *When $d = 2n+1$, $\sum \mu_p \leq (d-1)^2 - \lfloor \frac{d}{2} \rfloor$ with equality if and only if C is an odd Płoski curve.*

For semi-stable curves, $\sum \mu_p \leq (d-1)^2 - \lfloor \frac{d}{2} \rfloor - 1$.
For stable curves, $\sum \mu_p \leq (d-1)^2 - \lfloor \frac{d}{2} \rfloor - 1$.

From now on, we find a least upper bound for the Milnor sum of plane curves and that of semi-stable plane curves of even degree. So the remaining part is to lessen an upper bound for the Milnor sum of stable curves of even degree and that of (semi)-stable curves of odd degree. In order to do this, we need the following lemmas.

Lemma 3.10. *Let C be a plane curve of degree $2n$ whose all irreducible components are conics. If $\mathrm{pd}(C) \leq 2n-1$, then C is either a Płoski curve, $(*)$, or $(\star)$, where $(*)$, $(\star)$ are conics that intersect only at two points as the following figures show.*

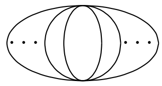

FIGURE 3. $(*)$

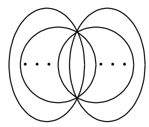

FIGURE 4. $(\star)$

Proof. For convenience, we denote the curve in Figure 3 and the curve in Figure 4 by $(*)$, $(\star)$, respectively. Let $\mathrm{pd}(C) \leq 2n-1$ and let C be not a Płoski curve. Then, we need to show that C is either $(*)$ or $(\star)$. For this, we need to show that there exists no such a form in Figure 5, where this is 3-conics that have common tangents with two intersection points. Suppose there exists such a curve. For

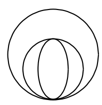

FIGURE 5. impossible conics

convenience, denote (1), (2), (3) from inside to outside conics. Since (2) $\cup$ (3) is a Płoski curve, we may let (2) to be $x^2 - yz$, (3) to be $x^2 - yz + z^2$. Clearly, (2) $\cap$ (3) $= \{[0,1,0]\}$ and their common tangent line at $[0,1,0]$ is $-z$. Since (1) is a conic, let (1) be $ax^2 + by^2 + cz^2 + dxy + eyz + fzx$. Since (1) passes $[0,1,0]$, $b = 0$. Now, we consider the following 2 cases:

Case 1) First, let $a \neq 0$. We may let (1) to be $x^2 + \alpha z^2 + \beta xy + \gamma yz + \delta xz$. Consider (2) $\cup$ (3) $= (x^2 - yz)(x^2 - yz + z^2)$. Since the tangent line of (1) at $[0,1,0]$ is $\beta x + \gamma z$ and it has the same tangent with (2), (3), $\beta x + \gamma z = -z$, i.e., $\beta = 0$, $\gamma = -1$, i.e., (1): $x^2 + \alpha z^2 - yz + \delta xz$. Since there exists another point in (1) $\cap$ (2) except $[0,1,0]$, we consider (1) $\cap$ (2), i.e., (1) $-$ (2) $= z(\alpha z + \delta x)$. If $z = 0$, then it is $[0,1,0]$. So let $\alpha z + \delta x = 0$. If $\alpha = 0$, then $\delta \neq 0$, i.e., (1): $x^2 - yz + \delta xz$. However, it is easy that there exists $p \neq [0,1,0]$ such that

$p \in (1) \cap (3)$ in this case, which is a contradiction. So let $\alpha \neq 0$. Also, in this case $\delta \neq 0$. (This can be obtained by the following argument: If $\delta = 0$, (1): $x^2 + \alpha z^2 - yz$, so $(1) \cap (2) = \{[0, 1, 0]\}$, which is a contradiction.) Also, by some calculation, $[\frac{-(\alpha-1)}{\delta} z, \frac{(\alpha-1)^2}{\delta^2} z + z, z]$ is another common root of (1), (3). We get a contradiction again. So there exists no such a curve when $a \neq 0$.

Case 2) Now, let $a = 0$. We may let (1) to be $cz^2 + dxy + eyz + fzx$. Since the tangent of (1) at $[0, 1, 0]$ is $dx + ez$, $dx + ez = -z$, i.e., $d = 0$, $e = -1$, i.e., (1) is $cz^2 - yz + fzx$. However, it is reduced, and it gives a contradiction. So there exists no such (1), i.e., we get the following: when $C_1 \cdots C_k$ is a Płoski curve but $C_1 \cdots C_k C_{k+1}$ is not, if C_{k+1} meet at some point of C_i that is not a common point of $C_1 \cdots C_k$, C_{k+1} meet at some point of all C_i that is not common. (It can be obtained by the following way: I proved that such (1) does not exist and also, by considering the intersection multiplicity, we can get it.) So suppose C is neither a Płoski curve, $(*)$ nor $(\star)$. We assume that $C = C_1 \cdots C_k C_{k+1} \cdots C_n$, where $C_1 \cdots C_k$ are the maximal number of conics that forms a Płoski curve in C. Since C is not a Płoski curve and a conic is a Płoski curve, $1 \leq k < n$. Then, by the above argument, $\mathrm{pd}(C_1 \cdots C_n) \geq -k^2 + nk + n$ (because $\sharp(C_1 \cap C_2) = 1$, $\sharp(C_1 \cdots C_{k-1} \cap C_k) = 1$, and $\sharp(C_1 \cdots C_k \cap C_{k+1}) \geq 1 + k$, $\sharp(C_1 \cdots C_{n-1} \cap C_n) \geq 1 + k$). Since $1 \leq k < n$, minimum occurs when $k = 1$, i.e., when $k = 1$, $\mathrm{pd}(C) \geq 2n - 1$,and $\mathrm{pd}(C) > 2n - 1$, otherwise. However, when $k = 1$, it is clear that $\mathrm{pd}(C) = 2n - 1$ if and only if C is either $(*)$ or $(\star)$, which is a contradiction. Therefore, $\mathrm{pd}(C) > 2n - 1$ if C is neither a Płoski curve, $(*)$, or $(\star)$. □

Lemma 3.11. *Let C be a stable plane curve of* $\deg C = 2n$ *whose all irreducible components are conics. Then* $\mathrm{pd}(C) > 2n - 1$.

Proof. We use the same notation in the previous lemma. It is easy that $\mathrm{pd}(*) = \mathrm{pd}(\star) = 2n - 1$. So we need to check the stability of $(*)$, $(\star)$. Since $(*)$ is $(x^2 - yz)(x^2 - 2yz) \cdots (x^2 - nyz)$ and $(\star)$ is $(x^2 - yz + xz) \cdots (x^2 - yz + nxz)$, by Hilbert-Mumford Criterion, they are strictly semi-stable. So, since a Płoski curve, $(*)$, and $(\star)$ are strictly-semistable, if C is stable, $\mathrm{pd}(C) > 2n - 1$. □

The following lemma is an immediate consequence of the previous lemmas.

Lemma 3.12. *Let C be as in Lemma 3.11. Then*

1) *C is a Płoski curve if and only if* $\mathrm{pd}(C) = n$.
2) *C is either $(*)$ or $(\star)$ in Lemma 3.10 if and only if* $\mathrm{pd}(C) = 2n - 1$.
3) $\mathrm{pd}(C) > 2n - 1$*, otherwise.*

Now, we are ready to get an upper bound for the Milnor sum of (semi)stable curves.

Proposition 3.13. *Let C be a plane curve with* $\deg C = d \geq 5$ *that has either a line or a conic as an irreducible component. Suppose C is stable. Then* $\mathrm{pd}(C) \geq d - 2$.

Proof. We consider the following 3 cases:

Case 1) First, let $C = C_1 \cdots C_m C_{m+1} \cdots C_k$, where $\deg C_i = 1$ for $1 \leq i \leq m$, $\deg C_i = 2$ for $m+1 \leq i \leq k$. For convenience, let $D = C_1 \cdots C_m$, $E = C_{m+1} \cdots C_k$. If $D = \emptyset$, we already proved it. So let $D \neq \emptyset$. Also, let us consider the case when $E = \emptyset$, i.e., $C = D$. By reordering, if necessary, let $D = C_1 \cdots C_t C_{t+1} \cdots C_d$, where t is the maximal number of concurrent lines in D and $C_1 \cdots C_t$ is concurrent lines. By the stability condition, $2 \leq t \leq \frac{2d}{3}$ (see Proposition 2.2). Then, $\mathrm{pd}(D) \geq -t^2 + (d+1)t - d$ (because $\sharp(C_1 \cdots C_i \cap C_{i+1}) = 1$ for all $i = 1, \ldots, t-1$ and $\geq t$ for all $i \geq t$). So the minimum occurs when $t = 2$. So $\mathrm{pd}(D) \geq d-2$. So we also let $E \neq \emptyset$. First, let us consider the case when E is a Płoski curve. If D is non-concurrent lines, then we can easily get the result. So suppose that D is concurrent lines. If common points of D and E coincide, then by using $\sharp(D \cap E) \geq 1 + (m-1)(k-m)$, $k \leq \frac{2d}{3}$, and $2k - m = d$, we can easily get that $\mathrm{pd}(C) \geq d-2$. Also, if they do not coincide, it is easy that $\mathrm{pd}(C) \geq d-2$ when $m = 1, \ldots, 5$. For $m \geq 6$, by using $\sharp(D \cap E) \geq 1 + (m-1)(2(k-m)-1)$, $m \leq \frac{2d}{3}$, and $k = \frac{d+m}{2} \leq \frac{5d}{6}$, we can get $\mathrm{pd}(C) \geq d-2$. If E is not a Płoski curve, by Lemma 3.12, we get the result.

Case 2) Next, let $C = C_1 \cdots C_m C_{m+1} \cdots C_k$, where $\deg C_i = 2$ for $1 \leq i \leq m$, $\deg C_i \geq 3$ for $m+1 \leq i \leq k$. Let $E = C_1 \cdots C_m$, $F = C_{m+1} \cdots C_k$. By the given condition, $E \neq \emptyset$. Also, by Lemma 3.12, we also let $F \neq \emptyset$, i.e., $1 \leq m < k$. First, we suppose that E is a Płoski curve. So we assume that $E : (x^2 - yz) \cdots (x^2 - yz + (m-1)z^2)$. We claim that $\sharp(E \cap F) \geq m+1$. If one of C_i, $m+1 \leq i \leq k$, does not pass $[0,1,0]$, we are done. So let all C_i's pass through $[0,1,0]$. The case when m=1 is obtained automatically by proof of the case when $m \geq 2$. So let $m \geq 2$. Fix $m+1 \leq i \leq k$. Suppose $C_i \cap E = \{[0,1,0]\}$. Then, since $C_i \cap C_1 = \{[0,1,0]\}$, $C_i : (x^2 - yz)f + z^{l_i}$, or $(x^2 - yz)f + x^{l_i}$, where f is a homogeneous polynomial of degree $l_i - 2$ in $k[x,y,z]$. Second one can be proven similarly as the first one, so we assume that $C_i : (x^2 - yz)f + z^{l_i}$. Since $C_i \cap C_2 = \{[0,1,0]\}$, $(x^2 - yz)f + z^{l_i} = z^2 f + z^{l_i} = z^2(f + z^{l_i - 2})$ has $z = 0$ as a unique root. Since base field is algebraically closed, $f = az^{l_i - 2}$, where $a \in k$, base field. So $C_i : (x^2 - yz)(az^{l_i-2}) + z^{l_i}$, which is a contradiction since C_i is irreducible. So C_i has another intersection point with C_1, which means that $i(C_i, C_j; [0,1,0]) < (\deg C_i)(\deg C_j) = 2 \deg C_i$, where $1 \leq j \leq m$, and $i(C_i, C_j; [0,1,0])$ is the intersection multiplicity of C_i and C_j at $[0,1,0]$. So $\sharp(C_i \cap C_j) \geq 2$, all $1 \leq j \leq m$. Therefore, $\sharp(E \cap F) \geq m+1$, which proves the claim. By the claim, $\sharp(E \cap C_i) \geq m+1$ for all $i \geq m+1$. So $\mathrm{pd}(C_1 \cdots C_k) = \mathrm{pd}(E) + \mathrm{pd}(C_{m+1}) + \cdots + \mathrm{pd}(C_k) + (\sharp(E \cap C_{m+1}) + \cdots + \sharp(EC_{m+1} \cdots C_{k-1} \cap C_k)) - (k-m) \geq d-1 \geq d-2$ since $k - m \geq 1$. So let us consider the case when E is not a Płoski curve. However, by using $\sharp(E \cap C_j) \geq 2$ for all j with $m+1 \leq j \leq k$ and Lemma 3.12, we easily get $\mathrm{pd}(C) \geq d-1 \geq d-2$.

Case 3) In general, let $C = C_1 \cdots C_m C_{m+1} \cdots C_t C_{t+1} \cdots C_k$, where $\deg C_i = 1$ for $1 \le i \le m$, $\deg C_i = 2$ for $m+1 \le i \le t$, $\deg C_i \ge 3$ for $t+1 \le i \le k$. For convenience, let $D = C_1 \cdots C_m$, $E = C_{m+1} \cdots C_t$, $F = C_{t+1} \cdots C_k$. If $D = \emptyset$, C is in Case 2), so let $D \ne \emptyset$. If $F = \emptyset$, then it is Case 1), so let $F \ne \emptyset$. Therefore, we need to deal with E. First, let $E \ne \emptyset$. If D is concurrent lines, then $\mathrm{pd}(C) = \mathrm{pd}(D) + \mathrm{pd}(EF) + \sharp(D \cap EF) - 1 \ge (d - m - 1) + (1 + (t-m)(m-1)) - 1 \ge d - 2$ because $t - m \ge 1$ and Case 2) always holds without the stability condition. If D is not concurrent lines, since $\mathrm{pd}(D) \ge m - 2$ and $\mathrm{pd}(EF) \ge d - m - 1$, $\mathrm{pd}(C) \ge d - 2$. Finally, we suppose that $E = \emptyset$. If D is concurrent lines, $\mathrm{pd}(C) = \mathrm{pd}(D) + \mathrm{pd}(C_{m+1}) + \cdots + \mathrm{pd}(C_k) + (\sharp(D \cap C_{m+1}) + \cdots + \sharp(DC_{m+1} \cdots C_{k-1} \cap C_k)) - (k - m) \ge \sum_{i=m+1}^{k} (\deg C_i - 1) + (\sum_{p \in C_{m+1}} (r_p - 1) + \sharp(D \cap C_{m+1})) + \cdots + (\sum_{p \in C_k} (r_p - 1) + \sharp(DC_{m+1} \cdots C_{k-1} \cap C_k)) - (k - m) \ge d - 2$. If D is not concurrent lines, then by using $\sum_{p \in C_{i+1}} (r_p - 1) + \sharp(DC_{m+1} \cdots C_i \cap C_{i+1}) \ge m$ for all $i \ge m+1$, $\mathrm{pd}(D) \ge m - 2$ and the above argument, $\mathrm{pd}(C) \ge (d-2) + (m-2)(k-m)$. Since D is not concurrent lines and $F \ne \emptyset$, $m \ge 3$ and $k - m \ge 1$. So $\mathrm{pd}(C) \ge d - 1 \ge d - 2$. □

By the previous proposition, we get a bound for the polar degree of stable curves. If C is of odd degree, since a Płoski curve is not semi-stable, by the same argument, we can get the same result for semi-stable curve C as the following proposition says.

Proposition 3.14. *Let C be a plane curve with $\deg C = d \ge 5$ that has either a line or a conic as an irreducible component, where d is odd. Suppose C is semi-stable. Then $\mathrm{pd}(C) \ge d - 2$.*

So we need to consider the case when all irreducible components of C are of $\deg \ge 3$. The following lemma gives a better bound of such a curve.

Lemma 3.15. *Let $\deg C = d \ge 5$. Suppose all irreducible components of C are of $\deg \ge 3$. Then $\mathrm{pd}(C) \ge \lceil \frac{2d}{3} \rceil$, where $\lceil \frac{2d}{3} \rceil$ is a round up integer of $\frac{2d}{3}$.*

Proof. Let $C = C_1 \cdots C_m C_{m+1} \cdots C_k$, where C_i's are irreducible, plane curves with $\deg C_i \ge 4$ for $1 \le i \le m$, $\deg C_j = 3$ for $m+1 \le j \le k$. Let $D = C_1 \cdots C_m$, $E = C_{m+1} \cdots C_k$. Then $\mathrm{pd}(C) = \mathrm{pd}(D) + \mathrm{pd}(E) + \sharp(D \cap E) - 1 \ge \sum_{i=1}^{m} (\deg C_i - 1) + 2(k - m) = (d - 3(k-m)) - m + 2(k-m) = d - k$. Since $3k \le d$ by degree consideration, $k \le \frac{d}{3}$. So $\mathrm{pd}(C) \ge d - k \ge \frac{2d}{3}$, i.e., $\mathrm{pd}(C) \ge \lceil \frac{2d}{3} \rceil$. □

So we get the following result:

Theorem 3.16. *Let $\deg C = d \ge 5$. Then we have the followings*:

1) *Suppose C is a stable curve that has either a line or a conic as an irreducible component. Then $\sum \mu_p \le (d-1)^2 - (d-2)$.*
2) *Let d be odd. Suppose C is a semi-stable curve that has either a line or a conic as an irreducible component. Then $\sum \mu_p \le (d-1)^2 - (d-2)$.*

3) *Suppose all irreducible components of* C *are of* $\deg \geq 3$. *Then* $\sum \mu_p \leq (d-1)^2 - \lceil \frac{2d}{3} \rceil$.

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