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SMOOTH HOROSPHERICAL VARIETIES OF PICARD NUMBER ONE AS LINEAR SECTIONS OF RATIONAL HOMOGENEOUS VARIETIES

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Abstract. We construct projective embeddings of horospherical varieties of Picard number one by means of Fano varieties of cones over rational homogeneous varieties. Then we use them to give embeddings of smooth horospherical varieties of Picard number one as linear sections of rational homogeneous varieties.

1. Introduction

Let G be a connected reductive algebraic group over $\mathbb{C}$ and let H be a closed subgroup of G. A homogeneous space G/H is said to be *horospherical* if H contains the unipotent radical of a Borel subgroup of G, or equivalently, G/H is isomorphic to a torus bundle over a rational homogeneous variety G/P. A normal G-variety is called *horospherical* if it contains an open dense G-orbit isomorphic to a horospherical homogeneous space G/H. Toric varieties and rational homogeneous varieties are horospherical. The cone over a rational homogeneous variety G/P is horospherical because it has an open G-orbit isomorphic to a $\mathbb{C}^\times$-bundle over G/P.

As can be seen from the latter example, horospherical varieties are not necessarily smooth and it is not easy to classify all smooth horospherical varieties. If we assume that the Picard number is one, such varieties are classified.

Theorem 1.1 (Theorem 0.1 and Theorem 1.7 of Pasquier [8])**.** *Let G be a connected reductive algebraic group. Let X be a smooth projective horospherical G-variety of Picard number one. Then X is either homogeneous or one of the following.*

(1) $(B_n, \varpi_{n-1}, \varpi_n)$, $n \geq 3$;
(2) $(B_3, \varpi_1, \varpi_3)$;

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(3) $(C_n, \varpi_{i+1}, \varpi_i)$, *$n \geq 2$ and $i \in \{1, 2, \ldots, n-1\}$;*
(4) $(F_4, \varpi_2, \varpi_3)$;
(5) $(G_2, \varpi_2, \varpi_1)$.

For the notations see Section 2.

Some of the above-mentioned varieties can be embedded into rational homogeneous varieties as linear sections. For example, the horospherical varieties $(C_n, \varpi_{i+1}, \varpi_i)$ are also known as odd symplectic Grassmannians. They are linear sections of symplectic Grassmannians (Mihai [7]). Pasquier [8] asked whether the smooth horospherical varieties listed in Theorem 1.1 can be embedded as linear sections into rational homogeneous varieties as in the case of $(C_n, \varpi_{i+1}, \varpi_i)$. In this paper, we investigate this problem.

First, we give geometric descriptions of the horospherical varieties listed in Theorem 1.1. For example, we prove the following results.

Proposition 5.2 *Let $\mathbb{S}_n$ be the rational homogeneous variety (D_{n+1}, ϖ_{n+1}).*

(1) *The horospherical variety $(B_n, \varpi_{n-1}, \varpi_n)$ is a linear section of the horospherical variety $(D_{n+1}, \varpi_{n-1}, \varpi_{n+1})$.*
(2) *The Fano variety $F_1(\widehat{\mathbb{S}_n})$ of lines in the cone $\widehat{\mathbb{S}_n}$ over $\mathbb{S}_n$ is isomorphic to the horospherical variety $(D_{n+1}, \varpi_{n-1}, \varpi_{n+1})$.*

Proposition 5.3 *Let $\mathbb{S}$ be the rational homogeneous variety (E_6, ϖ_6).*

(1) *The horospherical variety $(F_4, \varpi_2, \varpi_3)$ is a linear section of the horospherical variety $(E_6, \varpi_4, \varpi_5)$.*
(2) *The Fano variety $F_2(\widehat{\mathbb{S}})$ of planes in the cone $\widehat{\mathbb{S}}$ over $\mathbb{S}$ is isomorphic to the horospherical variety $(E_6, \varpi_4, \varpi_5)$.*

Using these descriptions we show that:

Theorem 1.2. *A smooth horospherical variety X of Picard number one can be embedded as a linear section into a rational homogenous variety S of Picard number one except when X is $(B_n, \varpi_{n-1}, \varpi_n)$ for $n \geq 7$.*

For an explicit description of an embedding of X into S, see Proposition 5.1 and its proof.

In the case of $(B_n, \varpi_{n-1}, \varpi_n)$ for $n \geq 7$, it is still open whether we can embed it into a rational homogeneous variety as a linear section. The reason why we cannot apply our method to the horospherical variety $(B_n, \varpi_{n-1}, \varpi_n)$ for $n \geq 7$, is that there does not exist a rational homogeneous variety $\mathcal{S}$ whose variety $\mathcal{C}_x(\mathcal{S})$ of minimal rational tangents at $x \in \mathcal{S}$ is isomorphic to the rational homogeneous variety (D_{n+1}, ϖ_{n+1}) for $n \geq 7$. Even though this does not imply that the horospherical variety $(B_n, \varpi_{n-1}, \varpi_n)$ for $n \geq 7$ cannot be embedded as a linear section into a rational homogeneous variety of Picard number one, we expect it cannot be.

It would be interesting to investigate the Fano varieties $F_k(\widehat{\mathbb{S}})$ of the cones $\widehat{\mathbb{S}}$ over rational homogeneous varieties $\mathbb{S}$ from the viewpoint of horospherical varieties, which we expect to be useful for understanding horospherical varieties geometrically as in the case of rational homogeneous varieties (Landsberg-Manivel [6]).

The remainder of this paper is organized as follows. In Section 2, we introduce the notations used in subsequent sections, and we review and prove some properties of horospherical varieties. In Section 3, we consider a way to embed the cone over a rational homogenous variety into another rational homogeneous variety as a linear section (Proposition 3.5). We review the results on Fano varieties of rational homogeneous varieties in Section 4. In Section 5, we describe each horospherical variety listed in Theorem 1.1 as a linear section of the Fano variety of the cone over a rational homogenous manifold (Proposition 5.2 and Proposition 5.3), and we employ the results obtained in Section 3 to complete the proof of Theorem 1.2 (Proposition 5.1).

2. Horospherical varieties

Let G be a connected reductive algebraic group over $\mathbb{C}$. For a dominant weight ϖ, let $V_G(\varpi)$ denote the irreducible representation space of G with the highest weight ϖ. Fix a Borel subgroup of G. Let $\{\varpi_1, \dots, \varpi_n\}$ be the system of fundamental weights of G. Take a highest weight vector v_i in $V_G(\varpi_i)$ for $i = 1, \dots, n$. Then the G-orbit of $[v_i]$ in $\mathbb{P}(V_G(\varpi_i))$ is a rational homogeneous variety. We denote it by (G, ϖ_i). The isotropy group P_i of G at $[v_i]$ is called the parabolic subgroup associated with ϖ_i. More generally, the G-orbit of $[v_{i_1} \otimes \cdots \otimes v_{i_k}]$ in $\mathbb{P}(V_G(\varpi_{i_1}) \otimes \cdots \otimes V_G(\varpi_{i_k}))$ is a rational homogeneous variety, which is denoted by $(G, \{\varpi_{i_1}, \dots, \varpi_{i_k}\})$.

Let H be a closed subgroup of G. A homogeneous space G/H is said to be *horospherical* if H contains the unipotent radical of a Borel subgroup of G. In this case, the normalizer $N_G(H)$ of H in G is a parabolic subgroup P of G and P/H is a torus $(\mathbb{C}^\times)^r$. Thus G/H has a structure of $(\mathbb{C}^\times)^r$-bundle over G/P. A normal G-variety is called *horospherical* if it contains an open dense G-orbit isomorphic to a horospherical homogeneous space G/H.

Proposition 2.1. *For any $i \neq j$, the closure of the G-orbit of $[v_i + v_j]$ in $\mathbb{P}(V_G(\varpi_i) \oplus V_G(\varpi_j))$ is a horospherical G-variety.*

Proof. The isotropy group H of G at $[v_i + v_j]$ contains the unipotent radical U of the Borel subgroup B of G and thus the homogeneous space $G.[v_i + v_j]$ is horospherical. It suffices to show that the closure X of $G.[v_i+v_j]$ in $\mathbb{P}(V_G(\varpi_i) \oplus V_G(\varpi_j))$ is a normal variety.

The normalizer P of H in G is the parabolic subgroup $P_i \cap P_j$ and P/H is the one-dimensional torus $\mathbb{C}^\times := \mathbb{C} \backslash \{0\}$. The P/H-orbit of $[v_i + v_j]$ is isomorphic to $\mathbb{C}^\times$ and the G-orbit of $[v_i + v_j]$ is isomorphic to the $\mathbb{C}^\times$-bundle $G \times_P \mathbb{C}^\times$ over G/P.

The closure of the P/H-orbit of $[v_i + v_j]$ is a line $\mathbb{P}^1$ in $\mathbb{P}(V_G(\varpi_i) \oplus V_G(\varpi_j))$. The line $\mathbb{P}^1$ has three P/H-orbits: one open orbit $\mathbb{C}^\times = P/H.[v_i + v_j]$, and two fixed points $[v_i]$ and $[v_j]$. Let X' be the fiber bundle $G \times_P \mathbb{P}^1$ over G/P with fiber $\mathbb{P}^1$. Define a map

$$f : X' = G \times_P \mathbb{P}^1 \to X$$

by $f(g,x) = gx$ for $g \in G$ and $x \in \mathbb{P}^1$. Then f is a birational morphism and is restricted to an isomorphism of $G \times_P \mathbb{C}^\times$ onto $G.[v_i + v_j]$. The image $X = f(X')$ of f is normal because the fibers of $X' = G \times_P \mathbb{P}^1 \to G/P$ are normal (Proposition 1 of [5]). Hence, X is normal, and thus, it is a horospherical G-variety. □

We will denote the closure of $G.[v_i+v_j]$ in $\mathbb{P}(V_G(\varpi_i)\oplus V_G(\varpi_j))$ by (G,ϖ_i,ϖ_j).[1] It has three G-orbits: one open orbit $G.[v_i+v_j]$ and two closed orbits, $G.[v_i]$ and $G.[v_j]$. The horospherical variety (G,ϖ_i,ϖ_j) is smooth if and only if $P_j.[v_i]$ and $P_i.[v_j]$ are linear ([8]). By showing that any non-linear smooth horospherical variety of Picard number one is of the form (G,ϖ_i,ϖ_j), Pasquier obtained the classification described in Theorem 1.1.

We end this section by giving a characterization of (G,ϖ_i,ϖ_j) by means of the structure of G-orbits.

For a dominant weight ϖ of G, let v_ϖ be a highest weight vector of $V_G(\varpi)$ and let P_ϖ be the isotropy group of G at $[v_\varpi] \in \mathbb{P}(V_G(\varpi))$. For two distinct dominant weights ϖ', ϖ'' of G, let (G,ϖ',ϖ'') denote the closure of the G-orbit of $[v_{\varpi'} \oplus v_{\varpi''}]$ in $\mathbb{P}(V_G(\varpi') \oplus V_G(\varpi''))$. Then the isotropy group H of G at $[v_{\varpi'} \oplus v_{\varpi''}] \in \mathbb{P}(V_G(\varpi')\oplus V_G(\varpi''))$ is the kernel of $\varpi' - \varpi'' : P_{\varpi'} \cap P_{\varpi''} \to \mathbb{C}^\times$, and the normalizer P of H in G is $P_{\varpi'} \cap P_{\varpi''}$. By the same arguments as those in the proof of Proposition 2.1, (G,ϖ',ϖ'') is a horospherical G-variety, and it has three G-orbits: one open orbit $G.[v_{\varpi'} \oplus v_{\varpi''}]$, and two closed orbits $G.[v_{\varpi'}]$ and $G.[v_{\varpi''}]$.

For example, $(G, 2\varpi_i, \varpi_i+\varpi_j)$ is also a projective embedding of G/H, where G/H is the open G-orbit in (G,ϖ_i,ϖ_j). Moreover, $(G,2\varpi_i,\varpi_i+\varpi_j)$ is the closure of the G-orbit of $[(v_i \circ v_i)\oplus(v_i\otimes v_j)]$ in $\mathbb{P}(S^2(V(\varpi_i))\oplus(V(\varpi_i)\otimes V(\varpi_j)))$, and $[(v_i\circ v_i)\oplus(v_i\otimes v_j)]$ is $[v_i\otimes(v_i\oplus v_j)]$ as an element in $\mathbb{P}(V(\varpi_i)\otimes(V(\varpi_i)\oplus V(\varpi_j))$. Define

$$\begin{array}{rcl} \tau : (G,2\varpi_i,\varpi_i+\varpi_j) & \to & \mathbb{P}(V(\varpi_i)\oplus V(\varpi_j)) \quad \text{by} \\ g[v_i\otimes(v_i\oplus v_j)] & \mapsto & g[v_i\oplus v_j], \end{array}$$

where $g \in G$. Then, the image of τ is (G,ϖ_i,ϖ_j).

Similarly, we get two more projective embeddings of G/H.

Proposition 2.2. *For $i \neq j$, let H be the isotropy group of G at $[v_i \oplus v_j] \in \mathbb{P}(V_G(\varpi_i)\oplus V_G(\varpi_j))$. Then, there are exactly 4 projective embeddings of G/H,*

[1]In [8] (G,ϖ_i,ϖ_j) is the notation for the unique smooth horospherical variety of Picard number one such that the isotropy group at a generic point is given by the kernel of $\varpi_i-\varpi_j : P_i \cap P_j \to \mathbb{C}^\times$. Afterwards, it is realized as a projective variety, the closure of the G-orbit of $[v_i \oplus v_j]$ in $\mathbb{P}(V(\varpi_i) \oplus \mathbb{P}(\varpi_j))$. In this paper we adopt the latter as the definition of (G,ϖ_i,ϖ_j) regardless of its smoothness.

(G,ϖ_i,ϖ_j), $(G,\varpi_i+\varpi_j,2\varpi_j)$, $(G,2\varpi_i,\varpi_i+\varpi_j)$, and $(G,2\varpi_i+\varpi_j,\varpi_i+2\varpi_j)$, and we have the following commutative diagram:

$$\begin{array}{ccccc} & & (G,2\varpi_i+\varpi_j,\varpi_i+2\varpi_j) & & \\ & \overset{\sigma_2}{\swarrow} & & \overset{\tau_2}{\searrow} & \\ (G,\varpi_i+\varpi_j,2\varpi_j) & & & & (G,2\varpi_i,\varpi_i+\varpi_j) \\ & \underset{\sigma_1}{\searrow} & & \underset{\tau_1}{\swarrow} & \\ & & (G,\varpi_i,\varpi_j) & & \end{array}$$

Proof. The horospherical homogeneous space $G.[v_i+v_j]=G/H$ is of rank one and has two colors. Thus, the classification of G/H-embeddings in terms of colored fans tells us that it has exactly 4 projective embeddings and that they are determined by the set of colors (See Section 1.3 of [8]). (G,ϖ_i,ϖ_j) has two colors, $(G,\varpi_i+\varpi_j,2\varpi_j)$ and $(G,2\varpi_i,\varpi_i+\varpi_j)$ have one color, and $(G,2\varpi_i+\varpi_j,\varpi_i+2\varpi_j)$ has no color. □

By Proposition 2.2, we can distinguish 4 projective embeddings of G/H by the types of their closed orbits. In particular, the following three properties uniquely determine (G,ϖ_i,ϖ_j).

Corollary 2.3. *Let X be a normal G-variety. Assume the following properties.*

(1) *There are three G-orbits in X : one open orbit and two closed orbits.*
(2) *The open orbit is isomorphic to a $\mathbb{C}^\times$-bundle over $(G,\{\varpi_i,\varpi_j\})$.*
(3) *One closed orbit is isomorphic to (G,ϖ_i) and the other closed orbit is isomorphic to (G,ϖ_j).*

Then X is the horospherical variety (G,ϖ_i,ϖ_j).

3. Cones over rational homogeneous varieties

Let $X\subset\mathbb{P}(V)$ be a projective variety. By a linear space in X we mean a (projective) linear space $\mathbb{P}^k$ in $\mathbb{P}(V)$ contained in X. We call a linear space of dimension one (two, respectively) a line (a plane, respectively). Let $F_k(X)$ denote the Fano variety of $\mathbb{P}^k$'s in X.

Proposition 3.1. *Let $X\subset\mathbb{P}(V)$ be a projective variety and let Z be a linear section of X. Then the Fano variety $F_k(Z)$ is also a linear section of $F_k(X)$.*

Proof. Suppose that Z is the intersection $X\cap\mathbb{P}(W)$ for some linear space $\mathbb{P}(W)$ in $\mathbb{P}(V)$. Then $\mathbb{P}(\wedge^{k+1}W)$ is a linear space in $\mathbb{P}(\wedge^{k+1}V)$ and $F_k(Z)$ is the intersection $F_k(X)\cap\mathbb{P}(\wedge^{k+1}W)$. □

Proposition 3.2. *Let $X\subset\mathbb{P}(V)$ be a projective variety. Let $\widehat{X}$ be the cone over X with vertex a point x_0. For a line L in X, let $\widehat{L}$ denote the cone over L with vertex x_0. Then, $\widehat{L}$ is a plane in $\widehat{X}$. Furthermore, any line in $\widehat{X}$ is contained in $\widehat{L}$ for some line L in X.*

In general, any linear space in $\widehat{X}$ is contained in the cone over a linear space in X with vertex x_0.

Proposition 3.3. *Let G be a connected reductive algebraic group. Let V_ϖ be an irreducible representation space of G and let v_ϖ be a highest weight vector of V_ϖ. Let P denote the isotropy group of G at $[v_\varpi] \in \mathbb{P}(V_\varpi)$. Let V_0 be the one-dimensional trivial representation space of G and let v_0 be a non-zero vector in V_0. Then, the closure $\widehat{S}$ of the G-orbit $G.[v_\varpi \oplus v_0]$ in $\mathbb{P}(V_\varpi \oplus V_0)$ is the cone over the rational homogeneous variety $S = G/P$, and it is horospherical.*

Proof. Consider the map $\pi : G.[v_\varpi \oplus v_0] \to \mathbb{P}(V_\varpi)$ defined by $g.[v_\varpi \oplus v_0] \mapsto g.[v_\varpi]$, where $g \in G$. Then, the image of π is the rational homogeneous variety $S = G/P$ embedded in $\mathbb{P}(V_\varpi)$, and the fiber over the base point $[v_\varpi]$ of S consists of $[v_\varpi \oplus cv_0]$, where $c \in \mathbb{C}^\times$. Thus, the G-orbit $G.[v_\varpi \oplus v_0]$ is horospherical. The closure $\widehat{S}$ of $G.[v_\varpi \oplus v_0]$ contains two more G-orbits: $G.[v_\varpi \oplus 0]$ and $G.[0 \oplus v_0]$. The former is $S = G/P$ and the latter is the point $[0 \oplus v_0]$. Then, $\widehat{S}$ is the cone over S with vertex $[0 \oplus v_0]$. □

We will retain the same notations and embeddings of Proposition 3.3 whenever we mention the cone $\widehat{S}$ over the rational homogeneous variety $S = G/P$. The cone over a rational homogeneous variety appears naturally as a linear section of another rational homogeneous variety. For clarity, we first recall several definitions.

Let $X \subset \mathbb{P}(V)$ be a projective variety and let $x \in X$. The *projective tangent space* $\mathbb{T}_x X$ at $x \in X$ is the projectivization of the Zariski tangent space $\widetilde{T}_{\tilde{x}}\tilde{X} \subset V$ of the affine cone $\tilde{X} \subset V$ of X at a point $\tilde{x}$ with $[\tilde{x}] = x$. The projective tangent space $\mathbb{T}_x X$ is a linear space in $\mathbb{P}(V)$. The Zariski tangent space $T_x X$ can be identified with $(\widetilde{T}_{\tilde{x}}\tilde{X}/\mathbb{C}\tilde{x}) \otimes (\mathbb{C}\tilde{x})^*$. The *projectivized tangent space* $\mathbb{P}(T_x X)$ is the projectivization of the Zariski tangent space $T_x X$.

When X is uniruled by lines in X, the *variety $\mathcal{C}_x(X)$ of minimal rational tangents* of X at a smooth point $x \in X$ is, by definition, the subvariety of $\mathbb{P}(T_x X)$ consisting of tangent directions to lines in X passing through x (Hwang-Mok [2], [3], [4]).

Proposition 3.4. *Let $S = G/P \subset \mathbb{P}(V(\varpi_i))$ be the rational homogeneous variety (G, ϖ_i). Let x be the base point $[v_{\varpi_i}]$. Then, the intersection $\mathbb{T}_x S \cap S$ of S with the projective tangent space $\mathbb{T}_x S$ of S is the cone over the variety of minimal rational tangents $\mathcal{C}_x(S)$ with vertex x.*

Proof. We will repeat the arguments presented in Section 2.1 of Landsberg-Manivel [6]. Since the ideal of S is generated in degree two, any line L osculating to order two at x is contained in S. Hence, if $y \in \mathbb{T}_x S \cap S$ and $y \neq x$, then the line passing through x and y is contained in S. Therefore, the intersection $\mathbb{T}_x S \cap S$, of S with the projective tangent space $\mathbb{T}_x S$ of S, is the locus of lines in S passing through x. Now, the locus of lines in S passing through x is the cone over $\mathcal{C}_x(S)$ with vertex x. □

Proposition 3.5. *Let $S = G/P \subset \mathbb{P}(V_{\varpi_i})$ be a rational homogeneous variety (G, ϖ_i). Let x be the base point $[v_{\varpi_i}]$ and let L denote the reductive part of P. Let $\mathbb{S}$ be a linear section of $\mathcal{C}_x(S)$ by a linear space invariant under the action of a reductive subgroup $\mathbb{G}$ of L. Then there is a $\mathbb{G}$-equivariant embedding of the cone $\widehat{\mathbb{S}}$ over $\mathbb{S}$ as a linear section, into S.*

In this case the Fano variety $F_k(\widehat{\mathbb{S}})$ is also a linear section of $F_k(S)$.

Proof. The first statement follows from Proposition 3.4. The second statement follows from Proposition 3.1. □

Now, our strategy is

(I) to describe the horospherical variety X listed in Theorem 1.1 as (a linear section of) $F_k(\widehat{\mathbb{S}})$ for some $\mathbb{G}$-homogeneous variety $\mathbb{S}$,

(II) and to find a $\mathbb{G}$-equivariant embedding of $\mathbb{S}$ as a linear section into $\mathcal{C}_x(\mathcal{S})$ for some rational homogeneous variety $\mathcal{S}$.

Then, we can embed X as a linear section into $S = F_k(\mathcal{S})$ by Proposition 3.5.

4. Fano varieties of rational homogeneous varieties

To achieve our goal, more precisely, (I) and (II) at the end of Section 3, we will use the results on varieties of minimal rational tangents or Fano varieties of rational homogeneous varieties, obtained by Hwang and Mok [2], [3], [4] and Landsberg-Manivel [6].

In the following, we identify the fundamental weights of G with the nodes of the Dynkin diagram of G.

Proposition 4.1 (Hwang and Mok [2], [3], [4] and Landsberg-Manivel [6])**.** *Let $S = G/P \subset \mathbb{P}(V_{\varpi_i})$ be a rational homogeneous variety (G, ϖ_i). Let x be the base point $[v_{\varpi_i}]$ and let L denote the reductive part of P. Then, $\mathcal{C}_x(S)$ has at most two orbits under the action of P, and $\mathcal{C}_x(S)$ has two P-orbits if and only if ϖ_i is associated to a short root.*

Furthermore, the closed P-orbit in $\mathcal{C}_x(S)$ is the rational homogeneous variety $(L^{ss}, N(\varpi_i))$ and is a linear section of $\mathcal{C}_x(S)$, where L^{ss} is the semisimple part of L and $N(\varpi_i)$ is the set of nodes connected to ϖ_i by an edge in the Dynkin diagram of G.

We remark that the Dynkin diagram of L^{ss} is obtained by deleting ϖ_i in the Dynkin diagram of G.

Proposition 4.2 (Landsberg-Manivel [6])**.** *Let $S = G/P \subset \mathbb{P}(V_{\varpi_i})$ be a rational homogeneous variety (G, ϖ_i). Then, $F_1(S)$ has at most two orbits under the action of G, and $F_1(S)$ has two G-orbits if and only if ϖ_i is associated to a short root.*

Furthermore, the closed G-orbit in $F_1(S)$ is the rational homogeneous variety $(G, N(\varpi_i))$ and is a linear section of $F_1(S)$, where $N(\varpi_i)$ is the set of nodes connected to ϖ_i by an edge in the Dynkin diagram of G.

In general, $F_k(S)$ can be described using subdiagrams of type (A_k, ϖ_1), of the marked Dynkin diagram (G, ϖ_i). We call such a subdiagram a *chain of length k*. For a chain $\mathcal{A}$, $N(\mathcal{A})$ is defined by the set of nodes connected to $\mathcal{A}$ by an edge in the Dynkin diagram of G.

Proposition 4.3 (Landsberg-Manivel [6])**.** *Let $S = G/P \subset \mathbb{P}(V_{\varpi_i})$ be a rational homogeneous variety (G, ϖ_i).*

(1) *If ϖ_i is not associated to a short root, then $F_k(S)$ is the disjoint union of rational homogeneous varieties $(G, N(\mathcal{A}))$, where $\mathcal{A}$ is a chain of length k.*
(2) *If ϖ_i is associated to a short root, then $F_k(S)$ has $(G, N(\mathcal{A}))$ as linear sections, where $\mathcal{A}$ is a chain of length k.*

We will apply Proposition 4.2 and Proposition 4.3 to $S = (G, \varpi_i)$ where ϖ_i is an end of the Dynkin diagram of G. A node of the Dynkin diagram of G is called an *end* if it is connected to only one other node. For an end ϖ, a *branch* of ϖ is a series of nodes

$$\varpi^{(r)}, \dots, \varpi^{(1)}, \varpi$$

that satisfy:

(1) Each $\varpi^{(i)}$ $(i = 1, \dots, r-1)$ is connected only to $\varpi^{(i-1)}$ and $\varpi^{(i+1)}$.
(2) The subdiagram of the Dynkin diagram of G with nodes $\varpi^{(r)}, \dots, \varpi^{(1)}$, ϖ is either of type A_{r+1} or of type C_{r+1}.
(3) The series $\varpi^{(r)}, \dots, \varpi^{(1)}, \varpi$ is maximal with respect to properties (1) and (2).

Example. (1) The Dynkin diagram of B_n has two ends: ϖ_1 and ϖ_n. The branch of ϖ_1 is $\varpi_{n-1}, \dots, \varpi_2, \varpi_1$ and the branch of ϖ_n is ϖ_{n-1}, ϖ_n.
(2) The Dynkin diagram of F_4 has two ends: ϖ_1 and ϖ_4. The branch of ϖ_1 is ϖ_2, ϖ_1 and the branch of ϖ_4 is $\varpi_2, \varpi_3, \varpi_4$.
(3) The Dynkin diagram of G_2 has two ends: ϖ_1 and ϖ_2. The branch of ϖ_1 is ϖ_1 and the branch of ϖ_2 is ϖ_2.

For the index of fundamental weights, we follow the conventions in [1].

Proposition 4.4. *Let $S \subset \mathbb{P}(V_\varpi)$ be a rational homogeneous variety (G, ϖ). Assume that ϖ is an end of the Dynkin diagram of G. Let $\varpi^{(r)}, \dots, \varpi^{(1)}, \varpi$ be the branch of ϖ and let $r \geq k \geq 1$.*

(1) *If ϖ is not associated to a short simple root, then $F_k(S)$ is the rational homogeneous variety $(G, \varpi^{(k)})$.*
(2) *If ϖ is associated to a short simple root, then $F_k(S)$ has the rational homogeneous variety $(G, \varpi^{(k)})$ as a linear section.*

Proof. It follows from Proposition 4.2 and Proposition 4.3. □

5. Embeddings as linear sections

In this section we will show that there is an embedding of the horospherical variety X into the rational homogeneous variety S for the following pairs.

Proposition 5.1. *For the following pair (X, S) of a smooth horospherical variety X of Picard number one and a rational homogeneous variety S of Picard number one, X can be embedded as a linear section into S:*

(1) $X = (B_3, \varpi_2, \varpi_3)$ *and* $S = (F_4, \varpi_3)$ *or* $S = (D_5, \varpi_2)$;
(2) $X = (B_4, \varpi_3, \varpi_4)$ *and* $S = (E_6, \varpi_3)$;
(3) $X = (B_5, \varpi_4, \varpi_5)$ *and* $S = (E_7, \varpi_3)$;
(4) $X = (B_6, \varpi_5, \varpi_6)$ *and* $S = (E_8, \varpi_3)$;
(5) $X = (F_4, \varpi_2, \varpi_3)$ *and* $S = (E_7, \varpi_5)$;
(6) $X = (C_2, \varpi_2, \varpi_1)$ *and* $S = (C_3, \varpi_2)$ *or* $S = (F_4, \varpi_3)$;
(7) $X = (C_n, \varpi_{i+1}, \varpi_i)$, $n \geq 3$ *and* $1 \leq i \leq n-1$ *and* $S = (C_{n+1}, \varpi_{i+1})$;
(8) $X = (G_2, \varpi_2, \varpi_1)$ *and* $S = (B_4, \varpi_2)$;
(9) $X = (B_3, \varpi_1, \varpi_3)$ *and* $S = (B_4, \varpi_4) = (D_5, \varpi_5)$.

For the index of fundamental weights, we follow the conventions in [1].

For example, we will show that there are embeddings (Fig. 1).

(3) $(B_5, \varpi_4, \varpi_5) \overset{\text{Prop. 5.2}}{\longrightarrow} (D_6, \varpi_4, \varpi_6) \overset{\text{Prop. 5.4 et al.}}{\longrightarrow} (E_7, \varpi_3)$.

(5) $(F_4, \varpi_2, \varpi_3) \overset{\text{Prop. 5.3}}{\longrightarrow} (E_6, \varpi_4, \varpi_5) \overset{\text{Prop. 5.5 et al.}}{\longrightarrow} (E_7, \varpi_5)$.

Proposition 5.2. *Let $\mathbb{S}_n$ be the rational homogeneous variety (D_{n+1}, ϖ_{n+1}).*

(1) *The horospherical variety $(B_n, \varpi_{n-1}, \varpi_n)$ is a linear section of the horospherical variety $(D_{n+1}, \varpi_{n-1}, \varpi_{n+1})$.*
(2) *$F_1(\widehat{\mathbb{S}_n})$ is isomorphic to the horospherical variety $(D_{n+1}, \varpi_{n-1}, \varpi_{n+1})$.*

Proof. (1) Let $\mathbb{S}_n$ be the rational homogeneous variety (D_{n+1}, ϖ_{n+1}). Then, $\mathbb{S}_n$ is isomorphic to (B_n, ϖ_n). Furthermore, (B_n, ϖ_{n-1}) is a linear section of $F_1(\mathbb{S}_n) = (D_{n+1}, \varpi_{n-1})$ (Proposition 4.4). Therefore, $X = (B_n, \varpi_{n-1}, \varpi_n)$ is a linear section of the horospherical variety $(D_{n+1}, \varpi_{n-1}, \varpi_{n+1})$.

(2) Let V_0 be the one-dimensional trivial representation space of D_{n+1} and let v_0 be a nonzero vector in V_0. By Proposition 3.3 the closure of the D_{n+1}-orbit $D_{n+1}.[v_n \oplus v_0]$ is the cone $\widehat{\mathbb{S}_n}$ over $\mathbb{S}_n$ with vertex $[0 \oplus v_0]$.

We first show the following properties.

(1) $F_1(\widehat{\mathbb{S}_n})$ has three D_{n+1}-orbits: one is open and the others are closed.
(2) The open D_{n+1}-orbit is isomorphic to a $\mathbb{C}^\times$-bundle over $(D_{n+1}, \{\varpi_{n-1}, \varpi_{n+1}\})$.
(3) One closed orbit is isomorphic to (D_{n+1}, ϖ_{n-1}) and the other closed orbit is isomorphic to (D_{n+1}, ϖ_{n+1}).

The variety of lines in $\widehat{\mathbb{S}_n}$ passing through the vertex $[0 \oplus v_0]$ is isomorphic to the base $\mathbb{S}_n$ of the cone $\widehat{\mathbb{S}_n}$, and thus, it is isomorphic to (D_{n+1}, ϖ_{n+1}). The variety of lines in the base $\mathbb{S}_n$ is isomorphic to (D_{n+1}, ϖ_{n-1}) (Proposition 4.4). These two varieties are closed orbits in $F_1(\widehat{\mathbb{S}_n})$.

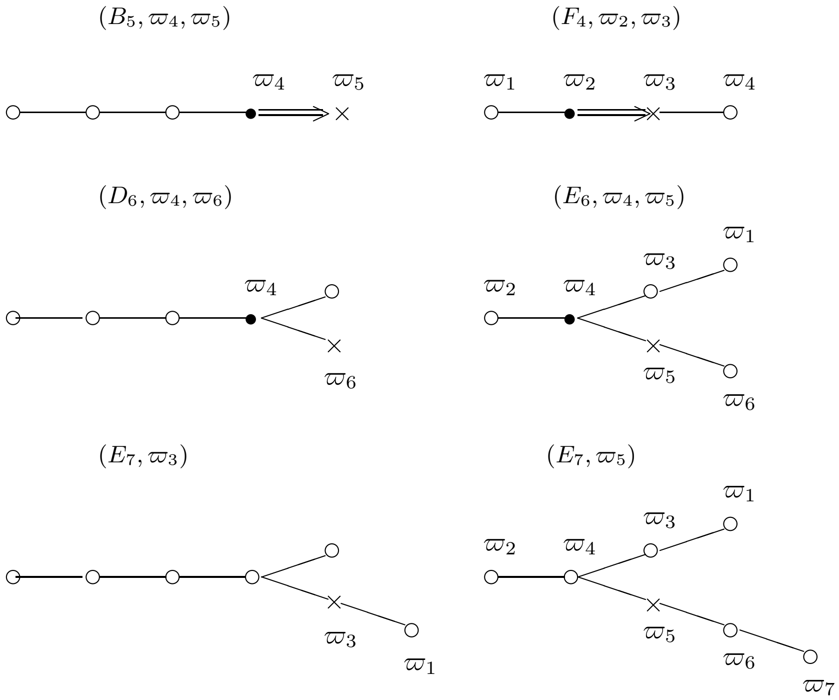

FIGURE 1. Embeddings of (3) $(B_5,\ \varpi_4,\ \varpi_5)$ and (5) $(F_4,\ \varpi_2,\ \varpi_3)$ in Proposition 5.1.

Let Ω denote the complement of these two closed orbits. Let ℓ be a line in $\widehat{\mathbb{S}_n}$ corresponding to a point in Ω. By Proposition 3.2, there is a line L in $\mathbb{S}_n$ such that ℓ is contained in the cone $\widehat{L}$ over L with vertex. If there are two such lines L, L', then ℓ is contained in the intersection $\widehat{L} \cap \widehat{L'}$. However, $\widehat{L} \cap \widehat{L'}$ is a line passing through the vertex, which is a contradiction. Thus, there is a unique L with $\ell \subset \widehat{L}$. Since ℓ and L are lines in the plane $\widehat{L}$, they intersect at a point. Define a map

$$\begin{array}{rcl} \pi : \Omega & \rightarrow & (D_{n+1}, \{\varpi_{n-1}, \varpi_{n+1}\}) \\ {[\ell]} & \mapsto & ([L], [L \cap \ell]). \end{array}$$

Then, π is a $\mathbb{C}^{\times}$-bundle over $(D_{n+1}, \{\varpi_{n-1}, \varpi_{n+1}\})$ and Ω is a horospherical homogeneous space.

For any line $\ell \in \widehat{\mathbb{S}_n}$, there is a line L in $\widehat{\mathbb{S}_n}$ such that $\ell \subset \widehat{L}$. Since the variety of lines in $\widehat{L}$ is $\mathbb{P}^2$, we have the following collapsing:

$$f : \mathbf{G} \times_{\mathbf{P}} \mathbb{P}^2 \rightarrow F_1(\widehat{\mathbb{S}_n}),$$

where $\mathbf{G}/\mathbf{P} = (D_{n+1}, \varpi_{n-1})$ is the variety $F_1(\mathbb{S}_n)$ of lines in $\mathbb{S}_n$. By Proposition 1 of [5], $F_1(\widehat{\mathbb{S}_n})$ is normal.

By Corollary 2.3, $F_1(\widehat{\mathbb{S}_n})$ is isomorphic to $(D_{n+1}, \varpi_{n-1}, \varpi_{n+1})$. □

Proposition 5.3. *Let $\mathbb{S}$ be the rational homogeneous variety (E_6, ϖ_6).*

(1) *The horospherical variety $(F_4, \varpi_2, \varpi_3)$ is a linear section of the horospherical variety $(E_6, \varpi_4, \varpi_5)$.*
(2) *$F_2(\widehat{\mathbb{S}})$ is isomorphic to the horospherical variety $(E_6, \varpi_4, \varpi_5)$.*

Proof. (1) Let $\mathbb{S}$ be the rational homogeneous variety (E_6, ϖ_6). Then, $\mathbb{S}' := (F_4, \varpi_4)$ is a linear section of $\mathbb{S}$ (Section 6.3 of Landsberg-Manivel [6]). By Proposition 3.1, $F_1(\mathbb{S}')$ is a linear section of $F_1(\mathbb{S})$. Furthermore, (F_4, ϖ_3) is a linear section of $F_1(\mathbb{S}')$ by Proposition 4.4. Hence, (F_4, ϖ_3) is a linear section of $F_1(\mathbb{S}) = (E_6, \varpi_5)$. Similarly, (F_4, ϖ_2) is a linear section of (E_6, ϖ_4)(Proposition 4.4). Therefore, $X = (F_4, \varpi_2, \varpi_3)$ is a linear section of the horospherical variety $(E_6, \varpi_4, \varpi_5)$.

(2) Let V_0 be the one-dimensional trivial representation of E_6 and let v_0 be a nonzero vector in V_0. By Proposition 3.3 the closure of the E_6-orbit $E_6.[v_n \oplus v_0]$ is the cone $\widehat{\mathbb{S}}$ over $\mathbb{S}$ with vertex $[0 \oplus v_0]$.

We first show the following properties.

(1) $F_2(\widehat{\mathbb{S}})$ has three E_6-orbits: one is open and the others are closed.
(2) The open E_6-orbit is isomorphic to a $\mathbb{C}^\times$-bundle over $(E_6, \{\varpi_4, \varpi_5\})$.
(3) One closed orbit is isomorphic to (E_6, ϖ_4) and the other closed orbit is isomorphic to (E_6, ϖ_5).

The variety of planes in $\widehat{\mathbb{S}}$ passing through the vertex $[0 \oplus v_0]$ is isomorphic to the Fano variety $F_1(\mathbb{S})$ of lines in the base of the cone $\widehat{\mathbb{S}}$, and thus, it is isomorphic to (E_6, ϖ_5). The variety of planes in the base $\mathbb{S}$ is isomorphic to (E_6, ϖ_4) (Proposition 4.4). These two varieties are closed orbits in $F_2(\widehat{\mathbb{S}})$.

Let Ω denote the complement of these two closed orbits. Let $\wp$ be a plane in $\widehat{\mathbb{S}}$ corresponding to a point in Ω. By Proposition 3.2, there is a plane E in $\mathbb{S}$ such that $\wp$ is contained in the cone $\widehat{E}$ over E with vertex. If there are two such planes E, E', then $\wp$ is contained in the intersection $\widehat{E} \cap \widehat{E'}$. If E and E' intersect at a point, then $\widehat{E} \cap \widehat{E'} = \widehat{E \cap E'}$ is a line, and it cannot contain the plane $\wp$, which is a contradiction. If E and E' intersect in a line, then $\widehat{E} \cap \widehat{E'} = \widehat{E \cap E'}$ is a plane passing through the vertex and is equal to $\wp$, which contradicts to the assumption that $\wp$ does not contain the vertex. Thus, there is a unique E with $\wp \subset \widehat{E}$. Since $\wp$ and E are planes in the linear space $\widehat{E}$ of dimension 3, they intersect in a line. Define a map

$$\begin{aligned} \pi : \Omega &\rightarrow (E_6, \{\varpi_4, \varpi_5\}) \\ [\wp] &\mapsto ([E], [E \cap \wp]). \end{aligned}$$

Then π is a $\mathbb{C}^\times$-bundle over $(E_6, \{\varpi_4, \varpi_5\})$ and Ω is a horospherical homogeneous space.

As in the proof of Proposition 5.2(2), there is a collapsing

$$g : \mathbf{G} \times_{\mathbf{P}} \mathbb{P}^3 \rightarrow F_2(\widehat{\mathbb{S}}),$$

where $\mathbf{G}/\mathbf{P} = (E_6, \varpi_4)$ is the variety $F_2(\mathbb{S})$ of planes in $\mathbb{S}$. Thus, $F_2(\widehat{\mathbb{S}})$ is normal.

By Corollary 2.3, $F_2(\widehat{\mathbb{S}})$ is isomorphic to $(E_6, \varpi_4, \varpi_5)$. □

Proposition 5.4. *Let $\mathbb{S}_n$ be the rational homogeneous variety (B_n, ϖ_n). Then, there is an embedding of $\mathbb{S}_n$ into the variety $\mathcal{C}_x(\mathcal{S})$ of minimal rational tangents of the rational homogeneous variety $\mathcal{S}$ at the base point x of $\mathcal{S}$ in the following cases:*

(1) *$n = 3$ and $\mathcal{S} = (F_4, \varpi_4)$ or $\mathcal{S} = (D_5, \varpi_1)$;*
(2) *$n = 4$ and $\mathcal{S} = (E_6, \varpi_1)$;*
(3) *$n = 5$ and $\mathcal{S} = (E_7, \varpi_1)$;*
(4) *$n = 6$ and $\mathcal{S} = (E_8, \varpi_1)$.*

Proof. The variety $\mathcal{C}_x(\mathcal{S})$ of minimal rational tangents of $\mathcal{S} = (F_4, \varpi_4)$ has (B_3, ϖ_3) as a linear section (Proposition 4.1).

The variety $\mathcal{C}_x(\mathcal{S})$ of minimal rational tangents of $\mathcal{S}$ respectively equal to (D_5, ϖ_1), (E_6, ϖ_1), (E_7, ϖ_1) and (E_8, ϖ_1) are respectively (D_4, ϖ_4), (D_5, ϖ_5), (D_6, ϖ_6) and (D_7, ϖ_7) (Proposition 4.1). Furthermore, (B_n, ϖ_n) is isomorphic to (D_{n+1}, ϖ_{n+1}). □

Proposition 5.5. *Let $\mathbb{S}$ be the rational homogeneous variety (E_6, ϖ_6) and let $\mathcal{S}$ be the rational homogeneous variety (E_7, ϖ_7). Then there is an embedding of $\mathbb{S}$ into the variety $\mathcal{C}_x(\mathcal{S})$ of minimal rational tangents of $\mathcal{S}$ at the base point x.*

Proof. The variety $\mathcal{C}_x(\mathcal{S})$ of minimal rational tangents of $\mathcal{S} = (E_7, \varpi_7)$ is (E_6, ϖ_6) (Proposition 4.1). □

Proof of Proposition 5.1. (1)–(4) By Proposition 5.2, the horospherical variety $(B_n, \varpi_{n-1}, \varpi_n)$ is a linear section of the Fano variety $F_1(\widehat{\mathbb{S}_n})$ of lines in the cone $\widehat{\mathbb{S}_n}$ over $\mathbb{S}_n$. By Proposition 5.4, there is an embedding of $\mathbb{S}_n$ into the variety $\mathcal{C}_x(\mathcal{S})$ of minimal rational tangents of the rational homogeneous variety $\mathcal{S}$ at the base point x in cases (1) – (4). By Proposition 3.5, the cone $\widehat{\mathbb{S}_n}$ can be embedded into $\mathcal{S}$ as a linear section, and $F_1(\widehat{\mathbb{S}_n})$ is again a linear section of $F_1(\mathcal{S})$. Put $S = F_1(\mathcal{S})$. Then $X = (B_n, \varpi_{n-1}, \varpi_n)$ can be embedded into S as a linear section.

(5) The proof is similar to that of (1)–(4). In this case, use Proposition 5.3, Proposition 5.5, and Proposition 3.5, and then, put $S = F_2(\mathcal{S})$.

(6)–(7) We already know that there is an embedding of $X = (C_n, \varpi_{i+1}, \varpi_i)$ into $S = (C_{n+1}, \varpi_{i+1})$. One can prove this using the method described above. In particular, this new method provides another embedding: the embedding of $X = (C_2, \varpi_2, \varpi_1)$ into $S = (F_4, \varpi_3)$.

(8) Let $\mathbb{S}$ be the rational homogeneous variety (G_2, ϖ_1). Then, $\mathbb{S}$ is the hyperquadric $\mathbb{Q}^5$ of $\mathbb{P}^6$, and thus, it is isomorphic to (B_3, ϖ_1). Furthermore, (G_2, ϖ_2) is a linear section of $F_1(\mathbb{S}) = (B_3, \varpi_2)$ (Proposition 4.4). Therefore,

$X = (G_2, \varpi_2, \varpi_1)$ is a linear section of the horospherical variety $(B_3, \varpi_2, \varpi_1)$. Using the same method described above, starting with the embedding of $\mathbb{S} = (B_3, \varpi_1)$ into $\mathcal{C}_x(\mathcal{S})$ where $\mathcal{S} = (B_4, \varpi_1)$, we get an embedding of $(B_3, \varpi_2, \varpi_1) = F_1(\widehat{\mathbb{S}})$ into $S = F_1(\mathcal{S})$ as a linear section. Consequently, $X = (G_2, \varpi_2, \varpi_1)$ can be embedded into $S = (B_4, \varpi_2)$ as a linear section.

(9) (B_3, ϖ_3) is isomorphic to (D_4, ϖ_4), and (B_3, ϖ_1) is a linear section of (D_4, ϖ_1). Thus, the horospherical variety $X = (B_3, \varpi_1, \varpi_3)$ is a linear section of the horospherical variety $(D_4, \varpi_1, \varpi_4)$. By the triviality of D_4, $(D_4, \varpi_1, \varpi_4)$ is isomorphic to $(D_4, \varpi_3, \varpi_4)$. Now, $(D_4, \varpi_3, \varpi_4)$ is isomorphic to $(B_4, \varpi_4) = (D_5, \varpi_5)$. □

Now Theorem 1.2 follows from Theorem 1.1 and Proposition 5.1.

Remark. (B_n, ϖ_n) is isomorphic to $(B_n, 2\varpi_n)$, but the embeddings are different: the latter is the Veronese re-embedding of the former. Thus, $(B_n, \varpi_{n-1}, \varpi_n)$ is not isomorphic to $(B_n, \varpi_{n-1}, 2\varpi_n)$. Using the method described above, one can show that $(B_n, \varpi_{n-1}, 2\varpi_n)$ is isomorphic to $F_{n-1}(\widehat{\mathbb{Q}^{2n-1}})$, which is singular because $\widehat{\mathbb{Q}^{2n-1}}$ is singular.

In Proposition 5.2(2) we prove that the Fano variety $F_1(\widehat{\mathbb{S}})$ of the cone $\widehat{\mathbb{S}}$ over the rational homogeneous variety $\mathbb{S} = (D_{n+1}, \varpi_{n+1})$ is the horospherical variety $(D_{n+1}, \varpi_{n-1}, \varpi_{n+1})$. While proving this, we use the fact that $F_1(\mathbb{S})$ is (D_{n+1}, ϖ_{n-1}). The same arguments can be applied to $\mathbb{S} = (\mathbb{G}, \varpi)$ for any end ϖ of the Dynkin diagram of $\mathbb{G}$.

Proposition 5.6. *Let $\mathbb{S}$ be a rational homogeneous variety $(\mathbb{G}, \varpi)$. Assume that ϖ is an end of the Dynkin diagram of $\mathbb{G}$ and is not associated to a short simple root. Then, the Fano variety $F_1(\widehat{\mathbb{S}})$ of the cone $\widehat{\mathbb{S}}$ over $\mathbb{S}$ is the horospherical variety $(\mathbb{G}, \varpi^{(1)}, \varpi)$, where $\varpi^{(1)}$ is the node adjacent to ϖ.*

More generally, as we prove that the Fano variety $F_2(\widehat{\mathbb{S}})$ is the horospherical variety $(E_6, \varpi_4, \varpi_5)$ for $\mathbb{S} = (E_6, \varpi_6)$ in Proposition 5.3(2), one can prove the following.

Proposition 5.7. *Let $\mathbb{S}$ be a rational homogeneous variety $(\mathbb{G}, \varpi)$. Assume that ϖ is an end of the Dynkin diagram of $\mathbb{G}$ and that ϖ is not associated to a short simple root. Let $(\varpi^{(r)}, \dots, \varpi^{(1)}, \varpi)$ be the branch of the Dynkin diagram of $\mathbb{G}$ with the end ϖ. Then, for $r \geq k \geq 1$, the Fano variety $F_k(\widehat{\mathbb{S}})$ of the cone $\widehat{\mathbb{S}}$ over $\mathbb{S}$ is the horospherical variety $(\mathbb{G}, \varpi^{(k)}, \varpi^{(k-1)})$. Here, we set $\varpi^{(0)} = \varpi$.*

If, furthermore, there is another rational homogeneous variety $\mathcal{S}$ such that the variety $\mathcal{C}_x(\mathcal{S})$ of minimal rational tangents of $\mathcal{S}$ at the base point $x \in \mathcal{S}$ is isomorphic to $\mathbb{S}$, then $F_k(\mathcal{S}) =: S$ is a rational homogeneous variety of Picard number one, and the horospherical variety $(\mathbb{G}, \varpi^{(k)}, \varpi^{(k-1)})$ can be embedded into S as a linear section for $r \geq k \geq 1$.

Proof. For the first statement, use the same arguments as those in the proof of Proposition 5.3. For the second statement, use the same arguments as those in the proof of Proposition 5.1. □

Remark. If ϖ is an end of the Dynkin diagram of $\mathbb{G}$ and ϖ corresponds to a short simple root, then $(\mathbb{G}, \varpi)$ is either (B_n, ϖ_n), (C_n, ϖ_1), (F_4, ϖ_4), or (G_2, ϖ_1). In these cases, $(\mathbb{G}, \varpi)$ can be considered as (D_{n+1}, ϖ_{n+1}), (A_{2n-1}, ϖ_1), a hyperplane section of (E_6, ϖ_6), and a hyperplane section of (D_4, ϖ_4), respectively. Let $\mathbb{S}'$ be the latter homogeneous variety. Then, for the branch $(\varpi^{(r)}, \ldots, \varpi^{(1)}, \varpi)$ of the Dynkin diagram of $\mathbb{G}$ with the end ϖ, the horospherical variety $(\mathbb{G}, \varpi^{(k)}, \varpi^{(k-1)})$ is a linear section of $F_k(\widehat{\mathbb{S}'})$, and $F_k(\widehat{\mathbb{S}'})$ can be embedded into a rational homogeneous variety of Picard number one, if $\mathbb{S}'$ is isomorphic to the variety $\mathcal{C}_x(\mathcal{S})$ of minimal rational tangents of $\mathcal{S}$ at the base point $x \in \mathcal{S}$. These are the embeddings we constructed in Proposition 5.1, except the embedding of $(F_4, \varpi_3, \varpi_4)$ into (E_7, ϖ_6). This is not included in the list of Proposition 5.1 because $(F_4, \varpi_3, \varpi_4)$ is singular.

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