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THE INCOMPLETE GENERALIZED τ-HYPERGEOMETRIC AND SECOND τ-APPELL FUNCTIONS

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Abstract. Motivated mainly by certain interesting recent extensions of the generalized hypergeometric function [*Integral Transforms Spec. Funct.* 23 (2012), 659–683] and the second Appell function [*Appl. Math. Comput.* 219 (2013), 8332–8337] by means of the incomplete Pochhammer symbols $(\lambda;\kappa)_\nu$ and $[\lambda;\kappa]_\nu$, we introduce here the family of the incomplete generalized τ-hypergeometric functions ${}_2\gamma_1^\tau(z)$ and ${}_2\Gamma_1^\tau(z)$. The main object of this paper is to study these extensions and investigate their several properties including, for example, their integral representations, derivative formulas, Euler-Beta transform and associated with certain fractional calculus operators. Further, we introduce and investigate the family of incomplete second τ-Appell hypergeometric functions $\Gamma_2^{\tau_1,\tau_2}$ and $\gamma_2^{\tau_1,\tau_2}$ of two variables. Relevant connections of certain special cases of the main results presented here with some known identities are also pointed out.

1. Introduction, definitions and preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ denote the sets of positive integers, negative integers and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}\,.$$

The familiar *incomplete Gamma functions* $\gamma(s,\kappa)$ and $\Gamma(s,\kappa)$ defined by

$$\gamma(s,\kappa) := \int_0^\kappa t^{s-1}\, e^{-t}\, dt \quad \big(\Re(s) > 0;\ \kappa \geqq 0\big) \tag{1.1}$$

and

$$\Gamma(s,\kappa) := \int_\kappa^\infty t^{s-1}\, e^{-t}\, dt \quad \big(\kappa \geqq 0;\ \Re(s) > 0 \ \text{ when } \ \kappa = 0\big), \tag{1.2}$$

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respectively, satisfy the following decomposition formula:

$$\gamma(s,\kappa)+\Gamma(s,\kappa):=\Gamma(s) \quad \big(\Re(s)>0\big). \tag{1.3}$$

Each of these functions plays an important rôle in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, *e.g.*, [1, 3, 7, 11, 14, 15, 17, 20, 21, 22, 29, 30, 31, 40, 43]).

Recently, Srivastava *et al.* [28] introduced and studied in a rather systematic manner the following two families of generalized incomplete hypergeometric functions:

$${}_p\gamma_q\left[\begin{array}{r}(\alpha_1,\kappa),\alpha_2,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1;\kappa)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\,\frac{z^n}{n!} \tag{1.4}$$

and

$${}_p\Gamma_q\left[\begin{array}{r}(\alpha_1,\kappa),\alpha_2,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{[\alpha_1;\kappa]_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\,\frac{z^n}{n!}, \tag{1.5}$$

where, in terms of the incomplete Gamma functions $\gamma(s,\kappa)$ and $\Gamma(s,\kappa)$ defined by (1.1) and (1.2), respectively, the *incomplete* Pochhammer symbols $(\lambda;\kappa)_\nu$ and $[\lambda;\kappa]_\nu$ $(\lambda;\ \nu\in\mathbb{C};\ \kappa\geqq 0)$ are defined as follows:

$$(\lambda;\kappa)_\nu:=\frac{\gamma(\lambda+\nu,\kappa)}{\Gamma(\lambda)} \quad (\lambda,\nu\in\mathbb{C};\ \kappa\geqq 0) \tag{1.6}$$

and

$$[\lambda;\kappa]_\nu:=\frac{\Gamma(\lambda+\nu,\kappa)}{\Gamma(\lambda)} \quad (\lambda,\nu\in\mathbb{C};\ \kappa\geqq 0), \tag{1.7}$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda;\kappa)_\nu$ and $[\lambda;\kappa]_\nu$ satisfy the following decomposition relation:

$$(\lambda;\kappa)_\nu+[\lambda;\kappa]_\nu:=(\lambda)_\nu \quad (\lambda;\ \nu\in\mathbb{C};\ \kappa\geqq 0). \tag{1.8}$$

Here, and in what follows, $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$ denotes the Pochhammer symbol (or the shifted factorial) which is defined (in general) by

$$(\lambda)_\nu:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\})\\[2ex] \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\end{cases} \tag{1.9}$$

it being understood *conventionally* that $(0)_0:=1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [31, p. 21 *et seq.*]).

As already observed by Srivastava *et al.* [28], the definitions (1.4) and (1.5) readily yield the following decomposition formula:

$$\begin{aligned}&{}_p\gamma_q\left[\begin{array}{r}(\alpha_1,\kappa),\alpha_2,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]+{}_p\Gamma_q\left[\begin{array}{r}(\alpha_1,\kappa),\alpha_2,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]\\ &={}_pF_q\left[\begin{array}{r}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]\end{aligned} \tag{1.10}$$

for the familiar generalized hypergeometric function ${}_pF_q$ [26].

More recently, Çetinkaya [6] introduced and studied various properties of the following two families of the incomplete second Appell hypergeometric functions γ_2 and Γ_2:

$$\gamma_2[(\alpha,\kappa),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2] = \sum_{m,p=0}^{\infty} \frac{(\alpha;\kappa)_{m+p}(\beta_1)_m(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_p} \frac{x_1^m}{m!}\frac{x_2^p}{p!} \tag{1.11}$$

and

$$\Gamma_2[(\alpha,\kappa),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2] = \sum_{m,p=0}^{\infty} \frac{[\alpha;\kappa]_{m+p}(\beta_1)_m(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_p} \frac{x_1^m}{m!}\frac{x_2^p}{p!}. \tag{1.12}$$

In 2001, Virchenko *et al.* [42, p. 90, Eq. (5)] have studied and investigated (see also [12]) the following generalized τ-hypergeometric function:

$$ {}_2R_1^{\tau}\,(a,\,b;\,c;\,z) = {}_2R_1\,(a,\,b;\,c;\,\tau;\,z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!} \tag{1.13}$$

$$\big(\tau>0,\ |z|<1\,;\,\Re(c)>\Re(b)>0\big)$$

They gave the Euler type integral representation as follows [42, p. 91, Eq. (6)]:

$$ {}_2R_1\,(a,\,b;\,c;\,\tau;\,z) = \frac{1}{B(b,\,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,(1-zt^{\tau})^{-a}\,dt. \tag{1.14}$$

$$\big(\tau>0;\,|\arg(1-z)|<\pi;\ \Re(c)>\Re(b)>0\big)$$

The special case when $\tau=1$ in (1.13) and (1.14) yields the familiar representations of Gauss's hypergeometric function [23].

Moreover, Al-Shammery and Kalla [2] introduced and studied various properties of second τ-Appell's hypergeometric functions as:

$$\begin{aligned} &F_2^{\tau_1,\tau_2}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2] \\ &:= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)}\sum_{m_1,m_2=0}^{\infty}\frac{(\alpha)_{m_1+m_2}\Gamma(\beta_1+\tau_1m_1)\Gamma(\beta_2+\tau_2m_2)}{\Gamma(\gamma_1+\tau_1m_1)\Gamma(\gamma_2+\tau_2m_2)}\,\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!} \end{aligned} \tag{1.15}$$

$$\big(\tau_1,\tau_2>0;\ |x_1|+|x_2|<1\big).$$

Motivated essentially by the demonstrated potential for applications of these incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, and the incomplete second Appell hypergeometric functions γ_2 and Γ_2 in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [6, 28] and the references cited therein), we aim here at systematically investigating the family of the incomplete generalized τ-hypergeometric function ${}_2\gamma_1^{\tau}(z)$ and ${}_2\Gamma_1^{\tau}(z)$. For each of these incomplete generalized τ-hypergeometric function, we obtain integral representations, derivative formula, Euler-Beta transform and associated with the fractional calculus operators. Further, we introduce and investigate the family of incomplete second τ-Appell hypergeometric functions

$\Gamma_2^{\tau_1,\tau_2}$ and $\gamma_2^{\tau_1,\tau_2}$ of two variables. Some interesting special cases of our main results are also pointed out. For various other investigations involving generalizations of the hypergeometric function ${}_pF_q$ of p numerator and q denominator parameters, which were motivated essentially by the pioneering work of Srivastava *et al.* [28], the interested reader may be referred to several recent papers on the subject (see, *e.g.*, [6, 8, 9, 16, 27, 33, 35, 36, 37, 38, 39] and the references cited in each of these papers).

2. The incomplete generalized τ-hypergeometric function

In terms of the incomplete Pochhammer symbol $(\lambda;\kappa)_\nu$ and $[\lambda;\kappa]_\nu$ defined by (1.6) and (1.7), we introduce the families of the incomplete generalized τ-hypergeometric function ${}_2\gamma_1^\tau(z)$ and ${}_2\Gamma_1^\tau(z)$ as follows: For a, $b,\in\mathbb{C}$ and $c\in\mathbb{C}\setminus\mathbb{Z}_0^-$, we have

$$(2.1)\qquad {}_2\gamma_1^\tau(z) = {}_2\gamma_1^\tau((a,\kappa),\, b;\, c;\, z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a;\kappa)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}$$

$$(\kappa \geqq 0;\ \tau>0,\ |z|<1\,;\ \Re(c)>\Re(b)>0 \quad \text{when} \quad \kappa=0)$$

and

$$(2.2)\qquad {}_2\Gamma_1^\tau(z) = {}_2\Gamma_1^\tau((a,\kappa),\, b;\, c;\, z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;\kappa]_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}$$

$$(\kappa \geqq 0;\ \tau>0,\ |z|<1\,;\ \Re(c)>\Re(b)>0 \quad \text{when} \quad \kappa=0).$$

In view of (1.8), these families of incomplete generalized τ-hypergeometric function satisfy the following decomposition formula:

$$(2.3)\qquad {}_2\gamma_1^\tau((a,\kappa),\, b;\, c;\ z) + {}_2\Gamma_1^\tau((a,\kappa),\, b;\, c;\ z) = {}_2R_1^\tau\,(a,\, b;\, c;\, z),$$

where ${}_2R_1^\tau(z)$ is the generalized τ-hypergeometric function [12, 42].

It is noted in passing that, in view of the decomposition formula (2.3), it is sufficient to discuss the properties and characteristics of the incomplete generalized τ-hypergeometric function ${}_2\Gamma_1^\tau(z)$.

Remark 1. The special cases of (2.1) and (2.2) when $\tau=1$ are easily seen to reduce to the known families of the incomplete Gauss hypergeometric functions [28, p. 664, Eq. (3.1)] and [28, p. 664, Eq. (3.2)]:

$${}_2\gamma_1[(a,\kappa),b;c;z] = \sum_{n=0}^{\infty}\frac{(a;\kappa)_n\,(b)_n}{(c)_n}\frac{z^n}{n!} \quad \text{and}$$

$${}_2\Gamma_1[(a,\kappa),b;c;z] = \sum_{n=0}^{\infty}\frac{[a;\kappa]_n\,(b)_n}{(c)_n}\frac{z^n}{n!}$$

respectively. Also, the special cases of (2.1) and (2.2) when $\tau = 1$ and $\kappa = 0$ is seen to yield the classical Gauss's hypergeometric function (see, *e.g.*, [15, 23]):

$$ {}_2F_1(a,\, b;\, c;\, z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!} \quad (|z| < 1\,;\, \Re(c) > \Re(b) > 0)\,. $$

2.1. Integral representations

In this section, we present certain integral representations of the incomplete generalized τ-hypergeometric function ${}_2\Gamma_1^{\tau}(z)$ by applying (1.2) and (1.7).

Theorem 1. *The following integral representation for* ${}_2\Gamma_1^{\tau}(z)$ *in* (2.2) *holds true*:

$$ {}_2\Gamma_1^{\tau}[(a,\kappa),b;c;z] = \frac{1}{\Gamma(a)} \int_{\kappa}^{\infty} e^{-t}\, t^{a-1}\; {}_1\Phi_1^{\tau}(b;c;zt)dt, \tag{2.4} $$

$$ \big(\kappa \geqq 0;\ \Re(z) < 1,\ \Re(a) > 0 \text{ when } \kappa = 0\big) $$

where ${}_1\Phi_1^{\tau}\,(b;\,c;\,z)$ *is the* τ*-confluent hypergeometric function introduced by Virchenko* [41]:

$$ {}_1\Phi_1^{\tau}(z) = {}_1\Phi_1^{\tau}\,(b;\,c;\,z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \tag{2.5} $$

$$ (\tau > 0,\ \Re(c) > \Re(b) > 0)\,. $$

Proof. Using the definition of the incomplete Pochhammer symbol $[a;\kappa]_n$ in (2.2) by considering the integral representation resulting from (1.2) and (1.7) and using (2.5), we are led to the desired result (2.4) asserted by Theorem 1. □

Theorem 2. *The following integral representation for* ${}_2\Gamma_1^{\tau}(z)$ *in* (2.2) *holds true*:

$$ {}_2\Gamma_1^{\tau}[(a,\kappa),b;c;z] = \frac{1}{B(b,\, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\; {}_1\Gamma_0\left[(a,\kappa)\,;\, \text{——}\,;\, zt^{\tau}\right]\, dt \tag{2.6} $$

$$ \big(\tau > 0,\, \Re(c) > \Re(b) > 0\,;\ \kappa \geqq 0\big). $$

Proof. Considering the following elementary identity involving the Beta function:

$$ \begin{aligned} \frac{\Gamma(c)}{\Gamma(b)} \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} = \frac{(b)_{\tau n}}{(c)_{\tau n}} &= \frac{B(b+\tau n, c-b)}{B(b, c-b)} \\ &= \frac{1}{B(b,\, c-b)} \int_0^1 t^{b+\tau n-1}(1-t)^{c-b-1}\, dt \end{aligned} $$

$$ \big(\tau > 0,\, \Re(c) > \Re(b) > 0\big) $$

in (2.2) and interchanging summation and integration under the stated conditions, we have

$$ {}_2\Gamma_1^{\tau}[(a,\kappa),b;c;z]=\frac{1}{B(b,\,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\sum_{n=0}^{\infty}(a;\kappa)_n\frac{(zt)^{\tau n}}{n!}. $$

Finally, using the definition (1.5), we get the desired integral representation (2.6) asserted by Theorem 2. □

Theorem 3. *The following relationship with the incomplete gamma function* $\Gamma(s,\kappa)$ *defined by* (1.2) *holds true*:

$$ {}_2\Gamma_1^{\tau}[(a,\kappa),b;b;z]=\frac{(1-z)^{-a}}{\Gamma(a)}\Gamma(a,\kappa(1-z)) \qquad (|z|<1;\ \kappa\geqq 0). \tag{2.7} $$

Proof. Putting $c=b$ in (2.4), we immediately obtain the following simplified form:

$$ {}_2\Gamma_1^{\tau}[(a,\kappa),b;b;z]=\frac{1}{\Gamma(a)}\int_{\kappa}^{\infty} e^{-t(1-z)}\ t^{a-1}dt. \tag{2.8} $$

Now by setting $t=\frac{u}{1-z}$ and $dt=\frac{du}{1-z}$ in (2.8), we have

$$ {}_2\Gamma_1^{\tau}[(a,\kappa),b;b;z]=\frac{(1-z)^{-a}}{\Gamma(a)}\int_{\kappa(1-z)}^{\infty} e^{-u}\ u^{a-1}du \quad (|z|<1), \tag{2.9} $$

which is precisely the assertion (2.7) asserted by Theorem 3. □

Theorem 4. *The following relationship with the complementary error function* $\mathrm{erfc}(z)$ *holds true*:

$$ {}_2\Gamma_1^{\tau}\left(\left(\frac{1}{2},\kappa\right),b;b;1-z\right)=\frac{1}{\sqrt{z}}\ \mathrm{erfc}(\sqrt{\kappa z}) \quad (\kappa\geqq 0). \tag{2.10} $$

Proof. Upon replacing z by $1-z$ and putting $a=\frac{1}{2}$ in (2.7), we find that the incomplete generalized τ-hypergeometric function ${}_2\Gamma_1^{\tau}(z)$ reduces to the complementary error function $\mathrm{erfc}(z)$ (see, *e.g.*, [22, p. 726]) as follows:

$$ {}_2\Gamma_1^{\tau}\left(\left(\frac{1}{2},\kappa\right),b;b;1-z\right)=\frac{z^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\Gamma(\frac{1}{2},\kappa z)=\frac{1}{\sqrt{z}}\ \mathrm{erfc}(\sqrt{\kappa z}) $$ □

Remark 2. The special cases of (2.6) and (2.4) when $\tau=1$ are easily seen to reduce to the known integral representations of the incomplete Gauss hypergeometric functions [28, p. 665, Eq. (3.6)] and [28, p. 672, Eq. (3.53)]:

$$ {}_2\Gamma_1[(a,\kappa),b;c;z]=\frac{1}{\Gamma(a)}\int_{\kappa}^{\infty} e^{-t}\ t^{a-1}\ {}_1F_1(b;c;zt)\,dt \tag{2.11} $$

and

(2.12)

$$ {}_2\Gamma_1[(a,\kappa),b;c;z]=\frac{1}{B(b,\,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\ {}_1\Gamma_0\left[(a,\kappa);\underline{\quad};zt\right]\,dt $$

respectively. Also, the special cases of (2.6) and (2.4) when $\tau = 1$ and $\kappa = 0$ are seen to yield the classical integral representations of Gauss's hypergeometric function (see, *e.g.*, [23]):

$$ {}_2F_1[a, b; c; z] = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t}\, t^{a-1}\, {}_1F_1(b; c; zt)dt \tag{2.13}$$

and

$$ {}_2F_1[a, b; c; z] = \frac{1}{B(b,\, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\, (1-zt)^{-a}\, dt. \tag{2.14}$$

respectively.

2.2. Derivative formula

Theorem 5. *Each of the following derivative formula for* ${}_2\Gamma_1^\tau(z)$ *holds true*:

$$\begin{aligned} &\frac{d^n}{dz^n}\left[{}_2\Gamma_1^\tau((a,\kappa), b; c; z)\right] \\ &= \frac{(a)_n \Gamma(c)\Gamma(b+\tau n)}{\Gamma(b)\Gamma(c+\tau n)}\, {}_2\Gamma_1^\tau[(a+n,\kappa), b+\tau n; c+\tau n; z] \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\left(\frac{d}{dz}\right)^m \left[z^{c-1}\, {}_2\Gamma_1^\tau((a,\kappa), b; c; \omega z^\tau)\right] \\ &= \frac{z^{c-m-1}\Gamma(c)}{\Gamma(c-m)}\, {}_2\Gamma_1^\tau((a,\kappa), b; c-m; \omega z^\tau) \quad (\Re(c-m) > 0,\ m \in \mathbb{N}), \end{aligned} \tag{2.16}$$

where $a,\ b,\ c,\ \omega \in \mathbb{C}$*;* $\Re(\tau) > 0$, $\Re(a) > 0$, $\Re(b) > 0$, $\Re(c) > 0$; $\Re(\kappa) \geq 0$.

Proof. Differentiating n times both sides of (2.2) with respect to z, we can easily obtain a derivative formula for the incomplete generalized τ-hypergeometric function ${}_2\Gamma_1^\tau(z)$ asserted by (2.15).

Next, according to the uniform convergence of the series (2.2), differentiating term by term under the sign of summation, we have

$$\begin{aligned} &\left(\frac{d}{dz}\right)^m \left[z^{c-1}\, {}_2\Gamma_1^\tau((a,\kappa), b; c; \omega z^\tau)\right] \\ &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{[a;\kappa]_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{\omega^n}{n!} \left(\frac{d}{dz}\right)^m \left[z^{c+\tau n-1}\right] \\ &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{[a;\kappa]_n \Gamma(b+\tau n)}{\Gamma(c+\tau n-m)} \frac{\omega^n}{n!}\, z^{c+\tau n-m-1} \\ &= z^{c-m-1} \frac{\Gamma(c)\Gamma(c-m)}{\Gamma(b)\Gamma(c-m)} \sum_{n=0}^\infty \frac{[a;\kappa]_n \Gamma(b+\tau n)}{\Gamma(c+\tau n-m)} (\omega z^\tau)^n, \end{aligned}$$

which, in view of the definition (2.2), yields the desired representation (2.16). □

2.3. Euler-Beta transform

The Euler-Beta transform of the function $f(z)$ is defined, as usual, by

$$\mathcal{B}\{f(z);\mu,\nu\} = \int_0^1 z^{\mu-1}(1-z)^{\nu-1} f(z)\,dz. \tag{2.17}$$

Theorem 6. *The following Euler-Beta transform representation for the* ${}_2\Gamma_1^{\tau}(z)$ *in* (2.2) *holds true*:

$$\mathcal{B}\left\{{}_2\Gamma_1^{\tau}[(a,\kappa),b;c;\omega z^{\tau}] : c,\nu\right\} := B(c,\nu){}_2\Gamma_1^{\tau}[(a,\kappa),b;c+\nu;\omega] \tag{2.18}$$
$$(\tau>0;\ \Re(a)>0,\ \Re(\nu)>0,\ \Re(b)>0,\ \Re(c)>0).$$

Proof. Using the definition (2.17) of the Euler-Beta transform, we find from (2.2)

$$\begin{aligned} &\mathcal{B}\left\{{}_2\Gamma_1^{\tau}[(a,\kappa),b;c;\omega z^{\tau}] : c,\nu\right\} \\ &:= \int_0^1 z^{c-1}(1-z)^{\nu-1}\ {}_2\Gamma_1^{\tau}[(a,\kappa),b;c;\omega z^{\tau}]dz \\ &= \int_0^1 z^{c-1}(1-z)^{\nu-1}\left(\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;\kappa]_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{\omega^n z^n}{n!}\right). \end{aligned} \tag{2.19}$$

Upon interchanging the order of integration and summation in (2.19), which can easily be justified by uniform convergence under the constraints stated with (2.17), we get

$$\begin{aligned} &\mathcal{B}\left\{{}_2\Gamma_1^{\tau}[(a,\kappa),b;c;\omega z^{\tau}] : c,\nu\right\} \\ &:= \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;\kappa]_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{\omega^n}{n!}\left(\int_0^1 z^{c+\tau n-1}(1-z)^{\nu-1}dz\right) \\ &= \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;\kappa]_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{\omega^n}{n!}\frac{\Gamma(c+\tau n)\Gamma(\nu)}{\Gamma(c+\nu+\tau n)} \\ &= \frac{\Gamma(c)\Gamma(\nu)\Gamma(c+\nu)}{\Gamma(b)\Gamma(c+\nu)}\frac{\omega^n}{n!}\sum_{n=0}^{\infty}\frac{[a;\kappa]_n\Gamma(b+\tau n)}{\Gamma(c+\nu+\tau n)}. \end{aligned}$$

Using the definition (2.2), we get the desired representation (2.18). □

2.4. Fractional calculus approach

In this section, we derive certain interesting properties of the incomplete generalized τ-hypergeometric function ${}_2\Gamma_1^{\tau}(z)$ in (2.2) associated with *right-sided* Riemann-Liouville fractional integral operator $I^{\mu}_{\alpha+}$ and the *right-sided* Riemann-Liouville fractional derivative operator $D^{\mu}_{\alpha+}$, which are defined as follows (see, *e.g.*, [14, 19, 25]):

$$\left(I^{\mu}_{\alpha+}\varphi\right)(x) = \frac{1}{\Gamma(\mu)}\int_{\alpha}^{x}\frac{\varphi(t)}{(x-t)^{1-\mu}}dt \quad \left(\mu\in\mathbb{C},\ \Re(\mu)>0\right) \tag{2.20}$$

and

$$(2.21)\quad \left(D_{\alpha+}^{\mu}\varphi\right)(x)=\left(\frac{d}{dx}\right)^{n}\left(I_{\alpha+}^{n-\mu}\varphi\right)(x)\ \left(\mu\in\mathbb{C},\ \Re(\mu)>0;\ n=[\Re(\mu)]+1\right),$$

where $[x]$ means the greatest integer not exceeding real x.

Another generalization of Riemann-Liouville fractional derivative operator $D_{\alpha+}^{\mu}$ in (2.21) by introducing a *right-sided* Riemann-Liouville fractional derivative operator $D_{\alpha+}^{\mu,\nu}$ of order $0<\mu<1$ and type $0\leqq\nu\leqq 1$ with respect to x by Hilfer (see, *e.g.*,[13]) is given as follows:

$$(2.22)\qquad \left(D_{\alpha+}^{\mu,\nu}\varphi\right)(x)=\left(I_{\alpha+}^{\nu(1-\mu)}\frac{d}{dx}\right)\left(I_{\alpha+}^{(1-\nu)(1-\mu)}\varphi\right)(x)$$
$$\left(\mu\in\mathbb{C},\ \Re(\mu)>0;\ n=[\Re(\mu)]+1\right).$$

The generalization (2.22) yields the classical Riemann-Liouville fractional derivative operator $D_{\alpha+}^{\mu}$ when $\nu=0$.

Theorem 7. *Let* $\alpha\in\mathbb{R}_{+}=[0,\infty)$, a, b, c, μ, $\omega\in\mathbb{C}$ *and* $\Re(a)>0$, $\Re(b)>0$, $\Re(c)>0$, $\Re(\mu)>0$, $\tau>0$. *Then, for* $x>\alpha$, *the following relations hold true*:

$$(2.23)\qquad \left(I_{\alpha+}^{\mu}\left[(t-\alpha)^{c-1}\,{}_{2}\Gamma_{1}^{\tau}((a,\kappa),b;c;\omega(t-\alpha)^{\tau})\right]\right)(x)$$
$$=\frac{(x-\alpha)^{c+\mu-1}\Gamma(c)}{\Gamma(c+\mu)}\,{}_{2}\Gamma_{1}^{\tau}((a,\kappa),b;c+\mu;\omega(x-\alpha)^{\tau}),$$

$$(2.24)\qquad \left(D_{\alpha+}^{\mu}\left[(t-\alpha)^{c-1}\,{}_{2}\Gamma_{1}^{\tau}((a,\kappa),b;c;\omega(t-\alpha)^{\tau})\right]\right)(x)$$
$$=\frac{(x-\alpha)^{c-\mu-1}\Gamma(c)}{\Gamma(c-\mu)}\,{}_{2}\Gamma_{1}^{\tau}[(a,\kappa),b;c-\mu;\omega(x-\alpha)^{\tau})$$

and

$$(2.25)\qquad \left(D_{\alpha+}^{\mu,\nu}\left[(t-\alpha)^{c-1}\,{}_{2}\Gamma_{1}^{\tau}((a,\kappa),b;c;\omega(t-\alpha)^{\tau})\right]\right)(x)$$
$$=\frac{(x-\alpha)^{c-\mu-1}\Gamma(c)}{\Gamma(c-\mu)}\,{}_{2}\Gamma_{1}^{\tau}[(a,\kappa),b;c-\mu;\omega(x-\alpha)^{\tau}).$$

Proof. By virtue of the formulas (2.20) and (2.2), the term-by-term fractional integration and the application of the relation [25]:

$$(2.26)\qquad \left(I_{a+}^{\alpha}[(t-a)^{\beta-1}]\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}\ \left(\alpha,\beta\in\mathbb{C},\ \Re(\alpha)>0,\ \Re(\beta)>0\right)$$

yields, for $x>\alpha$,

$$(2.27)\qquad \left(I_{\alpha+}^{\mu}[(t-\alpha)^{c-1}\,{}_{2}\Gamma_{1}^{\tau}((a,\kappa),b;c;\omega(t-\alpha)^{\tau})]\right)(x)$$
$$=\left(I_{\alpha+}^{\mu}\left[\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;\kappa]_{n}\Gamma(b+\tau n)}{\Gamma(c+\tau n)n!}\omega^{n}\,(t-\alpha)^{c+\tau n-1}\right]\right)$$

$$= \frac{(x-\alpha)^{c+\mu-1}\Gamma(c)}{\Gamma(c+\mu)} \, {}_2\Gamma_1^{\tau}[(a,\kappa),b;c+\mu;\omega(x-\alpha)^{\tau}).$$

Next, by (2.21) and (2.2), we find that

$$\text{(2.28)} \quad \left(D_{\alpha+}^{\mu}[(t-\alpha)^{c-1} \, {}_2\Gamma_1^{\tau}((a,\kappa),b;c;\omega(t-\alpha)^{\tau})]\right)(x))$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{\alpha+}^{n-\mu}[(t-\alpha)^{c-1} \, {}_2\Gamma_1^{\tau}((a,\kappa),b;c;\omega(t-\alpha)^{\tau})]\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n \left[\frac{(x-\alpha)^{c+n-\mu-1}\Gamma(c)}{\Gamma(c-\mu+n)} \, {}_2\Gamma_1^{\tau}[(a,\kappa),b;c-\mu+n;\omega(x-\alpha)^{\tau})\right].$$

Applying (2.16), we are led to the desired result (2.24).

Finally, by (2.22) and (2.2), we have

$$\text{(2.29)} \quad \left(D_{\alpha+}^{\mu,\nu}\left[(t-\alpha)^{c-1} \, {}_2\Gamma_1^{\tau}((a,\kappa),b;c;\omega(t-\alpha)^{\tau})\right]\right)(x))$$

$$= \left(D_{\alpha+}^{\mu,\nu}\left[\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;\kappa]_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)n!}\omega^n\,(t-\alpha)^{c+\tau n-1}\right]\right)(x)$$

$$= \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;\kappa]_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)n!}\omega^n\left(D_{\alpha+}^{\mu,\nu}\left[(t-\alpha)^{c+\tau n-1}\right]\right)(x).$$

Using the known relation of Srivastava and Tomovski [34, p. 203, Eq. (2.18)]

$$\text{(2.30)} \quad \left(D_{\alpha+}^{\mu,\nu}\left[(t-\alpha)^{\lambda-1}\right]\right)(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)}(x-a)^{\lambda-\mu-1}$$

$$\left(x>\alpha;\ 0<\mu<1;\ 0\leqq\nu\leqq 1;\ \Re(\lambda)>0\right)$$

in (2.29), we are led to the desired result (2.25). □

Remark 3. The special cases of (2.23)-(2.25) when $\tau = 1$ yield the corresponding known relations for the generalized τ-hypergeometric function [24].

3. The incomplete second τ-Appell functions

Further, we introduce the incomplete second Appell τ-hypergeometric functions $\gamma_2^{\tau_1,\tau_2}$ and $\Gamma_2^{\tau_1,\tau_2}$ of two variables as follows: For $a, b_1, b_2 \in \mathbb{C}$ and $c_1, c_2 \in \mathbb{C}\setminus\mathbb{Z}_0^-$, we have

$$\text{(3.1)} \quad \gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;x_1,x_2]$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)}\sum_{m_1,m_2=0}^{\infty}\frac{(a;\kappa)_{m_1+m_2}\Gamma(b_1+\tau_1 m_1)\Gamma(b_2+\tau_2 m_2)}{\Gamma(c_1+\tau_1 m_1)\Gamma(c_2+\tau_2 m_2)}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}$$

$$\left(\kappa\geqq 0;\ \tau_1,\tau_2>0;\ |x_1|+|x_2|<1 \ \text{ when } \ \kappa=0\right)$$

and

$$\text{(3.2)} \quad \Gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;x_1,x_2]$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)} \sum_{m_1,m_2=0}^{\infty} \frac{[a;\kappa]_{m_1+m_2}\Gamma(b_1+\tau_1 m_1)\Gamma(b_2+\tau_2 m_2)}{\Gamma(c_1+\tau_1 m_1)\Gamma(c_2+\tau_2 m_2)} \frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}$$

$$\big(\kappa \geqq 0;\ \tau_1,\tau_2>0;\ |x_1|+|x_2|<1 \ \text{ when } \ \kappa=0\big).$$

In view of (1.8), these families of incomplete second τ-Appell function satisfy the following decomposition formula:

$$\gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;x_1,x_2] + \Gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;x_1,x_2] = F_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;x_1,x_2], \tag{3.3}$$

where $F_2^{\tau_1,\tau_2}$ is the second τ-Appell function [2].

Remark 4. The special cases of (3.1) and (3.2) when $\tau_1=1=\tau_2$ are easily seen to reduce to the known families of the incomplete second Appell functions (1.11) and (1.12), respectively.

Also the special cases of (3.1) and (3.2) when $\tau_1=1=\tau_2$ and $\kappa=0$ are easily seen to reduce to the classical second Appell functions [4, 5].

In view of the decomposition formula (3.3), it is sufficient to discuss the properties and characteristics of the incomplete second τ-Appell function $\Gamma_2^{\tau_1,\tau_2}$.

3.1. Integral representations

Theorem 8. *The following integral representation for* $\Gamma_2^{\tau_1,\tau_2}$ *in* (3.2) *holds true*:

$$\Gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;x_1,x_2] = \frac{1}{\Gamma(a)}\int_\kappa^\infty e^{-t}\, t^{a-1}\, {}_1\Phi_1^{\tau_1}\left[\begin{array}{c} b_1; \\ c_1; \end{array} x_1 t\right] {}_1\Phi_1^{\tau_2}\left[\begin{array}{c} b_2; \\ c_2; \end{array} x_2 t\right] dt \tag{3.4}$$

$$\big(\kappa \geqq 0;\ \tau_1,\tau_2>0;\ \Re(x_1+x_2)<1,\ \Re(a)>0 \ \textit{ when } \ \kappa=0\big).$$

Proof. Using the integral representation of the Pochhammer symbol $(\alpha)_{m_1+m_2}$ and the definition of τ-confluent hypergeometric function (2.5) in (3.2), we are led to the desired result (3.4) asserted by Theorem 8. ☐

Theorem 9. *The following integral representation for* $\Gamma_2^{\tau_1,\tau_2}$ *in* (3.2) *holds true*:

$$\begin{aligned} &\Gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;x_1,x_2] \\ &= \frac{1}{B(b_1,c_1-b_1)B(b_2,c_2-b_2)} \\ &\times \int_0^1\int_0^1 t^{b_1-1}s^{b_2-1}(1-t)^{c_1-b_1-1}(1-s)^{c_2-b_2-1} \\ &\times (1-x_1t^{\tau_1}-x_2s^{\tau_2})^{-a}\ \frac{\Gamma\left(a,\kappa(1-x_1t^{\tau_1}-x_2s^{\tau_2})\right)}{\Gamma(a)}\, dt\, ds \end{aligned} \tag{3.5}$$

$$\big(\kappa \geqq 0;\ \tau_1,\tau_2>0;\ \Re(c_j)>\Re(b_j)>0\ \ (j=1,2)\ \textit{when}\ \kappa=0\big).$$

Proof. Considering the following elementary identity involving the Beta function $B(\beta, \gamma)$:

$$\frac{(\beta)_\nu}{(\gamma)_\nu} = \frac{B(\beta+\nu, \gamma-\beta)}{B(\beta, \gamma-\beta)} = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta+\nu-1}(1-t)^{\gamma-\beta-1}\, dt \tag{3.6}$$

$$\big(\Re(\gamma) > \Re(\beta) > \max\{0, -\Re(\nu)\}\big)$$

in (3.2), we have

(3.7)

$$\begin{aligned}
&\Gamma_2^{\tau_1,\tau_2}[(a,\kappa), b_1, b_2; c_1, c_2; x_1, x_2] \\
&= \frac{1}{B(b_1, c_1-b_1)B(b_2, c_2-b_2)} \int_0^1 \int_0^1 t^{b_1-1} s^{b_2-1} (1-t)^{c_1-b_1-1}(1-s)^{c_2-b_2-1} \\
&\quad \times \Gamma_2[(a,\kappa), b_1, b_2; b_1, b_2; x_1 t^{\tau_1}, x_2 s^{\tau_2}]\, dt\, ds
\end{aligned}$$

and using the relation [6, p. 8334, Eq. (22)]:

$$\Gamma_2[(a,\kappa), b_1, b_2; b_1, b_2; x_1 t, x_2 s] = (1 - x_1 t - x_2 s)^{-a}\, \frac{\Gamma\left(a, \kappa(1 - x_1 t - x_2 s)\right)}{\Gamma(a)},$$

we get the desired multiple integral representation (3.5) asserted by Theorem 9. □

Remark 5. The special cases of (3.4) and (3.5) when $\tau_1 = 1 = \tau_2$ are easily seen to reduce to the known integral representations of the incomplete second Appell functions [6, p. 8333, Eq. (10)] and [6, p. 8335, Eq. (24)]:

$$\begin{aligned}
&\Gamma_2[(a,\kappa), b_1, b_2; c_1, c_2; x_1, x_2] \\
&= \frac{1}{\Gamma(a)} \int_\kappa^\infty e^{-t}\, t^{a-1}\; {}_1F_1\left[\begin{matrix} b_1; \\ c_1; \end{matrix}\; x_1 t\right] {}_1F_1\left[\begin{matrix} b_2; \\ c_2; \end{matrix}\; x_2 t\right] dt
\end{aligned} \tag{3.8}$$

$$\big(\kappa \geqq 0;\ \Re(x_1 + +x_2) < 1,\ \Re(a) > 0 \quad when \quad \kappa = 0\big)$$

and

(3.9)

$$\begin{aligned}
&\Gamma_2[(a,\kappa), b_1, b_2; c_1, c_2; x_1, x_2] \\
&= \frac{1}{B(b_1, c_1-b_1)B(b_2, c_2-b_2)} \int_0^1 \int_0^1 t^{b_1-1} s^{b_2-1} (1-t)^{c_1-b_1-1}(1-s)^{c_2-b_2-1} \\
&\quad \times (1 - x_1 t - x_2 s)^{-a}\, \frac{\Gamma\left(a, \kappa(1 - x_1 t - x_2 s)\right)}{\Gamma(a)}\, dt\, ds
\end{aligned}$$

$$\big(\kappa \geqq 0; \Re(c_j) > \Re(b_j) > 0\ \ (j = 1, 2) \text{ when } \kappa = 0\big),$$

respectively.

Also, the special cases of (2.6) and (2.4) when $\tau_1 = 1 = \tau_2$ and $\kappa = 0$ are seen to yield the classical integral representations of second Appell functions (see, *e.g.*, [4, 31]).

$$\begin{aligned} &F_2[(a,\kappa), b_1, b_2; c_1, c_2; x_1, x_2] \\ &= \frac{1}{\Gamma(a)} \int_o^\infty e^{-t}\, t^{a-1}\ {}_1F_1\left[\begin{array}{c} b_1; \\ c_1; \end{array} x_1 t\right] {}_1F_1\left[\begin{array}{c} b_2; \\ c_2; \end{array} x_2 t\right] dt \end{aligned} \tag{3.10}$$

$$\big(\Re(x_1 + x_2) < 1,\ \Re(a) > 0\big)$$

and

$$\begin{aligned} &F_2[(a,\kappa), b_1, b_2; c_1, c_2; x_1, x_2] \\ &= \frac{1}{B(b_1, c_1 - b_1) B(b_2, c_2 - b_2)} \\ &\int_0^1 \int_0^1 t^{b_1 - 1} s^{b_2 - 1} (1-t)^{c_1 - b_1 - 1} (1-s)^{c_2 - b_2 - 1} \left(1 - x_1 t - x_2 s\right)^{-a} \ dt\, ds \end{aligned} \tag{3.11}$$

$$\big(\Re(c_j) > \Re(b_j) > 0 \ \ (j = 1, 2)\big),$$

respectively.

3.2. Connections with certain known functions

Theorem 10. *The following relationship with the incomplete gamma function* $\Gamma(s,\kappa)$ *defined by* (1.2) *holds true*:

$$\Gamma_2^{\tau_1,\tau_2}[(a,\kappa), b_1, b_2; b_1, b_2; x_1, x_2] = \frac{(1 - x_1 - x_2)^{-a}}{\Gamma(a)} \Gamma(a, \kappa(1 - x_1 - x_2)) \tag{3.12}$$

$$(|x_1 + x_2| < 1;\ \kappa \geqq 0).$$

Proof. Putting $c_1 = b_1$ and $c_2 = b_2$ in (3.4), we obtain the following simplified form:

$$\Gamma_2^{\tau_1,\tau_2}[(a,\kappa), b_1, b_2; b_1, b_2; x_1, x_2] = \frac{1}{\Gamma(a)} \int_\kappa^\infty e^{-t(1 - x_1 - x_2)}\ t^{a-1} dt. \tag{3.13}$$

Now by setting $t = \frac{u}{1 - x_1 - x_2}$ and $dt = \frac{du}{1 - x_1 - x_2}$ in (3.13), we have

$$\begin{aligned} &\Gamma_2^{\tau_1,\tau_2}[(a,\kappa), b_1, b_2; b_1, b_2; x_1, x_2] \\ &= \frac{(1 - x_1 - x_2)^{-a}}{\Gamma(a)} \int_{\kappa(1 - x_1 - x_2)}^\infty e^{-u}\ u^{a-1} du \quad (|z| < 1), \end{aligned} \tag{3.14}$$

which is precisely the assertion (3.12) asserted by Theorem 10. □

Theorem 11. *The following relationship with the complementary error function* $\operatorname{erfc}(z)$ *holds true*:

$$\Gamma_2^{\tau_1,\tau_2}\left[\left(\frac{1}{2},\kappa\right),b_1,b_2;b_1,b_2;0,1-z\right]=\Gamma_2^{\tau_1,\tau_2}\left[\left(\frac{1}{2},\kappa\right),b_1,b_2;b_1,b_2;1-z,0\right]$$

$$=\frac{1}{\sqrt{z}}\operatorname{erfc}(\sqrt{\kappa z}) \qquad (\kappa \geqq 0). \tag{3.15}$$

Proof. Upon replacing $x_1=0$ and $x_2=1-z$ or $x_2=0$ and $x_1=1-z$ and putting $a=\frac{1}{2}$ in (3.12), we can easily find that the incomplete second τ-Appell function $\Gamma_2^{\tau_1,\tau_2}$ reduces to the complementary error function $\operatorname{erfc}(z)$ as follows:

$$\Gamma_2^{\tau_1,\tau_2}\left[\left(\frac{1}{2},\kappa\right),b_1,b_2;b_1,b_2;0,1-z\right]=\Gamma_2^{\tau_1,\tau_2}\left(\left(\frac{1}{2},\kappa\right),b_1,b_2;b_1,b_2;1-z,0\right)$$

$$=\frac{1}{\sqrt{z}}\operatorname{erfc}(\sqrt{\kappa z}).$$

□

Theorem 12. *Each of the following relationship with the incomplete generalized τ-hypergeometric function ${}_2\Gamma_1^{\tau}(z)$ in* (2.2) *holds true for $\kappa \geqq 0$:*

$$\Gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;x_1,0]={}_2\Gamma_1^{\tau_1}\left[(a,\kappa),\,b_1;\,c_1;\,x_1\right]; \tag{3.16}$$

$$\Gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;0,x_2]={}_2\Gamma_1^{\tau_2}\left[(a,\kappa),\,b_2;\,c_2;\,x_2\right]; \tag{3.17}$$

(3.18)
$$\Gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;b_1,c_2;x_1,x_2]=(1-x_1)^{-a}\,{}_2\Gamma_1^{\tau_2}\left[(a,\kappa(1-x_1)),b_2;c_2;\tfrac{x_2}{1-x_1}\right];$$

(3.19)
$$\Gamma_2^{\tau_1,\tau_2}[(a,\kappa),b_1,b_2;c_1,b_2;x_1,x_2]=(1-x_2)^{-a}\,{}_2\Gamma_1^{\tau_1}\left[(a,\kappa(1-x_2)),b_1;c_1;\tfrac{x_1}{1-x_2}\right].$$

Proof. The proofs of (3.16) to (3.19) are direct consequences of the definitions (3.2) and (3.4). □

Corollary 1. *Each of the following relationship with the incomplete Gauss hypergeometric function ${}_2\Gamma_1(z)$ in* (1.5) *holds true for $\kappa \geqq 0$:*

$$\Gamma_2^{1,\tau_2}[(a,\kappa),b_1,b_2;c_1,c_2;x_1,0]={}_2\Gamma_1\left[(a,\kappa),\,b_1;\,c_1;\,x_1\right]; \tag{3.20}$$

$$\Gamma_2^{\tau_1,1}[(a,\kappa),b_1,b_2;c_1,c_2;0,x_2]={}_2\Gamma_1\left[(a,\kappa),\,b_2;\,c_2;\,x_2\right]; \tag{3.21}$$

(3.22)
$$\Gamma_2^{\tau_1,1}[(a,\kappa),b_1,b_2;b_1,c_2;x_1,x_2]=(1-x_1)^{-a}\,{}_2\Gamma_1\left[(a,\kappa(1-x_1)),\,b_2;\,c_2;\,\tfrac{x_2}{1-x_1}\right];$$

(3.23)
$$\Gamma_2^{1,\tau_2}[(a,\kappa),b_1,b_2;c_1,b_2;x_1,x_2]=(1-x_2)^{-a}\,{}_2\Gamma_1\left[(a,\kappa(1-x_2)),\,b_1;\,c_1;\,\tfrac{x_1}{1-x_2}\right].$$

4. Concluding remarks and observations

In our present investigation, with the help of the incomplete Pochhammer symbols $(\lambda;\kappa)_\nu$ and $[\lambda;\kappa]_\nu$, we have introduced the incomplete generalized τ-hypergeometric function ${}_2\gamma_1^\tau(z)$ and ${}_2\Gamma_1^\tau(z)$ and investigated their diverse properties such mainly as integral representations, derivative formula, Euler-beta transform and obtain Riemann-Liouville fractional integration and differentiation formulas. Further, we have introduced and investigated the family of incomplete second τ-Appell hypergeometric functions $\Gamma_2^{\tau_1,\tau_2}$ and $\gamma_2^{\tau_1,\tau_2}$ of two variables. The special cases of the results presented here when $\kappa = 0$ would reduce to the corresponding well-known results for the generalized τ-hypergeometric function (see, for details, [10, 12, 18, 24, 41, 42]) and second τ-Appell function (see, for details, [2]).

The expressions of the integrals, which we have evaluated in this paper, are (presumably) new and generalize the results in the existing literature (see, for details, [4, 23, 31, 32]).

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