J. Korean Math. Soc. **53** (2016), No. 2, pp. 381–401
http://dx.doi.org/10.4134/JKMS.2016.53.2.381

# ON ANNIHILATIONS OF IDEALS IN SKEW MONOID RINGS

Rasul Mohammadi, Ahmad Moussavi, and Masoome Zahiri

Abstract. According to Jacobson [31], a right ideal is bounded if it contains a non-zero ideal, and Faith [15] called a ring strongly right bounded if every non-zero right ideal is bounded. From [30], a ring is strongly right $AB$ if every non-zero right annihilator is bounded. In this paper, we introduce and investigate a particular class of McCoy rings which satisfy Property $(A)$ and the conditions asked by Nielsen [42]. It is shown that for a u.p.-monoid $M$ and $\sigma : M \to \mathrm{End}(R)$ a compatible monoid homomorphism, if $R$ is reversible, then the skew monoid ring $R * M$ is strongly right $AB$. If $R$ is a strongly right $AB$ ring, $M$ is a u.p.-monoid and $\sigma : M \to \mathrm{End}(R)$ is a weakly rigid monoid homomorphism, then the skew monoid ring $R * M$ has right Property $(A)$.

## 1. Introduction

Throughout this article, all rings are associative with identity. Recall that a monoid $M$ is called a u.p.-monoid (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely presented in the form $ab$ where $a \in A$ and $b \in B$. Unique product monoids and groups play an important role in ring theory, for example providing a positive case in the zero-divisor problem for group rings (see also [6]), and their structural properties have been extensively studied (see [17]). The class of u.p.-monoids includes the right and the left totally ordered monoids, submonoids of a free group, and torsionfree nilpotent groups. Every u.p.-monoid $S$ is cancellative and has no non-unity element of finite order.

Let $R$ be a ring, let $M$ be a monoid and let $\sigma : M \to \mathrm{End}(R)$ a monoid homomorphism. For any $g \in M$, we denote the image of $g$ under $\sigma$ by $\sigma_g$. Then we can form a skew monoid ring $R*M$ (induced by the monoid homomorphism $\sigma$) by taking its elements to be finite formal combinations $\sum_{g\in M} a_g g$ with multiplication induced by $(a_g g)(b_h h) = a_g \sigma_g(b_h) gh$.

Received February 22, 2015.
2010 *Mathematics Subject Classification.* 16D25, 16D70, 16S34.
*Key words and phrases.* skew monoid ring, McCoy ring, strongly right $AB$ ring, nil-reversible ring, CN ring, rings with Property $(A)$, zip ring.

According to Jacobson [31], a right ideal of $R$ is *bounded* if it contains a non-zero ideal of $R$. From [18], a ring $R$ is *right* (*left*) *duo* if every right (left) ideal is an ideal, and Faith [13] said a ring would be called *strongly right bounded* if every non-zero right ideal were bounded. The class of strongly bounded rings has been observed by many authors (e.g. [7], [31], [48], [49]).

Due to H. Bell [5], a ring $R$ is said to have the *insertion of factors property* (simply, *IFP*) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Note that a ring $R$ has $IFP$ if and only if any right (or left) annihilator is an ideal. Rings with $IFP$ are also called *semicommutative*, see [11]. Right (resp. left) duo rings are both strongly right (resp. left) bounded and semicommutative.

In [30], S. U. Hwang, N. K. Kim and Y. Lee introduced a condition that is a generalization of strongly bounded rings and semicommutative rings, calling a ring *strongly right AB* if every non-zero right annihilator is bounded. An element $c$ of $R$ is called right regular if $r_R(c) = 0$, left regular if $l_R(c) = 0$ and regular if $r_R(c) = 0 = l_R(c)$.

According to [7], a ring $R$ is called 2-*primal* if the prime radical of $R$ and the set of nilpotent elements of $R$ coincide. Another property between commutative and 2-primal is what Cohn in [10] calls *reversible* rings: those rings $R$ with the property that $ab = 0 \Rightarrow ba = 0$ for all $a, b \in R$. We direct the reader to the excellent papers [1] and [38] for a nice introduction to some standard zero-divisor conditions.

There is another important ring theoretic condition common in the literature related to the zero divisor and annihilator conditions we have been studying. Neilsen in [42], calls a ring $R$ *right McCoy* (resp. *left McCoy*) if for each pair of non-zero polynomial $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, then there exists a non-zero element $r \in R$ with $f(x)r = 0$ (resp. $rg(x) = 0$). Neilsen [42] asked whether there is a natural class of McCoy rings which includes all reversible rings and all rings $R$ such that $R[x]$ is semicommutative. We use this to define a new class of rings strengthening the condition for reversible rings. This property between "reversible" and "McCoy" is what we call nil-reversible rings. We say a ring $R$ is *nil-reversible*, if $ab = 0 \Leftrightarrow ba = 0$, where $b \in \mathrm{nil}(R)$.

An important theorem in commutative ring theory, related to zero-divisor conditions, is that if $I$ is an ideal in a Noetherian ring and if $I$ consists entirely of zero divisors, then the annihilator of $I$ is nonzero. This result fails for some non-Noetherian rings, even if the ideal $I$ is finitely generated. Huckaba and Keller [29], say that a commutative ring $R$ has *Property* ($A$) if every finitely generated ideal of $R$ consisting entirely of zero divisors has nonzero annihilator. Many authors have studied commutative rings with Property ($A$) ([3], [22], [28], [29], [36], [45], etc.), and have obtained several results which are useful studying commutative rings with zero-divisors. Hong, Kim, Lee and Ryu [27] extended Property ($A$) to noncommutative rings, and study such rings and several extensions with Property ($A$).

In this paper, we investigate a particular class of McCoy rings which satisfy Property ($A$) and the conditions asked by Nielsen [42]. Whenever the skew monoid ring $R * M$ is strongly right $AB$ and $r_{R*M}(Y) \neq 0$, then $r_R(Y) \neq 0$, for any $Y \subseteq R * M$. We then conclude that, nil-reversible rings is a larger class than the class asked by Nielsen [42], and satisfies the conditions. Indeed, nil-reversible rings is a natural class of McCoy rings which includes reversible rings, all rings $R$ such that $R[x]$ is strongly right (or left) $AB$ (and hence all rings $R$ such that $R[x]$ is semicommutative).

We prove that for a u.p.-monoid $M$ and $\sigma : M \to \mathrm{End}(R)$ a compatible monoid homomorphism, if $R$ is nil-reversible, then the skew monoid ring $R*M$ is strongly right $AB$. If $R$ is strongly right $AB$, $M$ a u.p.-monoid and $\sigma : M \to \mathrm{End}(R)$ a weakly rigid monoid homomorphism, then $R*M$ has right Property $(A)$.

It is also shown that, when $M$ is a u.p.-group and $\sigma : M \to \mathrm{Aut}(R)$ is a group homomorphism such that the ring $R$ is $M$-compatible and right duo, then $R$ is right skew $M$-McCoy. Also, if $R * M$ is strongly right $AB$, then $R$ is right skew $M$-McCoy and $R * M$ has right Property $(A)$. Whenever $R$ is strongly right $AB$ and skew $M$-Armendariz, then $R * M$ is strongly right $AB$. Moreover, if $R$ is strongly right $AB$ and right skew $M$-McCoy, then $R*M$ has right Property $(A)$.

Whenever $R$ is a right duo ring and $\sigma_i$ is a compatible automorphism of $R$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$, then $R$ is right skew McCoy. This implies that, if $R$ is a right duo ring, then the rings $R[x_1, x_2, \ldots, x_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ have right Property $(A)$, which also gives an answer to a question asked in [27].

For any non-empty subset $X$ of $R$, annihilators will be denoted by $r_R(X)$ and $l_R(X)$. We write $Z_l(R)$, $Z_r(R)$ for the set of all left zero-divisors of $R$ and the set of all right zero-divisors of $R$. The set of all nilpotent elements of $R$ are denoted by $\mathrm{nil}(R)$. For any $\alpha = a_1g_1 + \cdots + a_mg_m \in R * M$ ($a_i \neq 0$ for each $i$), we call $m$, the *length* of $\alpha$ and we denote by $C_\alpha$ the set of all coefficients of $\alpha$.

## 2. Rings whose right annihilators are bounded

According to Jacobson [31], a right ideal of $R$ is *bounded* if it contains a non-zero ideal of $R$. This concept has been extended in several ways. From Faith [13], a ring $R$ is called *strongly right* (resp. *left*) *bounded* if every non-zero right (resp. left) ideal of $R$ contains a non-zero ideal. A ring is called *strongly bounded* if it is both strongly right and strongly left bounded. Right (resp. left) duo rings are strongly right (resp. left) bounded and semicommutative. Birkenmeier and Tucci [7, Proposition 6] showed that a ring $R$ is right duo if and only if $R/I$ is strongly right bounded for all ideals $I$ of $R$.

A ring $R$ is called *right* (resp. *left*) $AB$ if every essential right (resp. left) annihilator of $R$ is bounded.

**Definition 2.1** ([30])**.** A ring $R$ is called *strongly right* (resp. *left*) $AB$ if every non-zero right (resp. left) annihilator of $R$ is bounded; $R$ is called *strongly* $AB$ if $R$ is strongly right and strongly left $AB$.

Obviously strongly right bounded rings and semicommutative rings are both strongly right $AB$, but the converses need not be true (see [30]).

**Definition 2.2.** We say a ring $R$ is *nil-reversible*, if for every $a \in R$, $b \in \mathrm{nil}(R)$, $ab = 0 \Leftrightarrow ba = 0$.

Reversible rings are clearly nil-reversible. In [39] the authors called a ring $R$ *nil-semicommutative* if for every $a, b \in \mathrm{nil}(R), ab = 0$ implies $aRb = 0$. Obviously, every nil-reversible ring is nil-semicommutative, so nil-reversible rings form a proper subclass of the class of 2-primal rings, by [39, Lemma 2.7].

According to Krempa [34], an endomorphism $\alpha$ of a ring $R$ is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring $R$ is said to be $\alpha$-*rigid* if there exists a rigid endomorphism $\alpha$ of $R$. In [21], the authors introduced $\alpha$-compatible rings and studied their properties. A ring $R$ is $\alpha$-*compatible* if for each $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. Basic properties of rigid and compatible endomorphisms, proved by Hashemi and the second author in [21, Lemmas 2.2 and 2.1].

Let $R$ be a ring, $M$ a monoid and $\sigma : M \to \mathrm{End}(R)$ a monoid homomorphism. The ring $R$ is called $M$-*compatible* if $\sigma_g$ is compatible for every $g \in M$.

The following lemma which appeared in [21, Lemma 3.2] will be helpful in the sequel.

**Lemma 2.3.** *Let $R$ be an $\alpha$-compatible ring. Then we have the following*:

(1) *If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for all positive integers $n$.*
(2) *If $\alpha^k(a)b = 0$ for some positive integer $k$, then $ab = 0$.*
(3) *If $ab \in \mathrm{nil}(R)$, then $a\alpha(b) \in \mathrm{nil}(R)$ for all $a, b \in R$.*

**Lemma 2.4.** *Let $R$ be a ring, $M$ a u.p.-monoid and $\sigma : M \to \mathrm{End}(R)$ a compatible monoid homomorphism. Then we have the following*:

(1) *$ab \in \mathrm{nil}(R) \Leftrightarrow a\sigma_g(b) \in \mathrm{nil}(R)$ for all $a, b \in R$ and all $g \in M$;*
(2) *$abc = 0 \Leftrightarrow a\sigma_g(b)c = 0$ for all $a, b, c \in R$ and all $g \in M$.*

*Proof.* The proof is similar to the proof of [44, Lemma 2.4]. □

**Lemma 2.5** ([19, Theorem 4.4])**.** *Let $R$ be a 2-primal ring, let $M$ be a u.p-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. Then $\mathrm{nil}(R * M) = \mathrm{nil}(R) * M$.*

**Theorem 2.6.** *Let $R$ be a nil-reversible ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. Then $R * M$ is a strongly $AB$ ring.*

*Proof.* We prove the right case, the proof of the left case is similar. Suppose $X \subseteq R * M$ and $r_{R*M}(X) \neq 0$. Let $X\beta = 0$, for some $\beta = b_1h_1 + b_2h_2 + \cdots + b_nh_n \in R * M$ with minimal length.

*Case* 1: $\beta \in \mathrm{nil}(R * M)$. We show that $Xb_j = 0$ for every $1 \leq j \leq n$. Assume, on the contrary, that $Xb_k \neq 0$ for some $1 \leq k \leq n$. Then there exists $\alpha \in X$ such that $\alpha b_k \neq 0$, where $\alpha = a_1g_1 + \cdots + a_mg_m$. On the other hand we have $\alpha\beta = 0$. Since $M$ is a u.p.-monoid, there exist $i, j$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ such that $g_i h_j$ is uniquely presented by considering two subsets $A = \{g_1, \ldots, g_m\}$ and $B = \{h_1, \ldots, h_n\}$ of $M$. We may assume without loss of generality that $i = j = 1$. Then $a_1\sigma_{g_1}(b_1) = 0$, as $g_1h_1$ is uniquely presented. Since $R$ is $M$-compatible nil-reversible and by Lemma 2.4, $b_1a_1 = 0$. Now take $\overline{\beta}$ to be $\beta a_1$. But $\overline{\beta}$ has $n-1$ terms and $\alpha\overline{\beta} = 0$, which contradicts to our assumption that $\beta$ has minimal length such that $\alpha\beta = 0$, thus $\overline{\beta} = 0$. By nil-reversibility of $R$, $a_1\beta = 0$, and from $\alpha\beta = 0$ we get $(a_2g_2+\cdots+a_mg_m)(b_1h_1+b_2h_2+\cdots+b_nh_n) = 0$. Continuing in this way we can show that $a_i\beta = 0$ for each $1 \leq i \leq m$, which contradicts with our assumption that $\alpha b_k \neq 0$. Thus $Xb_j = 0$, $1 \leq j \leq n$, and this implies $XRb_j = 0$, as $\mathrm{nil}(R * M) = \mathrm{nil}(R) * M$ by Lemma 2.5 and $R$ is $M$-compatible nil-reversible. So for every $\gamma = c_1l_1+\cdots+c_kl_k \in R*M$, $Xc_t\sigma_{l_t}(b_j) = 0$, $1 \leq j \leq n$, $1 \leq t \leq k$. Therefore $X(R * M)b_j = 0$, and so $R * M$ is strongly right $AB$.

*Case* 2: $\beta \notin \mathrm{nil}(R * M)$. Then we have two subcases:

(i) $\beta C_X \neq 0$ (we denote by $C_X$ the set of all coefficients of elements of $X$). In this case there exists $a \in C_X$ such that $\beta a \neq 0$. Then there exists $\gamma = c_1l_1 + \cdots + c_kl_k \in X$ with $a \in C_\gamma$. From $X\beta = 0$, we get $\gamma\beta = 0$. Since $\mathrm{nil}(R)$ is an ideal of $R$, it is easy to see that $c_ib_j \in \mathrm{nil}(R)$, $1 \leq i \leq k$, $1 \leq j \leq n$. Hence $b_ja \in \mathrm{nil}(R)$ and that $\beta a \in \mathrm{nil}(R * M)$. As $X\beta = 0$, we have $X\beta a = 0$ and reduce to the previous case.

(ii) $\beta C_X = 0$. When $XRb_j = 0$ for some $1 \leq j \leq n$, there is nothing to prove. Now assume that $XRb_j \neq 0$ for all $1 \leq j \leq n$. Then for some $\alpha = a_1g_1+\cdots+a_mg_m \in X, \alpha Rb_j \neq 0$. So $a_kg_kr\beta \neq 0$ for some $r \in R$ and $1 \leq k \leq m$. Since $\beta C_X = 0$, we have $a_kg_kr\beta C_X = 0$. On the other hand $\beta a_k = 0$, as $\beta C_X = 0$. So $a_k\sigma_{g_k}(rC_\beta) \in \mathrm{nil}(R)$, and hence by nil-reversibility $C_Xa_kg_krC_\beta = 0$. Thus $C_XRa_krC_\beta = 0$, by nil-reversibility and $M$-compatibility of $R$. So for every $\gamma = c_1l_1+\cdots+c_kl_k \in R*M$, $C_Xc_t\sigma_{l_t}(a_krC_\beta) = 0$, $1 \leq t \leq k$. This shows that $X(R * M)a_kr\beta = 0$ and we are done. □

Clearly every nil-reversible ring is strongly $AB$.

**Corollary 2.7.** *Let $R$ be a nil-reversible ring and let $\sigma_i$ be a compatible endomorphism of $R$. Assume that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Then the skew polynomial rings $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}; \sigma_1, \ldots, \sigma_n]$ are strongly $AB$.*

**Corollary 2.8.** *If $R$ is a nil-reversible ring, then the rings $R[x_1, x_2, \ldots, x_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ are strongly $AB$.*

**Corollary 2.9.** *Let $R$ be a reversible ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. Then $R * M$ is a strongly $AB$ ring.*

Right (resp. left) duo rings are both strongly right (resp. left) bounded and semicommutative. By M. P. Darzin [12] a ring $R$ is a *$CN$-ring* whenever every nilpotent element of $R$ is central. D. Khurana et al. [33], introduced the notion of *unit-central* rings (i.e., every invertible element of it lies in center), and show that each unit-central ring is a $CN$-ring. It is clear that $CN$-rings and reversible rings are nil-reversible.

**Corollary 2.10.** *Let $R$ be a $CN$-ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. Then $R * M$ is a strongly AB ring.*

**Corollary 2.11.** *Let $R$ be a $CN$-ring, let $\sigma_i$ be a compatible endomorphism of $R$ such that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Then the rings $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}; \sigma_1, \ldots, \sigma_n]$ are strongly AB.*

**Corollary 2.12.** *If $R$ is a $CN$-ring, then the polynomial rings $R[x_1, x_2, \ldots, x_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ are strongly AB.*

A ring $R$ is called *Armendariz* if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $i, j$. This definition was given by Rege and Chhawchharia in [46] using the name Armendariz since E. Armendariz had proved in [2] that reduced rings satisfied this condition. Also, by Anderson and Camillo [1, Theorem 4], a ring $R$ is Armendariz if and only if so is $R[x]$.

**Definition 2.13** ([19, Definition 3.1])**.** Let $R$ be a ring, let $M$ be a monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism. A ring $R$ is called *skew $M$-Armendariz*, if whenever elements $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R * M$ satisfy $\alpha\beta = 0$, then $a_i\sigma_{g_i}(b_j) = 0$ for each $i, j$.

**Proposition 2.14.** *Let $R$ be a skew $M$-Armendariz ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism. Then $R$ is reversible and $M$-compatible if and only if $R * M$ is reversible.*

*Proof.* Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ and $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m$ be non-zero elements of $R * M$, such that $\alpha\beta = 0$. Since $R$ is $M$-Armendariz, we have $a_i\sigma_{g_i}(b_j) = 0$. By $M$-compatibility of $R$, $a_ib_j = 0$. Since $R$ is reversible and $M$-compatible, $b_j\sigma_{h_j}(a_i) = 0$. So $\beta\alpha = 0$ and $R * M$ is reversible. The converse is clear since reversible rings are closed under subring. It follows from [37, Lemma 4.4(ii)] that if $R * M$ is reversible, then $R$ is $M$-compatible. $\square$

**Corollary 2.15.** *Let $R$ be a $\sigma_i$-skew Armendariz ring and let $\sigma_i$ be a compatible endomorphism of $R$ for each $i$. Assume that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Then $R$ is reversible if and only if the skew polynomial ring $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ is reversible if and only if $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, \sigma_1, \ldots, \sigma_n]$ is reversible.*

**Definition 2.16.** Let $R$ be a ring, let $M$ be a monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism. We say a ring $R$ is *right skew $M$-McCoy* if whenever $0 \neq \alpha = a_1g_1 + \cdots + a_ng_n, 0 \neq \beta = b_1h_1 + \cdots + b_mh_m \in R * M$ with $g_i, h_j \in M$, $a_i, b_j \in R$ satisfy $\alpha\beta = 0$, then $\alpha r = 0$ for some nonzero $r \in R$. *Left skew $M$-McCoy* rings are defined similarly. If $R$ is both left and right skew $M$-McCoy, then we say $R$ is *skew $M$-McCoy.*

**Theorem 2.17.** *Let $R$ be a ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{Aut}(R)$ be a monoid homomorphism. If $S = R * M$ is strongly right AB and $r_S(Y) \neq 0$, then $r_R(Y) \neq 0$ for any $Y \subseteq S$.*

*Proof.* Suppose $Y \subseteq R * M$ and $r_{R*M}(Y) \neq 0$. Let $Y'$ be the right ideal of $R * M$ generated by $Y$. Since $R * M$ is strongly right $AB$ and $r_{R*M}(Y) \neq 0$; it follows that $r_{R*M}(Y') \neq 0$. Hence by [26, Theorem 3] we have $r_R(Y') \neq 0$ and thus $r_R(Y) \neq 0$. □

**Corollary 2.18.** *Let $R$ be a ring and let $\sigma_i$ be an automorphism of $R$ such that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Let $S$ be either $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ or $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}; \sigma_1, \ldots, \sigma_n]$. If $S$ is strongly right AB and $r_S(Y) \neq 0$, then $r_R(Y) \neq 0$ for any $Y \subseteq S$.*

**Corollary 2.19.** *Let $R$ be a ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{Aut}(R)$ be a monoid homomorphism. If $S = R*M$ is nil-reversible and $r_S(Y) \neq 0$, then $r_R(Y) \neq 0$ for any $Y \subseteq S$.*

**Corollary 2.20.** *Let $R$ be a ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{Aut}(R)$ be a monoid homomorphism. If $S = R * M$ is a reversible ring and $r_S(Y) \neq 0$, then $r_R(Y) \neq 0$ for any $Y \subseteq S$.*

**Corollary 2.21.** *Let $R$ be a ring and let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{Aut}(R)$ be a monoid homomorphism. If $R * M$ is a strongly right AB ring, then $R$ is a right skew $M$-McCoy ring.*

**Corollary 2.22.** *If the polynomial ring $R[x]$ is strongly right AB, then $R$ is a right McCoy ring.*

**Corollary 2.23.** *The class of McCoy rings includes nil-reversible rings and all rings $R$ such that $R[x]$ is strongly right AB.*

Therefore, we conclude that, nil-reversible rings is a larger class of rings which satisfy the conditions asked by Nielsen [42]. Indeed, nil-reversible rings is a natural class of McCoy rings which includes reversible rings, $CN$ rings, all rings $R$ such that $R[x]$ is strongly right (or left) $AB$ (and hence all rings $R$ such that $R[x]$ is semicommutative).

**Lemma 2.24.** *Let $M$ be a u.p.-monoid, $|M| \geq 2$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in R * M$. Then there exist $k_1, k_2, \ldots, k_n \in M$ such that $\alpha_1k_1, \ldots, \alpha_nk_n$ are distinct.*

*Proof.* Let $\alpha_i = a_{i1}h_1^{(i)} + a_{i2}h_2^{(i)} + \cdots + a_{ini}h_{ni}^{(i)}$ and put $A_i = \{h_1^{(i)}, h_2^{(i)}, \ldots, h_{ni}^{(i)}\}$ for $1 \leq i \leq n$. Let $1 \leq t \leq n$ be maximal number such that $A_1, A_2, \ldots, A_t$ are disjoint. In this case we take, $k_1 = k_2 = \cdots = k_t = e_M$. We need to assume that

$|A_i| \geq 2$ for each $i$. Now take $e_M \neq h_s^{(t+1)} \in A_{t+1}, 1 \leq s \leq n(t+1)$. We claim that there exists a positive integer $m_{t+1}$ such that $A_1, A_2, \ldots, A_{t+1}(h_s^{(t+1)})^{m_{t+1}}$ are disjoint, otherwise for some $h_j^{(t+1)} \in A_{t+1}$ we get $h_j^{(t+1)}(h_s^{(t+1)})^k \in \bigcup_{i=1}^t A_i$, for all positive integers $k$. This follows a contradiction, since by [6, Lemma 1.1], u.p.-monoids are cancellative. So $h_i^{(t+1)}(h_s^{(t+1)})^{pl_i} \notin \bigcup_{i=1}^t A_i$ for some positive integers $l_i$ and each $p \in \mathbb{N}$, $1 \leq i \leq n(t+1)$. Therefore $A_1, A_2, \ldots, A_t, A_{t+1}k_{t+1}$ are disjoint, where $k_{t+1} = (h_s^{(t+1)})^{m_{t+1}}$ with $m_{t+1} = l_1 l_2 \cdots l_{n(t+1)}$. By a similar method as above there exist $k_{t+2}, \ldots, k_n \in M$ such that $\alpha_1 k_1, \ldots, \alpha_n k_n$ are distinct. □

**Lemma 2.25.** *Let $R$ be a semicommutative ring, let $M$ be a u.p.-group and $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. Assume that $\alpha = a_1g_1 + a_2g_2 + \cdots + a_mg_m$ and $\beta = b_1h_1 + \cdots + b_nh_n$ are non-zero elements of $R * M$ such that $\alpha\beta = 0$. Then there exists $a_i \in C_\alpha$ such that $a_i^n\beta = 0$.*

*Proof.* Let $\alpha = a_1g_1 + \cdots + a_mg_m$ and $\beta = b_1h_1 + \cdots + b_nh_n$ be non-zero elements of $R * M$. Since $M$ is a u.p.-group, there exist $i, j, t, k$ with $1 \leq i \leq m$ and $1 \leq t, j, k \leq n$ such that $g_ih_th_j^{-1}h_k$ is uniquely presented by considering two subsets $A = \{g_1, \ldots, g_m\}$ and $B = \{h_th_j^{-1}h_k : 1 \leq t, j, k \leq n\}$ of $M$. We may assume without loss of generality that $i = j = k = t = 1$. Hence $a_1\sigma_{g_1}(b_1) = 0$. Since $R$ is $M$-compatible, we have $a_1b_1 = 0$. By semicommutativity of $R$, $0 = a_1\alpha\beta = (a_1^2g_1 + \cdots + a_1a_mg_m)(b_1h_1 + \cdots + b_nh_n) = (a_1^2g_1 + \cdots + a_1a_mg_m)(b_2h_2 + \cdots + b_nh_n)$. Since $M$ is u.p.-group, $0 = (a_1^2g_1 + \cdots + a_1a_mg_m)(b_2h_2 + \cdots + b_nh_n)h_2^{-1}h_1 = (a_1^2g_1 + \cdots + a_1a_mg_m)(b_2h_1 + \cdots + b_nh_nh_2^{-1}h_1)$. Since $g_1h_1$ is uniquely presented, $a_1^2b_2 = 0$. Then by a similar argument as above and since $R$ is semicommutative, we have $0 = a_1^2\alpha\beta = (a_1^3g_1 + \cdots + a_1^2a_mg_m)(b_1h_1 + \cdots + b_nh_n) = (a_1^3g_1 + \cdots + a_1^2a_mg_m)(b_3h_3 + \cdots + b_nh_n)$. Continuing this process, we can deduce that $a_1^n\beta = 0$. We consider $i = 1$, so there exist $a_i \in C_\alpha$ such that $a_i^n\beta = 0$. □

**Corollary 2.26.** *Let $R$ be a semicommutative ring and $\alpha$ be an automorphism of $R$. Assume that $R$ is $\alpha$-compatible. Also let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x \cdots + a_mx^m$ be non-zero elements of $R[x, x^{-1}; \alpha]$ or $R[x; \alpha]$, such that $f(x)g(x) = 0$. Then there exists $a_i \in C_f$ such that $a_i^m g(x) = 0$.*

In [27], Hong, Kim and Lee, asked a question that, if $R$ is a right duo ring does $R[x]$ have right Property $(A)$? In Corollary 3.12, we show that, if $R$ is a right duo ring, then the rings $R[x_1, x_2, \ldots, x_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ have right Property $(A)$. We need the following result which is a generalization of [8, Theorem 8.2] to the more general setting:

**Theorem 2.27.** *Let $R$ be a ring, let $M$ be a u.p.-group and let $\sigma : M \to \mathrm{Aut}(R)$ be a group homomorphism such that the ring $R$ is $M$-compatible. If $R$ is a right duo ring, then $R$ is right skew $M$-McCoy.*

*Proof.* We apply the method of Camillo and Nielsen in the proof of [8, Theorem 8.2]. For every $\gamma \in R * M$ we let $I_\gamma$ denote the right ideal generated by the coefficients of $\gamma$. Suppose $\alpha, \beta \in R * M$ with $\alpha\beta = 0$ and $\beta \neq 0$. We will prove, by induction on the length of $\alpha$, that there is some non-zero element in $I_\beta$ which annihilates $\alpha$ on the right. Write $\alpha = \sum_{i=1}^{m} a_i g_i$ and $\beta = \sum_{j=1}^{n} b_j h_j$. First, if $\alpha = a_1 g_1$, then

$$(a_1 g_1)(b_1 h_1 + b_2 h_2 + \cdots + b_n h_n) = 0.$$

Since $M$ is a u.p.-group, there exist $i, j, k$, such that $g_1 h_i h_j^{-1} h_k$ is uniquely presented by considering two subsets $A = \{g_1\}$ and $B = \{h_i h_j^{-1} h_k : 1 \leq i, j, k \leq n\}$ of $M$. We may assume without loss of generality that $i = j = k = 1$. Hence $a_1 \sigma_{g_1}(b_1) = 0$. So $0 = (a_1 g_1)(b_1 h_1 + b_2 h_2 + \cdots + b_n h_n) = (a_1 g_1)(b_2 h_2 + \cdots + b_n h_n)$. Since $M$ is a u.p.-group, $0 = (a_1 g_1)(b_2 h_2 + \cdots + b_n h_n) h_2^{-1} h_1 = (a_1 g_1)(b_2 h_1 + \cdots + b_n h_n h_2^{-1} h_1)$. Since $g_1 h_1$ is uniquely presented, $a_1 \sigma_{g_1}(b_2) = 0$. Continuing this process and since $R$ is $M$-compatible, we have $a_1 b_j = 0$, for each $j$. Let length$(\alpha) \geq 2$.

Since duo rings are semicommutative, there exists $1 \leq l \leq m$ satisfying $a_l^{n-1} b_j \neq 0 = a_l^n b_j$ by Lemma 2.25. We then have the following.

Case (i): Suppose $a_l \beta = 0$. This implies $a_l I_\beta = 0$. In this case we set $\alpha_1 = \alpha - a_l g_l$ and find $\alpha_1 \beta = 0$. But length$(\alpha_1) <$ length$(\alpha)$, and hence by induction there exists a non-zero element $b \in I_\beta$ satisfying $\alpha_1 b = 0$, whence $\alpha b = 0$.

Case (ii): Suppose $a_l \beta \neq 0$. Then $a_l b_t = 0$ for some $b_t \in C_\beta$. Let $a_l b_j \neq 0$. Since $R$ is right duo, there exists $r \in R$ with $a_l^{n-1} b_j = b_j r$. If we let $\beta_1 = \beta \sigma_{g_j}^{-1}(r)$, then clearly $\alpha \beta_1 = 0$, and $(0) \neq I_{\beta_1} \subseteq I_\beta$. This means we can replace $\beta$ by $\beta_1$ without any loss of generality. By construction, $a_l$ annihilates the first $j$ coefficients of $\beta_1$, so after repeating this process a finite number of times we reduce to the previous case. □

Let $S$ be either the skew polynomial ring $R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ or the skew Laurent polynomial ring $R[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, \sigma_1, \ldots, \sigma_n]$. We say the ring $R$ is *skew McCoy*, when the equation $f(x)g(x) = 0$ over $S$, where $f(x), g(x) \neq 0$ implies there exists a nonzero $r \in R$ with $f(x)r = 0$.

**Corollary 2.28.** *Let $R$ be a right duo ring, let $\sigma_i$ be a compatible automorphism of $R$ such that $\sigma_i \sigma_j = \sigma_j \sigma_i$ for each $i, j$. Then $R$ is skew McCoy.*

**Corollary 2.29** ([8, Theorem 8.2])**.** *Every right duo ring is right McCoy.*

**Theorem 2.30.** *Let $R$ be a $M$-compatible ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism. If $R$ is skew $M$-Armendariz and strongly right $AB$, then $R * M$ is strongly right $AB$.*

*Proof.* We adopt the proof of [30, Proposition 4.6]. Assume $R$ is strongly right $AB$ and $X \subseteq R * M$ with $r_{R*M}(X) \neq 0$ and let $C$ be the set of all coefficients of the elements of $X$. Take non-zero $\alpha = a_1 g_1 + a_2 g_2 + \cdots + a_n g_n \in r_{R*M}(X)$.

Then for any $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in X, \beta\alpha = 0$. Since $R$ is skew $M$-Armendariz and $M$-compatible, $b_ia_j = 0$ for all $i, j$. Thus $a_j \in r_R(C)$, $1 \leq j \leq n$, entailing $r_R(C) \neq 0$. Since $R$ is strongly right $AB$, there exists a non-zero ideal $I$ of $R$ such that $r_R(C) \supseteq I$. So $CRt = 0$ for each $0 \neq t \in I$. By $M$-compatibility of $R$, $X(R * M)t = 0$. Therefore $R * M$ is strongly right $AB$. □

**Proposition 2.31.** *Let $R$ be a ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. If $R * M$ is strongly right $AB$, then $R$ is right skew $M$-McCoy and strongly right $AB$.*

*Proof.* We adopt the proof of [30, Proposition 4.6]. Suppose that $S = R * M$ is strongly right $AB$. Let $X \subseteq R$ with $r_R(X) \neq 0$. Note that $r_R(X) = r_S(X) \cap R$. Since $r_R(X) \neq 0$, we get $r_S(X) \neq 0$. But $S$ is strongly right $AB$, so there is a non-zero ideal $L$ of $S$ such that $r_S(X) \supseteq L$. For every $\gamma = c_1l_1 + c_2l_2 + \cdots + c_tl_t \in L$, $S\gamma S \subseteq L$. So $XR\gamma \subseteq XS\gamma = 0$. This implies that $XRc_k = 0, 1 \leq k \leq t$. So $r_R(X) \supseteq Rc_kR$. Therefore $R$ is strongly right $AB$. □

**Corollary 2.32.** *Let $R$ be a $\sigma_i$-skew Armendariz ring for each $i$ and let $\sigma_i$ be a compatible endomorphism of $R$ such that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Then $R$ is strongly right $AB$ if and only if $R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ is strongly right $AB$ if and only if $R[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, \sigma_1, \ldots, \sigma_n]$ is strongly right $AB$.*

**Corollary 2.33** ([30, Proposition 4.6])**.** *Let $R$ be an Armendariz ring. Then $R$ is strongly right $AB$ if and only if $R[x]$ is strongly right $AB$.*

Let $R$ be a ring and $\sigma$ denotes an endomorphism of $R$ with $\sigma(1) = 1$. We denote the identity matrix and unit matrices in the full matrix ring $M_n(R)$, by $I_n$ and $E_{ij}$, respectively. In [35], T. K. Lee and Y. Zhou introduced a subring of the *skew triangular matrix* ring as a set of all triangular matrices $T_n(R)$, with addition pointwise and a new multiplication subject to the condition $E_{ij}r = \sigma^{j-i}(r)E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + \cdots + a_{ij}\sigma^{j-i}(b_{jj})$ for each $i \leq j$ and denoted it, by $T_n(R, \sigma)$.

Let $R$ be a ring with an endomorphism $\sigma$. Consider the following subset of triangular matrices $T_n(R, \sigma)$, denoted by

$$T(R, n, \sigma) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ 0 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 0 & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 \end{pmatrix} | a_i \in R \right\}$$

with $n \geq 2$. It is easy to see that $T(R, n, \sigma)$ is a ring with matrix addition and multiplication. We can denote elements of $T(R, n; \sigma)$ by $(a_1, a_2, \ldots, a_n)$, then $T(R, n; \sigma)$ is a ring with addition pointwise and multiplication given by $(a_0, \ldots, a_{n-1})(b_0, \ldots, b_{n-1}) = (a_0b_0, a_0 * b_1 + a_1 * b_0, \ldots, a_0 * b_{n-1} + \cdots + a_{n-1} * b_0)$, with $a_i * b_j = a_i\sigma^i(b_j)$ for each $i$ and $j$. In the special case, when $\sigma = id_R$, we use $T(R,n)$ instead of $T(R,n,\sigma)$. On the other hand, there is a ring isomorphism $\varphi : R[x;\sigma]/(x^n) \to T(R,n,\sigma)$, given by $\varphi(\sum_{i=0}^{n-1} a_i x^i) = (a_0, a_1, \ldots, a_{n-1})$. So $T(R,n,\sigma) \cong R[x;\sigma]/(x^n)$, where $(x^n)$ is the ideal generated by $x^n$.

**Lemma 2.34.** *Let $R$ be a ring and $0 \neq a \in R$. Then $a$ is right (resp. left) regular if and only if $(a, a_1, a_2, \ldots, a_{n-1}) \in T(R,n,\sigma)$ is right (resp. left) regular.*

*Proof.* We adopt the proof of [30, Lemma 2.1]. Suppose that $a \in R$ is right regular, and let

$$A = (a, a_1, a_2, \ldots, a_{n-1}) \in T(R,n,\sigma).$$

We proceed by induction on $n$. Put $(a, a_1)(b, b_1) = 0$ for some $(b, b_1) \in T(R,n,\sigma)$. Then $ab = 0$ and $ab_1 + a_1\sigma(b) = 0$. Since $a$ is right regular, $b = 0$, and so $b_1 = 0$; hence $(a, a_1)$ is right regular. Next let

$$AB = 0, B = (b, b_1, b_2, \ldots, b_{n-1}) \in T(R,n,\sigma).$$

Then we get

$$(a, a_1, a_2, \ldots, a_{n-2})(b, b_1, b_2, \ldots, b_{n-2}) = 0.$$

By the induction hypothesis we obtain $b = 0$ and $b_j = 0$ for $1 \leq j \leq n-2$; hence $ab_{n-1} + a_1\sigma(b_{n-2}) + \cdots + a_{n-2}\sigma^{n-2}(b_1) + a_{n-1}\sigma^{n-1}(b) = 0, ab_{n-2} + a_1\sigma(b_{n-3}) + \cdots + a_{n-2}\sigma^{n-2}(b) = 0, \ldots, ab = 0$. Inductively we obtain $b = 0$ and $b_j = 0$ for $j = 1, 2, \ldots, n-1$, concluding that $A$ is right regular.

Conversely assume that $A$ is right regular, and let $ab = 0$ for some $b \in R$. Then from $A(bE_{1(n-1)}) = 0$ we have $b = 0$. Thus $a$ is regular. The proof of left case is similar. □

**Proposition 2.35.** *A ring $R$ is strongly right (resp. left) AB if and only if $T(R,n,\sigma)$ is strongly right (resp. left) AB for any $n \geq 2$.*

*Proof.* We apply the method of Hwang et al. in the proof of [30, Theorem 2.2]. Let $R$ be strongly right $AB$ and $X \subseteq T(R,n,\sigma)$ with $r_{T(R,n,\sigma)}(X) \neq 0$. Then any diagonal in matrices in $X$ is not right regular by Lemma 2.34. Let $Y$ be the set of all elements in $R$, which occur as diagonal entries of elements in $X$. If $Y = 0$, then $r_{T(R,n,\sigma)}(X)$ contains a non-zero ideal $RE_{1(n-1)}$ of $T(R,n,\sigma)$. Next we suppose $Y \neq 0$ and let $a$ be in $Y$. Take

$$0 \neq (b, b_1, \ldots, b_{n-1}) \in r_{T(R,n,\sigma)}(X).$$

We will show $r_R(Y) \neq 0$. If $b \neq 0$, then $r_R(Y) \neq 0$. Assume $b = 0$, since $(b, b_1, \ldots, b_{n-1}) \neq 0$, there exist $b_j \neq 0$, $1 \leq j \leq n-1$ and $ab_j = 0$. So $r_R(Y) \neq 0$.

Now since $R$ is strongly right $AB$, there is a non-zero ideal $I$ of $R$ with $I \subseteq r_R(Y)$. Then $r_{T(R,n,\sigma)}$ contains a non-zero ideal $IE_{1(n-1)}$ of $T(R,n,\sigma)$. Thus $T(R,n,\sigma)$ is strongly right $AB$.

Conversely suppose that $T(R,n,\sigma)$ is strongly right $AB$ and $V \subseteq R$ with $r_R(V) \neq 0$. Let $W = \{aI_n \mid a \in V\} \subseteq T(R,n,\sigma)$, where $I_n$ is the $n \times n$ identity matrix. Then $r_{T(R,n,\sigma)}(W) \neq 0$ because $WU = 0$ for any non-zero matrix $U$ in $T(R,n,\sigma)$ with entries in $r_R(V)$. Since $T(R,n,\sigma)$ is strongly right $AB$, there exists a non-zero ideal $J$ of $T(R,n,\sigma)$ such that $r_{T(R,n,\sigma)}(W) \supseteq J$. Now set $K = \{c \in R \,|\, c \text{ is an entry of a matrix in } J\}$. Then $K$ is a non-zero ideal of $R$ from the computations $(aI_n)(rI_n)C = 0$ for $a \in V$, $r \in R$ and $C \in J$. Moreover $aK = 0$ for all $a \in V$ from $(aI_n)J = 0$, entailing $r_R(X) \supseteq K$. Thus $R$ is strongly right $AB$. The proof of the left case is similar. □

**Example 2.36.** Let $R$ be a ring. Assume $\alpha$ and $\sigma$ are rigid endomorphisms such that $\alpha\sigma = \sigma\alpha$. Since $R$ is reduced, $T(R,n,\sigma)$ is strongly right $AB$ by Proposition 2.35 and by [20, Theorem 2.8 and Corollary 2.5], $T(R,n,\sigma)$ is skew Armendariz and $\overline{\alpha}$-compatible. So $T(R,n,\sigma)[x;\overline{\alpha}]$ is strongly right $AB$, by Corollary 2.32.

**Definition 2.37** ([9])**.** A ring $R$ is said to have the *right finite intersection property* (simply, *right FIP*) if, for any subset $X$ of $R$, there exists a finite subset $X_0$ of $X$ such that $r_R(X) = r_R(X_0)$.

**Proposition 2.38.** *Let $R$ be a ring, let $M$ be a u.p.-group with $|M| \geq 2$ and let $\sigma : M \to \mathrm{Aut}(R)$ be a group homomorphism such that the ring $R$ is $M$-compatible. If $R$ is right duo and $R * M$ has right FIP, then $R * M$ is strongly right $AB$.*

*Proof.* Assume that $X \subseteq R * M$ and $r_{R*M}(X) \neq 0$. Then there exists a finite subset $X_0$ of $X$ such that $r_{R*M}(X) = r_{R*M}(X_0)$, as $R * M$ has right $FIP$. Assume that $X_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, where $\alpha_i = a_{i1}g_{i1} + a_{i2}g_{i2} + \cdots + a_{in_i}g_{in_i}$, $1 \leq i \leq k$ with positive integers $n_i$. By Proposition 2.24, there exist $h_1, \ldots, h_k \in M$ such that $\alpha_1 h_1, \alpha_2 h_2, \ldots, \alpha_k h_k$, are distinct. Take $\alpha = \alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_k h_k$. Then for some $0 \neq \beta \in R * M$ we have $\alpha\beta = 0$. By Theorem 2.27, right duo rings are right skew $M$-McCoy, so there exists $0 \neq c \in R$ such that $a_{ij}c = 0$, $1 \leq i \leq k$, $1 \leq j \leq n_i$, as $R$ is $M$-compatible. Since $R$ is strongly right $AB$, there exists an ideal $J$ such that $a_{ij}J = 0$. For each $0 \neq d \in J$, we have $a_{ij}Rd = 0$, $1 \leq i \leq k$, $1 \leq j \leq n_i$. Thus we have $X_0(R{*}M)d = 0$, by $M$-compatibility of $R$. So $0 \neq (R{*}M)d(R{*}M) \subseteq r_{R*M}(X_0)$ and hence $R * M$ is strongly right $AB$. □

**Corollary 2.39.** *Let $R$ be a right duo ring and let $\sigma_i$ be a compatible automorphism of $R$ with $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Let $S$ be either $R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ or $R[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}; \sigma_1, \ldots, \sigma_n]$. If $S$ has right $IFP$, then $S$ is strongly right $AB$.*

Faith [15] called a ring $R$ *right zip* provided that if the right annihilator $r_R(X)$ of a subset $X$ of $R$ is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$. The concept of zip rings was initiated by Zelmanowitz [50] and appeared in various papers. Zelmanowitz stated that any ring satisfying the descending chain condition on right annihilators is a right zip ring, but the converse does not hold. Extensions of zip rings were studied by several authors. Beachy and Blair [4] showed that if $R$ is a commutative zip ring, then the polynomial ring $R[x]$ over $R$ is a zip ring. In [47, Theorem 2.4], Singh and et al., show that, $R$ is right zip if and only if $R*M$ is right zip.

**Proposition 2.40.** *Let $R$ be a right skew $M$-McCoy ring, let $M$ be a u.p.-monoid with $|M| \geq 2$ and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. If $R$ is strongly right $AB$ and $R*M$ is right zip, then $R*M$ is strongly right $AB$.*

*Proof.* Let $r_{R*M}(X) \neq 0$, where $X \subseteq R*M$. Assume, on the contrary, that $r_{R*M}(X(R*M)) = 0$. Since $R*M$ is right zip there exists a finite subset $X_0 = \{\alpha_1\gamma_1, \alpha_2\gamma_2, \ldots, \alpha_k\gamma_k\} \subseteq X(R*M)$, where $\alpha_i = a_{i1}g_{i1}+a_{i2}g_{i2}+\cdots+a_{in_i}g_{in_i} \in X$, $\gamma_i \in R*M$, $1 \leq i \leq k$, such that $r_{R*M}(X_0) = 0$. Since $\{\alpha_1, \ldots, \alpha_k\} \subseteq X$, we have $0 \neq r_{R*M}(X) \subseteq r_{R*M}(\{\alpha_1, \ldots, \alpha_k\})$. So $\{\alpha_1, \ldots, \alpha_k\}\beta = 0$ for some $0 \neq \beta \in R*M$. By Proposition 2.24, there exist $h_1, \ldots, h_k \in M$ such that $\alpha_1 h_1, \alpha_2 h_2, \ldots, \alpha_k h_k$ are distinct. Put $\alpha = \alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_k h_k$. Since $\alpha_i\beta = 0$ for every $1 \leq i \leq k$, we have $\alpha\beta = 0$. Since $R$ is right skew $M$-McCoy, there exists $0 \neq c \in R$ such that $\alpha c = 0$. Therefore $a_{ij}c = 0$, $1 \leq i \leq k$, $1 \leq j \leq n_i$, as $R$ is $M$-compatible. Since $R$ is strongly right $AB$, there exists an ideal $J$ such that $a_{ij}J = 0$ for every $i, j$. For every $0 \neq d \in J$, $a_{ij}Rd = 0$, $1 \leq i \leq k$, $0 \leq j \leq n_i$. Since $R$ is $M$-compatible, $\alpha_i(R*M)d = 0$. But this is a contradiction as we assumed $r_{R*M}(X_0) = 0$. □

**Corollary 2.41.** *Let $R$ be a skew McCoy, right zip and strongly right $AB$ ring and let $\sigma_i$ be a compatible endomorphism of $R$ with $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Then the rings $R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ and $R[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}; \sigma_1, \ldots, \sigma_n]$ are strongly right $AB$.*

**Corollary 2.42.** *Let $R$ be a right McCoy, right zip and strongly right $AB$ ring. Then the rings $R[x_1, \ldots, x_n]$ and $R[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ are strongly right $AB$.*

By Proposition 2.35, a ring $R$ is strongly right (resp. left) $AB$ if and only if $T(R, n, \sigma)$ is strongly right (resp. left) $AB$ for any $n \geq 2$. Now we provide more examples of noncommutative zip rings.

**Proposition 2.43.** *Let $R$ be a commutative ring with an endomorphisms $\sigma$ and let $n \geq 2$. Then $R$ is a zip ring if and only if $T(R, n, \sigma)$ is a zip ring.*

*Proof.* Assume that $R$ is a zip ring and $X \subseteq T(R, n, \sigma)$ with $r_{T(R,n,\sigma)}(X) = 0$. Let $Y$ be the set of all elements in $R$, which occur as main diagonal entries of elements in $X$. If $Y = 0$, then $r_{T(R,n,\sigma)}(X)$ contains $E_{1n}$ which contradicts to our assumption. So $Y \neq 0$. We then have $r_R(Y) = 0$, as $a \in r_R(Y)$ implies $YaE_{1n} = 0$. Since $R$ is right zip, there exists a finite subset $Y_0 = \{r_1, r_2, \ldots, r_n\} \subseteq Y$, such that $r_R(Y_0) = 0$. For each $1 \leq i \leq n$, put $U_i$ to be the set of matrices whose main diagonal entries are $r_i$. Take $A_i \in U_i$, $1 \leq i \leq n$. Clearly $U = \{A_1, \ldots, A_n\}$, $r_{T(R,n,\sigma)}(U) = 0$. So $T(R, n, \sigma)$ is right zip. For the

Converse we adopt the proof of [25, Proposition 3]. Suppose that $T(R,n,\sigma)$ is a right zip ring and $X \subseteq R$ with $r_R(X)=0$. Let $Y=\{aI \mid a \in X\} \subseteq T(R,n,\sigma)$, where $I$ is the $n \times n$ identity matrix. If $B=(b_1,b_2,\ldots,b_n) \in r_{T(R,n,\sigma)}(Y)$, then $aI.B=0$ for all $a \in X$. Thus $ab_j=0$ for all $j$. Therefore $b_j \in r_R(X)=0$ and so $b_j=0$ for all $j$. Since $T(R,n,\sigma)$ is right zip, there exists a finite subset $Y_0=\{a_1I,a_2I,\ldots,a_mI\} \subseteq Y$ such that $r_{T(R,n,\sigma)}(Y_0)=0$. Let $X_0=\{a_1,a_2,\ldots,a_m\} \subseteq X$. If $b \in r_R(X_0)$, then $a_kI(0,0,\ldots,0,b)=0$ for all $k=1,2,\ldots,m$. Thus, $(0,0,\ldots,0,b) \in r_R(Y_0)=0$, so $b=0$. Hence $r_R(X_0)=0$ and therefore $R$ is right zip. Moreover, $R$ is zip because $R$ is commutative. $\square$

By Proposition 2.35, a ring $R$ is strongly right (resp. left) $AB$ if and only if $T(R,n,\sigma)$ is strongly right (resp. left) $AB$ for any $n \geq 2$. Also by Proposition 2.43, for each commutative ring $R$ with an endomorphisms $\sigma$ and for $n \geq 2$, $R$ is a zip ring if and only if $T(R,n,\sigma)$ is a zip ring. We can provide more examples of strongly right $AB$ rings.

**Example 2.44.** For any commutative domain $R$ with endomorphisms $\sigma,\alpha$ such that $\sigma\alpha=\alpha\sigma$, the ring $T(R,n,\sigma)$ is $\overline{\alpha}$-compatible and skew McCoy, by [20, Theorem 2.8 and Corollary 2.5]. Since $R$ is zip and strongly right $AB$, $T(R,n,\sigma)$ is zip and strongly right $AB$. So the ring $T(R,n,\sigma)[x,\overline{\alpha}]$ is also strongly right $AB$.

**Theorem 2.45.** *Let $R$ be a skew $M$-Armendariz ring, let $M$ be a u.p.-monoid and let $\sigma: M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. Then $R$ is nil-reversible if and only if $R*M$ is nil-reversible.*

*Proof.* Let $R$ be nil-reversible. Let $\alpha=\sum_{i=0}^{n} a_ig_i \in R*M$ and $\beta=\sum_{j=0}^{m} b_jh_j \in \mathrm{nil}(R*M)$ with $\alpha\beta=0$. By Lemma 2.5, each $b_j \in \mathrm{nil}(R), 0 \leq j \leq m$, as $R$ is 2-primal. Since $R$ is skew $M$-Armendariz and $M$-compatible, $a_ib_j=0$ for every $0 \leq i \leq n$, $0 \leq j \leq m$. Therefore $b_j\sigma_{h_j}(a_i)=0$, since $R$ is nil-reversible and $R$ is $M$-compatible. Consequently we have $\beta\alpha=0$. So $R*M$ is nil-reversible. $\square$

**Corollary 2.46.** *Let $R$ be a $\sigma_i$-skew Armendariz ring and let $\sigma_i$ be a compatible endomorphism of $R$ for each $i$. Assume that $\sigma_i\sigma_j=\sigma_j\sigma_i$ for each $i,j$. Then $R$ is nil-reversible if and only if the ring $R[x_1,x_2,\ldots,x_n;\sigma_1,\ldots,\sigma_n]$ is nil-reversible if and only if the ring $R[x_1,x_2,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1},\sigma_1,\ldots,\sigma_n]$ is nil-reversible.*

**Corollary 2.47.** *Let $R$ be an Armendariz ring. Then $R$ is nil-reversible if and only if the ring $R[x_1,x_2,\ldots,x_n]$ is nil-reversible if and only if the ring $R[x_1,x_2,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}]$ is nil-reversible.*

## 3. Rings with Property ($A$)

Huckaba and Keller [29] introduced the following: a commutative ring $R$ has *Property* ($A$) if every finitely generated ideal of $R$ consisting entirely of zero-divisors has a nonzero annihilator. Property ($A$) was originally studied by

Quentel [45]. Quentel used the term Condition $(C)$ for Property $(A)$. The class of commutative rings with Property $(A)$ is quite large. For example, Noetherian rings ([32], p. 56), rings whose prime ideals are maximal [22], the polynomial ring $R[x]$ and rings whose classical ring of quotients are von Neumann regular [22], are examples of rings with Property $(A)$. Using Property $(A)$, Hinkle and Huckaba [23] extend the concept Kronecker function rings from integral domains to rings with zero divisors. Many authors have studied commutative rings with Property $(A)$, and have obtained several results which are useful studying commutative rings with zero-divisors. C. Y. Hong, N. K. Kim, Y. Lee and S. J. Ryu [27] extended the notion of Property $(A)$ to noncommutative rings:

**Definition 3.1** ([27])**.** A ring $R$ has *right* (*left*) *Property* $(A)$ if for every finitely generated two-sided ideal $I \subseteq Z_l(R)$ (resp. $Z_r(R)$), there exists nonzero $a \in R$ (resp. $b \in R$) such that $Ia = 0$ (resp. $bI = 0$). A ring $R$ is said to have *Property* $(A)$ if $R$ has the right and left Property $(A)$.

According to [41], a ring $R$ with a monomorphism $\alpha$ is called $\alpha$-*weakly rigid* if for each $a, b \in R$, $aRb = 0$ if and only if $a\alpha(Rb) = 0$. For any positive integer $n$, a ring $R$ is $\alpha$-weakly rigid if and only if, the $n$-by-$n$ upper triangular matrix ring $T_n(R)$ is $\overline{\alpha}$-weakly rigid if and only if, the matrix ring $M_n(R)$ is $\overline{\alpha}$-weakly rigid, where $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ for each $(a_{ij}) \in M_n(R)$. Also if $R$ is a semiprime $\alpha$-weakly rigid ring, then the polynomial ring $R[x]$ is a semiprime $\overline{\alpha}$-weakly rigid ring, where $\overline{\alpha}(\sum_{i=0}^{n} r_i x^i) = \sum_{i=0}^{n} \alpha(r_i)x^i$. For every prime ring $R$ and any automorphism $\alpha$, the rings $M_n(R)$, $T_n(R)$, $R[X]$ and the power series ring $R[[X]]$, for $X$ an arbitrary nonempty set of indeterminates, are weakly rigid rings.

**Definition 3.2.** Let $R$ be a ring, let $M$ be a monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism. We say $R$ is $M$-*weakly rigid* if $\sigma_g$ is weakly rigid for every $g \in M$.

**Lemma 3.3.** *Let $R$ be a ring, let $M$ be a monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-weakly rigid. Then for every $a, b \in R$ and $g_i, g_j \in M$, $aRb = 0$ if and only if $\sigma_{g_i}(a)R\sigma_{g_j}(b) = 0$.*

*Proof.* Suppose that $aRb = 0$, so $\sigma_{g_i}(aRb) = 0$, and hence $\sigma_{g_i}(a)\sigma_{g_i}(Rb) = 0$. Since $R$ is $M$-weakly rigid, $\sigma_{g_i}(a)Rb = 0$. So for each $r \in R$, $\sigma_{g_i}(a)rRb = 0$, and hence $\sigma_{g_i}(a)r\sigma_{g_j}(Rb) = 0$. Thus $\sigma_{g_i}(a)r\sigma_{g_j}(b) = 0$, and for each $i, j$, $\sigma_{g_i}(a)R\sigma_{g_j}(b) = 0$. Now assume that $\sigma_{g_i}(a)R\sigma_{g_j}(b) = 0$, for each $i, j$. Since $R$ is $M$-weakly rigid, $\sigma_{g_i}(a)\sigma_{g_i}(R\sigma_{g_j}(b)) = 0$, so $\sigma_{g_i}(aR(\sigma_{g_j}(b)) = 0$. Since $\sigma_{g_j}$ is injective, $aR\sigma_{g_j}(b) = 0$. This implies that $aRb = 0$, as $R$ is $M$-weakly rigid. $\square$

Hirano in [24, Theorem 2.2], proved that, when $r_{R[x]}(f(x)R[x]) \neq 0$ then $r_{R[x]}(f(x)R[x]) \cap R \neq 0$ for $f(x) \in R[x]$.

**Proposition 3.4.** *Let $R$ be a ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-weakly rigid. For $\alpha \in S = R * M$, if $r_S(\alpha S) \neq 0$, then $r_S(\alpha S) \cap R \neq 0$.*

*Proof.* We apply the method of Hirano in the proof of [24, Theorem 2.2]. Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$. When $\alpha = 0$ or $\mathrm{length}(\alpha) = 1$, then the assertion is clear. So, let $\mathrm{length}(\alpha) = n$, $n > 1$. Assume, to the contrary, that $r_R(\alpha S) = 0$ and let $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in S$ be a nonzero element with minimal length in $r_S(\alpha S)$. Since $\alpha S\beta = 0$, $\alpha R\beta = 0$. Since $M$ is a u.p.-monoid, there exist $i, j$ with $1 \leq i \leq n$ and $1 \leq j \leq m$ such that $g_ih_j$ is uniquely presented by considering two subsets $A = \{g_1, \dots, g_n\}$ and $B = \{h_1, \dots, h_m\}$ of $M$. We may assume, without loss of generality, that $i = n$, $j = m$. Then $a_n\sigma_{g_n}(Rb_m) = 0$, as $g_nh_m$ is uniquely presented. Since $R$ is $M$-weakly rigid, $a_nRb_m = 0$. This implies

$$a_nS\beta = a_nS(b_{m-1}h_{m-1} + \cdots + b_1h_1)$$

and

$$0 = \alpha S\beta \supseteq \alpha S(a_nS\beta) = \alpha S(a_nS(b_{m-1}g_{m-1} + \cdots + b_1g_1)).$$

So $a_nR(b_{m-1}g_{m-1} + \cdots + b_1h_1) \subseteq r_S(\alpha S)$. Now take $\overline{\beta}$ to be $a_nR\beta$. Since $\mathrm{length}(\overline{\beta}) = n-1$, and $\alpha S\overline{\beta} = 0$ this contradicts with the assumption that $\beta$ has minimal length such that $\alpha S\beta = 0$. Thus $\overline{\beta} = 0$ and we have $a_nS(b_{m-1}h_{m-1} + \cdots + b_1h_1) = 0$. Therefore $a_nRb_j = 0, 1 \leq j \leq m$. Hence $(a_{n-1}g_{n-1} + \cdots + a_1g_1)S(b_mh_m + \cdots + b_1h_1) = 0$. Since $M$ is a u.p.-monoid there exist $i, j$ with $1 \leq i \leq n-1$ and $1 \leq j \leq m$ such that $g_ih_j$ is uniquely presented by considering two subsets $A = \{g_1, \dots, g_{n-1}\}$ and $B = \{h_1, \dots, h_m\}$ of $M$. We may assume without loss of generality that $i = n-1$, $j = m$, and so $a_{n-1}\sigma_{g_{n-1}}(Rb_m) = 0$. Thus we have $\alpha S(a_{n-1}S(b_{m-1}h_{m-1} + \cdots + b_1h_1)) = \alpha(Sa_{n-1}S)\beta = 0$. Since $\beta$ is a nonzero element with minimal length in $r_S(\alpha S)$, we have $a_{n-1}S(b_{m-1}h_{m-1} + \cdots + b_1h_1) = 0$. Therefore $a_{n-1}Rb_j = 0$, $1 \leq j \leq m$. Repeating this process, we have $a_iRb_j = 0$ for all $i, j$. So $a_ig_iRb_j = 0$ and $a_iRl_kb_j = 0, g_i, l_k \in M$, since $R$ is $M$-weakly rigid. This implies that $b_1, \dots, b_m \in r_R(\alpha S)$. This is also a contradiction and hence the result follows. □

**Corollary 3.5.** *Let $R$ be a ring and let $\sigma_i$ be a weakly rigid endomorphism of $R$ such that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Assume that $S$ is either the skew polynomial ring $R[x_1, \dots, x_n; \sigma_1, \dots, \sigma_n]$ or the skew Laurent polynomial ring $R[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}, \sigma_1, \dots, \sigma_n]$. If $r_S(fS) \neq 0$, then $r_S(fS) \cap R \neq 0$ for each $f \in S$.*

**Proposition 3.6.** *Let $R$ be a ring, let $M$ be a u.p.-monoid with $|M| \geq 2$ and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-weakly rigid. Then $S = R * M$ has right Property (A) if and only if whenever $\alpha S \subseteq Z_l(S)$, $r_S(\alpha S) \neq 0$.*

*Proof.* We adopt the proof of [27, Lemma 2.8]. Let $I = \sum_{i=1}^{k} S\alpha_i S \subseteq Z_l(S)$, where $\alpha_i = a_{i1}g_{i1} + a_{i2}g_{i2} + \cdots + a_{in_i}g_{in_i}$. By Proposition 2.24, there exist $l_1, \ldots, l_k \in M$ such that $\alpha_1 l_1, \alpha_2 l_2, \ldots, \alpha_k l_k$ are distinct. Put $\beta = \alpha_1 l_1 + \alpha_2 l_2 + \cdots + \alpha_k l_k \in I$. Thus $\beta S \subseteq I$. By hypothesis, $r_S(\beta S) = r_S(S\beta S) \neq 0$. So $r_S(S\beta S) \cap R \neq 0$, by Theorem 3.4. Thus for some nonzero $r \in R$, $S\beta S r = 0$. Since $R\beta R \subseteq S\beta S$ and $R$ is $M$-weakly rigid, we have $Ra_{ij}Rr = 0$. Thus $Rt_k a_{ij} g_j R h_m r = 0$ for $t_k, g_j, h_m \in M$. So $Ir = (\sum_{i=1}^{k} S\alpha_i S)r = 0$. Therefore $S$ has right Property $(A)$. The converse is clear. □

**Corollary 3.7.** *Let $R$ be a ring and let $\sigma_i$ be a weakly rigid endomorphism of the ring $R$ such that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Assume that $S$ is the ring $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ or $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}; \sigma_1, \ldots, \sigma_n]$. Then $S$ has right Property $(A)$ if and only if whenever $fS \subseteq Z_l(S)$, $r_S(fS) \neq 0$ for each $f \in S$.*

**Corollary 3.8** ([27, Lemma 2.8])**.** *For a ring $R$, $R[x]$ has right Property $(A)$ if and only if whenever $f(x)R[x] \subseteq Z_l(R[x])$, $r_{R[x]}(f(x)R[x]) \neq 0$.*

There exists a ring $R$ which does not have Property $(A)$ whose the polynomial ring $R[x]$ has Property $(A)$. For, the polynomial ring $R[x]$ over any commutative ring $R$ has Property $(A)$ [29, Theorem 1], and there is a commutative ring $R$ which does not have Property $(A)$.

**Theorem 3.9.** *Let $R$ be a ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. If $R$ is strongly right AB and right skew $M$-McCoy, then $R*M$ has right Property $(A)$.*

*Proof.* Put $S = R * M$ and let $X = \alpha S \subseteq Z_l(S)$, where $\alpha = a_0g_0 + a_1g_1 + \cdots + a_ng_n$. By hypothesis, there exists $\beta \in R * M$ such that $\alpha\beta = 0$. Since $R$ is $M$-compatible and right skew $M$-McCoy, there exists $0 \neq c \in R$ such that $a_ic = 0$ for each $i$. Since $R$ is strongly right $AB$, there exists an ideal $J$ such that $a_iJ = 0$ for each $i$. So for every $0 \neq d \in J$, $a_iRd = 0$ for each $i$. Since $R$ is $M$-compatible, we have $\alpha Sd = 0$. This implies that $R * M$ has right Property $(A)$, by Proposition 3.6. □

**Corollary 3.10.** *Let $R$ be a strongly right AB, skew McCoy ring and let $\sigma_i$ be a compatible endomorphism of $R$ for each $1 \leq i \leq n$. Assume that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $1 \leq i, j \leq n$. Then the rings $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, \sigma_1, \ldots, \sigma_n]$ have right Property $(A)$.*

**Corollary 3.11.** *Let $R$ be a right duo ring, let $M$ be a u.p.-group and let $\sigma : M \to \mathrm{Aut}(R)$ be a group homomorphism such that the ring $R$ is $M$-compatible. Then $R * M$ has right Property $(A)$.*

*Proof.* It is clear that the right duo ring $R$ is strongly right $AB$. Moreover, $R$ is right skew $M$-McCoy by Theorem 2.27. So the result follows from Theorem 3.9. □

**Corollary 3.12.** *Let $R$ be a right duo ring and let $\sigma_i$ be a compatible endomorphism of $R$ such that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $1 \leq i, j \leq n$. Then the rings $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, \sigma_1, \ldots, \sigma_n]$ have right Property $(A)$.*

**Corollary 3.13.** *If $R$ is a right duo ring, then the rings $R[x_1, x_2, \ldots, x_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ have right Property $(A)$.*

**Corollary 3.14.** *Let $R$ be a $CN$-ring and let $\sigma_i$ be a compatible endomorphism of $R$ such that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Then the rings $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, \sigma_1, \ldots, \sigma_n]$ have right Property $(A)$.*

**Corollary 3.15.** *Let $R$ be a semicommutative and right skew $M$-McCoy ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. Then $R * M$ has right Property $(A)$.*

**Corollary 3.16.** *Let $R$ be a semicommutative skew McCoy ring and let $\sigma_i$ be a compatible endomorphism of $R$ with $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $i, j$. Then the rings $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, \sigma_1, \ldots, \sigma_n]$ have right Property $(A)$.*

**Corollary 3.17.** *If $R$ is a semicommutative right McCoy ring, then the rings $R[x_1, x_2, \ldots, x_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ have right Property $(A)$.*

**Corollary 3.18** ([27, Proposition 2.10])**.** *If $R$ is a semicommutative and right McCoy ring, then $R[x]$ has right Property $(A)$.*

**Corollary 3.19.** *Let $R$ be a reversible ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{End}(R)$ be a monoid homomorphism such that $R$ is $M$-compatible. Then $R * M$ has right Property $(A)$.*

**Corollary 3.20.** *Let $R$ be a reversible ring and let $\sigma_i$ be a compatible endomorphism of $R$ such that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $1 \leq i, j \leq n$. Then the rings $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ and $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, \sigma_1, \ldots, \sigma_n]$ have right Property $(A)$.*

**Corollary 3.21** ([27, Corollary 2.11])**.** *If $R$ is a reversible ring, then $R[x]$ has right Property $(A)$.*

**Theorem 3.22.** *Let $R$ be a ring, let $M$ be a u.p.-monoid and let $\sigma : M \to \mathrm{Aut}(R)$ be a monoid homomorphism such that the ring $R$ is $M$-compatible. If $R * M$ is strongly right $AB$, then $R * M$ has right Property $(A)$.*

*Proof.* Let $X = \alpha S \subseteq Z_l(S)$, where $\alpha = a_0g_0 + a_1g_1 + \cdots + a_ng_n$. By hypothesis, there exists $\beta \in R * M$ such that $\alpha\beta = 0$. Since $R$ is $M$-compatible and right skew $M$-McCoy, there exists $0 \neq c \in R$ such that $a_ic = 0$ for each $i$. Since $R * M$ is strongly right $AB$, by 2.30, $R$ is strongly right $AB$, so there exists an ideal $J$ such that $a_iJ = 0$. So for every $0 \neq d \in J$, $a_iRd = 0$ for each $i$. Since $R$ is $M$-compatible, we have $\alpha Sd = 0$. This implies that $R * M$ has right Property $(A)$, by Proposition 3.6. $\square$

**Corollary 3.23.** *Let $S$ be either the ring $R[x_1, x_2, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ or the ring $R[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, \sigma_1, \ldots, \sigma_n]$ with $\sigma_i\sigma_j = \sigma_j\sigma_i$ for each $1 \leq i, j \leq n$. Assume that $S$ is a strongly right AB ring and $\sigma_i$ is a compatible automorphism of $R$, $1 \leq i \leq n$. Then $S$ has right Property $(A)$.*

**Corollary 3.24.** *If either $R[x]$ or $R[x, x^{-1}]$ is a strongly right AB ring, then $R[x]$ and $R[x, x^{-1}]$ have right Property $(A)$.*

## References

[1] D. D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), no. 7, 2265–2272.

[2] E. P. Armendariz, *A note on extensions of Baer and p.p.-rings*, J. Austral. Math. Soc. **18** (1974), 470–473.

[3] F. Azarpanah, O. A. S. Karamzadeh, and A. Rezai Aliabad, *On ideals consisting entirely of zero divisors*, Comm. Algebra **28** (2000), no. 2, 1061–1073.

[4] J. A. Beachy and W. D. Blair, *Rings whose faithful left ideals are cofaithful*, Pacific J. Math. **58** (1975), no. 1, 1–13.

[5] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368.

[6] G. F. Birkenmeier and J. K. Park, *Triangular matrix representations of ring extensions*, J. Algebra **265** (2003), no. 2, 457–477.

[7] G. F. Birkenmeier and R. P. Tucci, *Homomorphic images and the singular ideal of a strongly right bounded ring*, Comm. Algebra **16** (1988), no. 6, 1099–1122.

[8] V. Camillo and P. P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), no. 3, 599–615.

[9] J. Clark, Y. Hirano, H. K. Kim, and Y. Lee, *On a generalized finite intersection property*, Comm. Algebra **40** (2012), no. 6, 2151–2160.

[10] P. M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999), no. 6, 641–648.

[11] L. M. de Narbonne, *Anneaux semi-commutatifs et unis riels anneaux dont les id aux principaux sont idempotents*, In Proceedings of the 106th National Congress of Learned Societies, 71–73, Bibliotheque Nationale, Paris, 1982.

[12] M. P. Drazin, *Rings with central idempotent or nilpotent elements*, Proc. Edinburgh Math. Soc. **9** (1958), no. 2, 157–165.

[13] C. Faith, *Algebra II*, Springer-Verlag, Berlin., 1976.

[14] ______, *Commutative FPF rings arising as split-null extensions*, Proc. Amer. Math. Soc. **90** (1984), no. 2, 181–185.

[15] ______, *Rings with zero intersection property on annihilator: zip rings*, Publ. Math. **33** (1989), no. 2, 329–338.

[16] ______, *Annihilator ideals, associated primes and Kasch-McCoy commutative rings*, Comm. Algebra **19** (1991), no. 7, 1867–1892.

[17] S. P. Farbman, *The unique product property of groups and their amalgamated free products*, J. Algebra **178** (1995), no. 3, 962–990.

[18] E. H. Feller, *Properties of primary noncommutative rings*, Trans. Amer. Math. Soc. Publ. Math. **89** (1958), 79–91.

[19] M. Habibi and R. Manaviyat, *A generalization of nil-Armendariz rings*, J. Algebra Appl. **12** (2013), no. 6, 1350001, 30 pages.

[20] M. Habibi, A. Moussavi, and A. Alhevaz, *The McCoy condition on ore extensions*, Comm. Algebra **41** (2013), no. 1, 124–141.

[21] E. Hashemi and A. Moussavi,*Polynomial extensions of quasi-Baer rings*, Acta Math. Hungar. **3** (2005), no. 3, 207–224.

[22] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. **115** (1965), 110–130.
[23] G. Hinkle and J. A. Huckaba, *The generalized Kronecker function ring and the ring $R(X)$*, J. Reine Angew. Math. **292** (1977), 25–36.
[24] Y. Hirano, *On annihilator ideals of a polynomial ring over a noncommutative ring*, J. Pure Appl. Algebra **168** (2002), no. 1, 45–52.
[25] C. Y. Hong, N. K. Kim, T. K. Kwak, and Y. Lee, *Extensions of zip rings*, J. Pure Appl. Algebra **195** (2005), no. 3, 231–242.
[26] C. Y. Hong, N. K. Kim, and Y. Lee, *Extensions of McCoy's Theorem*, Glasg. Math. J. **52** (2010), no. 1, 155–159.
[27] C. Y. Hong, N. K. Kim, Y. Lee, and S. J. Ryu, *Rings with Property (A) and their extensions*, J. Algebra **315** (2007), no. 2, 612–628.
[28] J. A. Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker Inc., New York, 1988.
[29] J. A. Huckaba and J. M. Keller, *Annihilation of ideals in commutative rings*, Pacific J. Math. **83** (1979), no. 2, 375–379.
[30] S. U. Hwang, N. K. Kim, and Y. Lee, *On rings whose right annihilator are bounded*, Glasg. Math. J. **51** (2009), no. 3, 539–559.
[31] N. Jacobson, *The Theory of Rings*, Amer. Math. Soc., Providence, RI, 1943.
[32] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.
[33] D. Khurana, G. Marks, and K. Srivastava, *On unit-central rings*, Advances in ring theory, 205–212, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010.
[34] J. Krempa, *Some examples of reduced rings*, Algebra Colloq. **3** (1996), no. 4, 289–300.
[35] T. K. Lee and Y. Zhou, *A unified approach to the Armendariz property of polynomial rings and power series rings*, Colloq. Math. **113** (2008), no. 1, 151–169.
[36] T. G. Lucas, *Two annihilator conditions*: *Property* $(A)$ *and* $(a.c.)$, Comm. Algebra **14** (1986), no. 3, 557–580.
[37] G. Marks, *Reversible and symmetric rings* J. Pure Appl. Algebra **174** (2002), no. 3, 311–318.
[38] G. Marks, R. Mazurek, and M. Zimbowski, *A unified approach to various generalization of Armendariz rings* Bull. Aust. Math. Soc. **81** (2010), no. 3, 361–397.
[39] R. Mohammadi, A. Moussavi, and M. Zahiti, *On nil-semicommutative rings*, Int. Electron. J. Algebra **11** (2012), 20–37.
[40] A. Moussavi and E. Hashemi, *On* $(\alpha, \delta)$*-skew Armendariz rings*, J. Korean Math. Soc. **42** (2005), no. 2, 353–363.
[41] A. R. Nasr-Isfahani and A. Moussavi, *On weakly rigid rings*, Glasg. Math. J. **51** (2009), no. 3, 425–440.
[42] P. P. Nielsen,*Semi-commutativity and the McCoy condition*, J. Algebra **298** (2006), no. 1, 134–141.
[43] J. Okninski, *Semigroup Algebras*, Marcel Dekker, New York, 1991.
[44] L. Ouyang, *On weak annihilator ideals of skew monoid rings*, Comm. Algebra **39** (2011), no. 11, 4259–4272.
[45] Y. Quentel, *Sur la compacité du spectre minimal d'un anneau*, Bull. Soc. Math. France **99** (1971), 265–272.
[46] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17.
[47] A. B. Singh, M. R. Khan, and V. N. Dixit, *Skew monoid rings over zip rings*, Int. J. Algebra **4** (2010), no. 21-24, 1031–1036.
[48] W. Xue, *On strongly right bounded finite rings*, Bull. Austral. Math. Soc. **44** (1991), no. 3, 353–355.
[49] ———, *Structure of minimal noncommutative duo rings and minimal strongly bounded non-duo rings*, Comm. Algebra **20** (1992), no. 9, 2777–2788.

[50] J. M. Zelmanowitz, *The finite intersection property on annihilator right ideals*, Proc. Amer. Math. Soc. **57** (1976), no. 2, 213–216.

Rasul Mohammadi
Department of Pure Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
P.O.Box:14115-134, Tehran, Iran
*E-mail address*: mohamadi.rasul@yahoo.com

Ahmad Moussavi
Department of Pure Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
P.O.Box:14115-134, Tehran, Iran
*E-mail address*: moussavi.a@modares.ac.ir, moussavi.a@gmail.com

Masoome Zahiri
Department of Pure Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
P.O.Box:14115-134, Tehran, Iran
*E-mail address*: m.zahiri86@gmail.com