SATURATED STRUCTURES FROM PROBABILITY THEORY

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Abstract. In the setting of continuous logic, we study atomless probability spaces and atomless random variable structures. We characterize κ-saturated atomless probability spaces and κ-saturated atomless random variable structures for every infinite cardinal κ. Moreover, κ-saturated and strongly κ-homogeneous atomless probability spaces and κ-saturated and strongly κ-homogeneous atomless random variable structures are characterized for every infinite cardinal κ. For atomless probability spaces, we prove that $\aleph_1$-saturation is equivalent to Hoover-Keisler saturation. For atomless random variable structures whose underlying probability spaces are Hoover-Keisler saturated, we prove several equivalent conditions.

1. Introduction

The notion of saturation comes from model theory and is one of the central notions in model theory. Hoover and Keisler in [14] studied saturated probability spaces. Although classical first order logic is not suitable for applications to probabilistic structures, Keisler and his followers studied probability spaces with a model-theoretic flavor. The book [10] is a survey of this topic. Let Γ be a probability space and let $\mathcal{X}$ be a Polish metric space. Let $f, g \colon \Gamma \to \mathcal{X}$ be random variables. A probability space Ω is said to have the *saturation property for* $\mathrm{dist}(f, g)$ if for every random variable $f' \colon \Omega \to \mathcal{X}$ with $\mathrm{dist}(f) = \mathrm{dist}(f')$, there is a random variable $g' \colon \Omega \to \mathcal{X}$ such that $\mathrm{dist}(f, g) = \mathrm{dist}(f', g')$. A probability space Ω is said to be *Hoover-Keisler saturated* if for all random variables $f, g \colon \Gamma \to \mathcal{X}$, where Γ is an probability space and $\mathcal{X}$ is a Polish metric space, Ω has the saturation property for $\mathrm{dist}(f, g)$. It was shown in [10, 13, 14, 15] that many properties, such as saturation properties for random variables and stochastic processes, existence of solutions of stochastic integral equations, regularity properties for distributions of correspondences, and the

existence of pure strategy equilibria in games with many players, are not real-
ized in the standard Lebesgue space, but are realized in saturated probability
spaces. The earlier work used nonstandard methods, especially used the Loeb
probability spaces, and [15] used measure-theoretic probabilistic methods.

Continuous logic was developed recently; see [4] and [6] for references. In
continuous logic, the truth value table is the interval $[0, 1]$ instead of {True,
False} in classical logic. Continuous logic is better suited to study structures
from analysis than classical first order logic, and it has improved the effective-
ness of first order model theory for those structures; for instance, see [2, 3, 7].
In continuous logic, the notions that were investigated by Keisler and his fol-
lowers, such as universality, homogeneity, and saturation, are defined in an
analogous way as in first order model theory.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Naturally, we define a probability
algebra $(\hat{\mathcal{F}}, \mu)$ associated to $(\Omega, \mathcal{F}, \mu)$; see Section 2.1 for details. Also, we
define a ($[0, 1]$-valued) random variable structure $L^1\big((\Omega, \mathcal{F}, \mu), [0, 1]\big)$ based on
$(\Omega, \mathcal{F}, \mu)$; see more in Section 2.2. Ben Yaacov [1] studied probability algebras
in the setting of compact abstract theories. His work on probability algebras
was later carried into continuous logic by Berenstein and Henson as part of the
background of their paper [7]. The theory of probability algebras is axiomatized
by Pr and APr for its atomless counterpart. In [2], Ben Yaacov studied the
theory of ($[0, 1]$-valued) random variable structures (axiomatized by RV), and
its atomless counterpart (axiomatized by ARV). He showed that the theory Pr
and the theory RV are biinterpretable.

In this paper, we study probability spaces and random variable structures
under the framework of continuous logic. In the theory APr, we study notions
of homogeneity and saturation in the setting of continuous logic, and connect
them with analogous notions studied by Keisler, Hoover, and Fajardo. Our new
result here is Theorem 4.1, which characterizes κ-saturated models of APr for all
infinite cardinals κ. Our main tool in proving this result is Maharam's Theorem
2.4 for measure algebras. We prove that a probability space is Hoover-Keisler
saturated if and only if it is $\aleph_1$-saturated. Then κ-saturated and strongly
κ-homogeneous models of APr are described for all infinite cardinals κ. In
the theory ARV, Theorem 5.3 characterizes κ-saturated models of ARV for
all infinite cardinals κ. We give several equivalent conditions in Theorem 5.4
for Hoover-Keisler saturation. Then κ-saturated and strongly κ-homogeneous
models of ARV are described for all infinite cardinals κ.

We assume the reader is familiar with basics of continuous logic. This paper
is organized as follows. Section 2 gives the background and notations for mea-
sure algebras (*e.g.*, Maharam's Theorem), and the theories APr and ARV. We
also list some technical lemmas. In Section 3, several notions of homogeneity
coming from measure theory and model theory are discussed. In Section 4,
we describe κ-saturated models of APr for every infinite cardinal κ and then
we show that for atomless probability spaces, $\aleph_1$-saturation is equivalent to

Hoover-Keisler saturation. In Section 5, we characterize κ-saturated models of ARV for all infinite cardinals κ.

2. Background and notations

In this section, we give the background and notations needed in this paper. We also list some technical lemmas that will be used in later sections.

2.1. Measure algebras

In this subsection, we present the necessary background for measure algebras. Some properties of "atomless over" are given, which play an important role in the next sections. Also, we present Maharam's celebrated theorem (Theorem 2.4), which characterizes the structure of measure algebras. See [11] for more details.

A *measure space* is a triple (X, A, μ) where X is a set, A is a σ-algebra of subsets of X, and $\mu \colon A \to [0, \infty)$ is a countably additive finite-valued measure. A *measured algebra* is a pair (A, μ) where A is a σ-complete boolean algebra and $\mu \colon A \to [0, \infty)$ is a finite-valued function such that $\mu(a) = 0$ if and only if $a = \mathbf{0}$, and μ is countably additive. A boolean algebra A is said to be a *measure algebra* if there is a finite-valued μ for which (A, μ) is a measured algebra. A measured algebra (A, μ) is called a *probability algebra* if $\mu(\mathbf{1}) = 1$. Let (A, μ) be a measured algebra. For all $a, b \in A$, define

$$d(a, b) = \mu(a \triangle b),$$

where $\triangle$ is the symmetric difference of those two sets. In [12, Lemma 323F], it is shown that A is complete under this metric.

Let (X, A, μ) be a measure space. For all $a, b \in A$, we write $a \equiv_\mu b$ if $\mu(a \triangle b) = 0$. Note that $\equiv_\mu$ defines an equivalence relation. We denote the equivalence class of a under $\equiv_\mu$ by $[a]_\mu$. Let $\widehat{A}$ denote the set $\{[a]_\mu \mid a \in A\}$. Then it is routine to verify that the operations of complement, union and intersection on A induce operations on $\widehat{A}$, which make $\widehat{A}$ a σ-complete boolean algebra. Moreover, μ induces a countably additive, strictly positive measure on $\widehat{A}$. We call $(\widehat{A}, \mu)$ the *measured algebra associated to* (X, A, μ) and we call $\widehat{A}$ the *measure algebra associated to* (X, A, μ). If the measure μ on A is a probability measure, then we call $(\widehat{A}, \mu)$ the *probability algebra associated to* (X, A, μ).

Let A be a boolean algebra and let A^+ denote the set of nonzero elements in A. For every $a \in A^+$, *the relative algebra of A on a*, denoted by $A \restriction a$, is the set $\{a \cap b \mid b \in A\}$. Note that $A \restriction a$ is also a boolean algebra, where a becomes the $\mathbf{1}$.

Definition. Let A be a measure algebra. Then

(i) $\tau(A) := \inf\{|X| : X \subseteq A \text{ and } A \text{ is } \sigma\text{-generated by } X\}$.
(ii) (τ-homogeneous) We say that A is τ-*homogeneous* if $\tau(A \restriction a) = \tau(A)$ for all $a \in A^+$.

(iii) (homogeneous) We say that A is *homogeneous* if $A \cong A \upharpoonright a$ as a boolean algebra for all $a \in A^+$.

(iv) Suppose B is a σ-subalgebra of A. For every $a \in A$, we define

$$\tau_B(a) = \inf\big\{|X| : X \subseteq A \text{ and } A \upharpoonright a \text{ is } \sigma\text{-generated by } \{a \cap b \mid b \in X \cup B\}\big\}.$$

Definition. (i) A measure space (X, A, μ) is *atomless* if for every $a \in A$ of positive measure, there exists $b \in A$ such that $b \subseteq a$ and $0 < \mu(b) < \mu(a)$.

(ii) Let (X, A, μ) be a measure space and let B be a σ-subalgebra of A. We say that *A is atomless over B*, if for every $a \in A$ of positive measure, there exists $c \in A$ such that for all $b \in B$, we have $a \cap c \neq a \cap b$.

(iii) Let (X, A, μ) be a measure space and let B be a σ-subalgebra of A. Given an infinite cardinal κ, we say that *A is κ-atomless over B*, if for every σ-subalgebra B', which is σ-generated by $B \cup S$, where S is a set of cardinality $< \kappa$ in A, we have that A is atomless over B'. When B is trivial and A is κ-atomless over B, we say simply that *(X, A, μ) is κ-atomless*.

Analogously, we define these notions of atomlessness for measure algebras.

Definition. (i) A measure algebra A is *atomless* if for every $a \in A^+$, there exists $b \in A^+$ such that $b \cap a = b$ and $b \neq a$.

(ii) Let A be a measure algebra and let B be a σ-complete subalgebra of A. We say that *A is atomless over B*, if for every $a \in A^+$, there exists $c \in A$ such that for all $b \in B$, we have $a \cap c \neq a \cap b$.

(iii) Let A be a measure algebra and let B be a σ-complete subalgebra of A. Given an infinite cardinal κ, we say that *A is κ-atomless over B*, if for every σ-complete subalgebra B', which is σ-generated by $B \cup S$, where S is a set of cardinality $< \kappa$ in A, we have that A is atomless over B'. When B is trivial and A is κ-atomless over B, we say simply that *A is κ-atomless*.

Lemma 2.1 (Maharam's Lemma). *Let $(X, A, \mu) \supseteq (X, B, \mu)$ be measure spaces and let $\widehat{A}$ and $\widehat{B}$ be the measure algebras associated to (X, A, μ) and (X, B, μ) respectively. Then the following are equivalent*:

(i) *The measure space (X, A, μ) is atomless over (X, B, μ).*

(ii) *For every $a \in A$ of positive measure and for every B-measurable function $f : X \to \mathbb{R}$ such that $0 \leq f \leq \mathbb{E}(a \mid B)$, there is a set $b \in A$ such that $b \subseteq a$ and $\mathbb{E}(b \mid B) = f$.*

(iii) *The measure algebra $\widehat{A}$ is atomless over $\widehat{B}$.*

Proof. (i) $\Leftrightarrow$ (ii): This is [8, Theorem 1.3].
(i) $\Leftrightarrow$ (iii): This is easy and left to the readers. $\square$

For the notion of "atomless over", we have the following technical lemma:

Lemma 2.2. *Let $A \supseteq B$ be two probability algebras. If A is atomless over B, then $\tau_B(a) \geq \omega$ for all $a \in A^+$.*

Proof. This follows from [14, Lemma 4.4(ii)]. $\qquad\qquad\qquad\qquad\qquad\square$

Let $[0,1]^\kappa$ denote the product of κ copies of $[0,1]$ with Lebesgue measure and let μ_κ be the product measure on it. Let (A_κ, μ_κ) denote the measured algebra associated to the product measure space $([0,1]^\kappa, \mu_\kappa)$. For all subsets $\Delta \supseteq \Gamma$ of κ, let $\mathrm{pr}_\Gamma^\Delta \colon [0,1]^\Delta \to [0,1]^\Gamma$ be the canonical projection mapping. We write simply pr_Γ if $\Delta = \kappa$. The following proposition describes the Borel measurable subsets of $[0,1]^\kappa$.

Proposition 2.3. *Let κ be an infinite ordinal and let (A_κ, μ_κ) denote the measured algebra associated to the product measure space $([0,1]^\kappa, \mu_\kappa)$. Then*

(i) *The set of Borel measurable subsets of $[0,1]^\kappa$ is*

$$B_\kappa = \{\mathrm{pr}_\Gamma^{-1}(G) \mid \Gamma \subseteq \kappa \text{ is countable and } G \text{ is a Borel subset of } [0,1]^\Gamma\}.$$

(ii) *For all countable subsets Γ of κ and all Borel subsets G of the product measure space $([0,1]^\Gamma, \mu_\Gamma)$, we have*

$$\mu_\kappa(\mathrm{pr}_\Gamma^{-1}(G)) = \mu_\Gamma(G).$$

(iii) *For every $A \in A_\kappa$, there is $B \in B_\kappa$ such that $[B]_{\mu_\kappa} = A$, where $[B]_{\mu_\kappa}$ is the equivalence class of B under $\equiv_{\mu_\kappa}$.*

Proof. (i) Note that B_κ is a σ-algebra and it contains the basic open sets in the product topological space $[0,1]^\kappa$. Hence B_κ is the set of Borel measurable subsets of the product measure space $[0,1]^\kappa$.

(ii) and (iii) follow from [11, Theorem 1.11]. $\qquad\qquad\qquad\qquad\square$

The following Maharam's Theorem [16] gives us the characterization of measure algebras.

Theorem 2.4 (Maharam's Theorem [16, Theorem 2]; see also [11, Theorem 3.9])**.** *For all atomless probability spaces Ω, there is a countable set of distinct infinite cardinals $S = \{\kappa_i \mid i \in I\}$ such that the measure algebra of Ω is isomorphic to a convex combination of the homogeneous probability algebras $[0,1]^{\kappa_i}$. The set S is uniquely determined by Ω and is called the Maharam spectrum of Ω.* $\qquad\qquad\square$

Note that the measure algebra of $([0,1]^n, \mu_n)$ is isomorphic to the measure algebra of $([0,1]^\omega, \mu_\omega)$ for all $n \geq 1$.

Remark 2.5. The Maharam spectrum of $([0,1]^\kappa, \mu_\kappa)$ is $\{\max(\aleph_0, \kappa)\}$.

2.2. Theories APr and ARV

Here, we give notations and background for the theories APr and ARV. For more details, see [2] and [7].

First, we introduce the theory APr, which was first studied by Ben Yaacov in [1] under the setting of compact abstract theories. Later, his work was carried into continuous logic in the background sections of Berenstein and Henson in [7]. Here, we follow the notations in [7]. Let the signature L_{Pr} denote the set $\{\mathbf{0}, \mathbf{1}, \cdot^c, \cap, \cup, \mu\}$, where $\mathbf{0}$ and $\mathbf{1}$ are constant symbols, $\cdot^c$ is a unary function symbol, $\cap$ and $\cup$ are binary function symbols, and μ is a unary predicate symbol. Among those symbols, $\cdot^c$ and μ are 1-Lipschitz, and $\cap$ and $\cup$ are 2-Lipschitz. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $\widehat{\mathcal{F}}$ be the probability algebra associated to $(\Omega, \mathcal{F}, \mu)$. Now we interpret $\widehat{\mathcal{F}}$ as an L_{Pr}-structure $\mathcal{M} = (\widehat{\mathcal{F}}, \mathbf{0}, \mathbf{1}, \cdot^c, \cap, \cup, \mu, d)$. We interpret $\mathbf{0}$ as $[\emptyset]_\mu$, $\mathbf{1}$ as $[\Omega]_\mu$, the unary function $\cdot^c$ as the complement of equivalence classes of events, and $\cap$, $\cup$ as the intersection and union of equivalence classes of events. The predicate μ is interpreted as the measure of equivalence classes of events, and d, the metric on $\widehat{\mathcal{F}}$, is defined as $d(A, B) := \mu(A \triangle B)$ for all $A, B \in \widehat{\mathcal{F}}$. Note that the interpretations of all function and predicate symbols satisfy the moduli of uniform continuity of those symbols. Also $(\widehat{\mathcal{F}}, d)$ is a complete metric space. Hence $\mathcal{M}$ is an L_{Pr}-structure. The L_{Pr}-structure $\mathcal{M} = (\hat{\mathcal{F}}, \mathbf{0}, \mathbf{1}, \cdot^c, \cap, \cup, \mu)$ is called a *probability algebra*. It is called an *atomless probability algebra* if the probability space $(\Omega, \mathcal{F}, \mu)$ is atomless. The L_{Pr}-theory of probability algebra is axiomatized by the theory Pr and APr for its atomless counterpart; see [7] for axioms of Pr and APr. The theory APr is separably categorical, complete, admits quantifier elimination, and is ω-stable, by [1, Proposition 2.10].

Next, we introduce the theory ARV, which was first introduced by Ben Yaacov in [2] building on the work of APr. The theory ARV was further studied in [3, 18]. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let $L^1\big((\Omega, \mathcal{F}, \mu), [0, 1]\big)$, or simply $L^1(\mu, [0, 1])$, denote the L^1-space of classes of $[0, 1]$-valued $\mathcal{F}$-measurable functions equipped with L^1-metric. A ($[0, 1]$-valued) *random variable structure* is based on a set of the form $M = L^1\big((\Omega, \mathcal{F}, \mu), [0, 1]\big)$, where $(\Omega, \mathcal{F}, \mu)$ is a probability space. It is called an *atomless random variable structure*, if its underlying probability space is atomless. We consider the signature $L_{\mathsf{RV}} = \{\mathbf{0}, \neg, \dotminus, \frac{1}{2}, I\}$, where $\mathbf{0}$ is a constant symbol, $\dotminus$ is a binary function symbol, $\neg$ and $\frac{1}{2}$ are unary function symbols, and I is a unary predicate symbol. Among those symbols, $\neg$ is 1-Lipschitz, $\dotminus$ is 2-Lipschitz, $\frac{1}{2}$ is $\frac{1}{2}$-Lipschitz, and I is 1-Lipschitz. We interpret the symbols of L_{RV} in M as follows:

- $\mathbf{0}^{\mathcal{M}}(\omega) = 0$ for all $\omega \in \Omega$,
- $\neg^{\mathcal{M}}(f) = 1 - f$ for all $f \in M$,
- $(\dotminus)^{\mathcal{M}}(f, g) = f \dotminus g = \max(f - g, 0)$ for all $f, g \in M$,
- $(\frac{1}{2})^{\mathcal{M}} f = f/2$ for all $f \in M$,
- $I^{\mathcal{M}}(f) = \int_\Omega f d\mu$ for all $f \in M$,

- $d^{\mathcal{M}}(f,g) = \int_{\Omega} |f - g| d\mu$ for all $f, g \in M$.

Then $\mathcal{M} = \left(L^1((\Omega, \mathcal{F}, \mu), [0, 1]), \mathbf{0}, \neg, \dot{-}, \frac{1}{2}, I, d \right)$ is an L_{RV}-structure. Let $\mathcal{RV}$ denote the class of all random variable structures as L_{RV}-structures and let $\mathcal{ARV}$ denote the class of all atomless random variable structures as L_{RV}-structures. Ben Yaacov [2] axiomatized the class $\mathcal{RV}$ by the theory RV and its atomless counterpart by the theory ARV. Ben Yaacov's axiomatization is based on the completeness theorem for Lukasiewicz's $[0, 1]$-valued propositional logic. An elementary approach to axiomatizing the class $\mathcal{RV}$ is given in [19]. The theory ARV is separably categorical, complete, admits quantifier elimination, and is ω-stable, by [2, Theorem 2.17].

3. Homogeneous models of APr

Definition 2.1 gives several notions of homogeneity coming from measure theory. In this section, we will show the connection between those notions of homogeneity and model-theoretic homogeneity.

Proposition 3.1. *Let (M, μ, d) be a probability algebra. Then $\tau(M) + \aleph_0 = \|M\| + \aleph_0$ where $\|M\|$ is the density character of M.*

Proof. This is trivial when M is finite. Assume M is infinite, then $\tau(M)$ and $\|M\|$ are both infinite. Let X be a subset of M such that M is σ-generated by X. Then consider the smallest boolean algebra A containing X. Since X is infinite, $|X| = |A|$. Note that A is a dense subset of M. Thus $\|M\| \leq |A| = |X|$, whereby $\|M\| \leq \tau(M)$. Let Y be a dense subset of M. Since the σ-complete subalgebra $\sigma(Y)$ is a measure algebra, it is complete under the metric $d(a, b) = \mu(a \triangle b)$. Hence, we have $\sigma(Y) \supseteq \overline{Y} = M$. Thus $\tau(M) \leq |Y|$, whereby $\tau(M) \leq \|M\|$. Therefore, $\tau(M) = \|M\|$. $\square$

The following result shows the connection among several notions of homogeneity.

Proposition 3.2. *Let $\mathcal{A} = (A, \mu, d)$ be an atomless probability algebra. Then the following are equivalent:*

  (i) *A is τ-homogeneous as a measure algebra.*
  (ii) *A is homogeneous as a measure algebra.*
  (iii) *$\mathcal{A}$ is strongly $\|A\|$-homogeneous as an L_{Pr}-structure.*
  (iv) *$\mathcal{A}$ is strongly ω-homogeneous as an L_{Pr}-structure.*

Proof. The implication (i) $\Rightarrow$ (ii) is [11, Corollary 3.6].
   For (ii) $\Rightarrow$ (iii), there are two cases.
   (a) When $\|A\| = \tau(A) = \aleph_0$, the proof is easy, since the σ-subalgebra σ-generated by a finite set is still finite.
   (b) Suppose $\|A\| = \tau(A) > \aleph_0$. Let C be an infinite subset of A with $\mathrm{Card}(C) < \|A\|$, and let $a, b \in A$ such that $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$. Define $g \colon a \cup C \to b \cup C$ as $g(a) = b$ and $g(c) = c$ for all $c \in C$. Clearly, g extends to

a monomorphism $g' \colon \sigma(a, C) \to \sigma(b, C) \subseteq A$, where $\sigma(a, C)$ and $\sigma(b, C)$ are σ-generated σ-subalgebras. Note that $\tau(\sigma(a, C)) = \|\sigma(a, C)\| < \|A\|$. By [11, Corollary 3.19], g' extends to an automorphism of A fixing $\sigma(C)$ pointwise and sending a to b, which shows that A is strongly $\|A\|$-homogeneous.

The implication (iii) $\Rightarrow$ (iv) is trivial.

We complete the proof by showing (iv) $\Rightarrow$ (i). We argue as follows:

(1) For all $a, b \in A^+$, if $\mu(a) = \mu(b)$, then there exists an automorphism f sending a to b. Hence $\tau(A \restriction a) = \tau(A \restriction b)$.

(2) The function τ is an increasing function with respect to the measures, that is if $\mu(a) \leq \mu(b)$, then $\tau(A \restriction a) \leq \tau(A \restriction b)$. Because there always exists $b' \in A$ such that $a \subseteq b'$ and $\mu(b') = \mu(b)$, then by the above Fact (1), we have $\tau(A \restriction b') = \tau(A \restriction b)$. Since $a \subseteq b'$, we have $\tau(A \restriction a) \leq \tau(A \restriction b')$. Then we get $\tau(A \restriction a) \leq \tau(A \restriction b)$.

(3) If $a \cap b = 0$, then $\tau(A \restriction (a \cup b)) \leq \tau(A \restriction a) + \tau(A \restriction b) = \max\bigl(\tau(A \restriction a), \tau(A \restriction b)\bigr)$, when $\tau(A \restriction a)$ and $\tau(A \restriction b)$ are both infinite. Since A is atomless, for all $a \in A^+$ we have that $\tau(A \restriction a)$ is infinite, which is actually a special case of Lemma 2.2.

From Facts (1), (2), and (3), we get that $\tau(A \restriction a)$ has a constant value for all $a \in A^+$. Hence we have proved that A is τ-homogeneous. $\qquad\square$

By [11, Corollary 3.8], every homogeneous probability algebra (A, μ) is isomorphic to (A_κ, μ_κ) where $\kappa = \tau(A)$.

4. Saturated models of APr

In this section, we describe the κ-saturated models of APr for each infinite cardinal κ and then we show that for atomless probability spaces, $\aleph_1$-saturation is equivalent to Hoover-Keisler saturation.

Definition. Let κ be an infinite cardinal. A probability space Ω is κ-saturated if its probability algebra is κ-saturated as an L_{Pr}-structure.

Theorem 4.1. *If Ω is an atomless probability space and κ is an infinite cardinal, then the following are equivalent*:

(i) Ω *is κ-saturated.*

(ii) *Every cardinal in the Maharam spectrum of Ω is $\geq \kappa$.*

(iii) Ω *is κ-atomless.*

Proof. (i) $\Longrightarrow$ (ii): Let $(\mathscr{B}, \mu)$ be the probability algebra associated to Ω; hence $\mathscr{B} \models \mathsf{APr}$. Suppose the least cardinal λ in the Maharam spectrum of $\mathscr{B}$ is less than κ. By Theorem 2.4 (Maharam's Theorem), Ω can be decomposed into two parts: $(\mathscr{B}, \mu) \cong (A_\lambda, \alpha\mu_\lambda) \oplus (\mathscr{B}', \mu')$ where (A_λ, μ_λ) is the probability algebra of $([0, 1]^\lambda, \mu_\lambda)$, every cardinal in the Maharam spectrum of $\mathscr{B}'$ is $> \lambda$, the real number $\alpha \in (0, 1]$, and $\mu'(1_{\mathscr{B}'}) = 1 - \alpha$. Note that A_λ as a metric space with the metric $d_\lambda(a, b) := \mu_\lambda(a \triangle b)$ has a dense subset of cardinality λ.

Let $\{a_\gamma \mid \gamma \in \lambda\}$ be such a dense subset. Without loss of generality, let a_0 be $[0,1]^\lambda$. Let $p(x/\{a_\gamma \oplus \mathbf{0}_{\mathscr{B}'} \mid \gamma \in \lambda\})$ consist of the following conditions:

- $\frac{\mu(x)}{\alpha} = \frac{1}{2}$;
- $\frac{\mu(x \cap (a_\gamma \oplus \mathbf{0}_{\mathscr{B}'}))}{\alpha} = \frac{\mu(x)}{\alpha} \frac{\mu(a_\gamma \oplus \mathbf{0}_{\mathscr{B}'})}{\alpha}$ for all $\gamma \in \lambda$.

The second property implies that if $x = A \oplus B$, where $A \in A_\lambda$ and $B \in \mathscr{B}'$, then A is independent from A_λ and $B = \mathbf{0}_{\mathscr{B}'}$. Now, we consider the probability algebra of $([0,1]^\lambda \times [0,1], \alpha\mu_\lambda \times \mu_1) \oplus ([0,1], (1-\alpha)\mu_1)$. It is a model of APr. Let $a'_\gamma := (a_\gamma \times [0,1]) \oplus \mathbf{0}_{[0,1]}$ and $a := ([0,1]^\lambda \times [0, \frac{1}{2}]) \oplus \mathbf{0}_{[0,1]}$. Then we have that $\mathrm{tp}(a'_\gamma : \gamma \in \lambda) = \mathrm{tp}(a_\gamma \oplus \mathbf{0}_{[0,1]} : \gamma \in \lambda)$ and a realizes $p(x/\{a'_\gamma : \gamma \in \lambda\})$. Hence, $p(x/\{a_\gamma \oplus \mathbf{0}_{\mathscr{B}'} \mid \gamma \in \lambda\})$ is consistent. Since $\mathscr{B}$ is κ-saturated, $p(x/\{a_\gamma \oplus \mathbf{0}_{\mathscr{B}'} \mid \gamma \in \lambda\})$ is realized by an element a in $\mathscr{B}$. Say $a = A \oplus B$, where $A \in A_\lambda$ and $B \in \mathscr{B}'$. Then

$$\alpha\mu((A \oplus B) \cap (a_\gamma \oplus \mathbf{0}_{\mathscr{B}'})) = \mu(A \oplus B)\mu(a_\gamma \oplus \mathbf{0}_{\mathscr{B}'}), \forall \gamma \in \lambda.$$

Thus $\alpha\mu((A \cap a_\gamma) \oplus (\mathbf{0}_{\mathscr{B}'})) = (\alpha\mu_\lambda(A) + \mu'(B))\alpha\mu_\lambda(a_\gamma)$. Then, we get

$$\alpha\mu_\lambda(A \cap a_\gamma) = (\alpha\mu_\lambda(A) + \mu'(B))\mu_\lambda(a_\gamma), \forall \gamma \in \lambda.$$

Letting $\gamma = 0$ and recalling $a_0 = [0,1]^\lambda$, we have $\alpha\mu_\lambda(A) = \alpha\mu_\lambda(A) + \mu'(B)$. Thus $\mu'(B) = 0$. Then, $\mu(A \oplus B) = \alpha\mu_\lambda(A) + \mu'(B) = \alpha\mu_\lambda(A)$. Also $\mu(A \oplus B) = \mu(a) = \frac{\alpha}{2}$, whence $\mu_\lambda(A) = \frac{1}{2}$. Then

$$\alpha\mu_\lambda(A \cap a_\gamma) = \alpha\mu_\lambda(A)\mu_\lambda(a_\gamma), \forall \gamma \in \lambda,$$

whereby

$$\mu_\lambda(A \cap a_\gamma) = \mu_\lambda(A)\mu_\lambda(a_\gamma), \forall \gamma \in \lambda.$$

Note that the set $\{C \in A_\lambda \mid \mu_\lambda(A \cap C) = \mu_\lambda(A)\mu_\lambda(C)\}$ is a closed subset in (A_λ, d_λ). Since $\{a_\gamma \mid \gamma \in \lambda\}$ is dense in (A_λ, d_λ), we have that $\mu_\lambda(A \cap C) = \mu_\lambda(A)\mu_\lambda(C)$ for all $C \in A_\lambda$. Since $A \in A_\lambda$, we have $\mu_\lambda(A) = \mu_\lambda(A)^2$. Hence $\mu_\lambda(A) = 0$ or 1, which contradicts the fact that $\mu_\lambda(A) = \frac{1}{2}$.

(ii) $\implies$ (iii): First, consider the case where $(\mathscr{B}, \mu) \cong (A_\lambda, \mu_\lambda)$ for some $\lambda \geq \kappa$, where (A_λ, μ_λ) is the measured algebra of the product measure space $([0,1]^\lambda, \mu_\lambda)$. Since Ω is atomless, by Lemma 2.2 we know that Ω is $\aleph_0$-atomless. Suppose κ is uncountable. Take $A \subseteq \mathscr{B}$ with $|A| < \kappa$, and let $\mathscr{A} = \sigma(A)$. We will show that $\mathscr{B}$ is atomless over $\mathscr{A}$. Let B_λ denote the set of Borel subsets of the measure space $([0,1]^\lambda, \mu_\lambda)$. By Proposition 2.3, for each $a \in A$, there is $c(a) \in B_\lambda$ of the form $\mathrm{pr}_\Gamma^{-1}(c_\Gamma(a))$ such that $[c(a)]_{\mu_\lambda} = a$, where $\Gamma \subseteq \lambda$ is countable, $c_\Gamma(a)$ is a Borel subset of $[0,1]^\Gamma$, and $\mathrm{pr}_\Gamma : [0,1]^\lambda \to [0,1]^\Gamma$ is the canonical projection. We define $\mathrm{Supp}(c(a)) := \Gamma$. Let $\mathcal{C}$ denote the σ-algebra σ-generated by $\{c(a) \mid a \in A\}$ in B_λ. Note that $\mathscr{B}$ and $\mathscr{A}$ are probability algebras associated to $([0,1]^\lambda, B_\lambda, \mu_\lambda)$ and $([0,1]^\lambda, \mathcal{C}, \mu_\lambda)$ respectively. By Lemma 2.1(iii), to show $\mathscr{B}$ is atomless over $\mathscr{A}$, it suffices to show B_λ is atomless over $\mathcal{C}$. Define $\mathrm{Supp}(\mathcal{C}) := \bigcup_{a \in A} \mathrm{Supp}(c(a))$. Since $|\mathrm{Supp}(c(a))| \leq \aleph_0$ for each $a \in A$, we know that $\mathrm{Supp}(\mathcal{C})$ is of cardinality $|A| \times \aleph_0 < \kappa \leq \lambda$. Take any $\gamma \in \lambda \backslash \mathrm{Supp}(\mathcal{C})$. Let $\mathcal{B}$ denote the set of Borel subsets of $[0,1]^{\{\gamma\}} = [0,1]$.

Then $\mathrm{pr}_{\{\gamma\}}^{-1}(\mathcal{B})$ is an atomless σ-subalgebra of B_λ. Since elements in $\mathcal{C}$ only depend on the coordinates in $\mathrm{Supp}(\mathcal{C})$ and elements in $\mathrm{pr}_{\{\gamma\}}^{-1}(\mathcal{B})$ only depend on the coordinate γ which is not in $\mathrm{Supp}(\mathcal{C})$, we know that $\mathrm{pr}_{\{\gamma\}}^{-1}(\mathcal{B})$ and $\mathcal{C}$ are independent. By [14, Lemma 4.4], B_λ is atomless over $\mathcal{C}$.

For the general case, by Theorem 2.4 (Maharam's Theorem) $\mathcal{B}$ is isomorphic to a convex combination of probability algebras $(A_{\kappa_n})_{n\in\mathbb{N}}$, where $\{\kappa_n \mid n \in \mathbb{N}\}$ is the Maharam spectrum of Ω. Let $p_n\colon \mathcal{B} \to A_{\kappa_n}$ be the canonical projection map for each $n \in \mathbb{N}$. Take $A \subseteq \mathcal{B}$ with $|A| < \kappa$, and let $\mathscr{A} = \sigma(A)$. As shown above, for each κ_n in the Maharam spectrum of Ω, we have that A_{κ_n} is atomless over $p_n(\mathscr{A})$. Let $B \in \mathcal{B}^+$. Then for some $n \in \mathbb{N}$, we have $p_n(B) \in A_{\kappa_n}^+$. Since A_{κ_n} is atomless over $p_n(\mathscr{A})$, there is $C_n \in A_{\kappa_n}$ such that $p_n(B) \cap C_n \neq p_n(B) \cap D_n$ for all $D_n \in p_n(\mathscr{A})$. Hence there is $C \in \mathcal{B}$ such that $B \cap C \neq B \cap D$ for all $D \in \mathscr{A}$, whereby $\mathcal{B}$ is atomless over $\mathscr{A}$.

(iii) $\Longrightarrow$ (i): Fix $A \subseteq \mathcal{B}$ with $|A| < \kappa$; let $\mathscr{A} := \sigma(A)$. By assumption, $\mathcal{B}$ is atomless over $\mathscr{A}$. Let $p(x)$ be any type in $S_1(A)$. By [1, Proposition 2.10], for all a, b in any elementary extension of $\mathcal{B}$, we have that

$$\mathrm{tp}(a/A) = \mathrm{tp}(b/A) \quad \Longleftrightarrow \quad \mathbb{P}(a \mid \mathscr{A}) = \mathbb{P}(b \mid \mathscr{A}).$$

If $p(x)$ is realized by a in some elementary extension of $\mathcal{B}$, then $\mathbb{P}(a \mid \mathscr{A})$ is an $\mathscr{A}$-measurable function. By Lemma 2.1 (Maharam's Lemma), there exists $b \in \mathcal{B}$ such that $\mathbb{P}(a \mid \mathscr{A}) = \mathbb{P}(b \mid \mathscr{A})$. Therefore $\mathrm{tp}(b/A) = \mathrm{tp}(a/A) = p(x)$, which means that $p(x)$ is realized in $\mathcal{B}$. $\square$

Corollary 4.2. *An atomless probability space is Hoover-Keisler saturated if and only if it is $\aleph_1$-saturated.*

Proof. By [10, Theorem 3B.7], an atomless probability space Ω is Hoover-Keisler saturated if and only if every cardinal in its Maharam spectrum is uncountable. By Theorem 4.1, Ω is $\aleph_1$-saturated if and only if every cardinal in its Maharam spectrum is uncountable. $\square$

Theorem 4.3. *For all infinite cardinals κ, the probability algebra of $[0,1]^\kappa$ is strongly κ-homogeneous and κ-saturated. Moreover, it is the unique model of* APr *of density character κ with these properties.*

Proof. The first part follows directly from Proposition 3.2, Theorem 2.4 (Maharam's Theorem) and Theorem 4.1. The uniqueness follows from the following argument: for a complete theory T and $\kappa \geq \mathrm{Card}(T)$, by a standard back-and-forth argument, any two κ-saturated models of T of density character κ are isomorphic to each other. $\square$

5. Saturated models of ARV

In this section, we characterize the κ-saturated models of ARV for all infinite cardinals κ. The theory ARV is a separably categorical theory, then by the Ryll-Nardzewski theorem for continuous logic due to Henson (see [5, Fact 1.14]), we

know that the unique separable model of ARV is approximately $\aleph_0$-saturated. However, it is not $\aleph_0$-saturated.

Theorem 5.1. *No separable model of* ARV *is* $\aleph_0$*-saturated.*

Proof. By [2, Theorem 2.17], the theory ARV is $\aleph_0$-categorical. We may assume the separable model $\mathcal{M}$ is of the form $L^1(\lambda, [0,1])$, where $([0,1], \mathscr{B}, \lambda)$ is the standard Lebesgue space. Then $(\Omega, \mathcal{F}, \mu) = ([0,1] \times \{0,1\}, \mathscr{B} \otimes P(\{0,1\}), \lambda \otimes \mu_0)$ is also a separable probability space, where $\mu_0(\{0\}) = \mu_0(\{1\}) = \frac{1}{2}$ and $P(\{0,1\})$ is the power set of $\{0,1\}$. Let $\{C_n \mid n \in \mathbb{N}\}$ be a countable open basis for $\mathscr{B}$ and let α be a measure-preserving isomorphism between probability spaces $\prod_{n \in \mathbb{N}}(\{0,1\}, P(\{0,1\}), \mu_0)$ and $([0,1], \mathscr{B}, \lambda)$. Define $f \colon [0,1] \to [0,1]$ as follows:

$$f(t) = \alpha\Big(\big(\chi_{C_1}(t), \chi_{C_2}(t), \ldots, \chi_{C_n}(t), \ldots\big)\Big) \text{ for all } t \in [0,1].$$

Then we know $f \in L^1(\lambda, [0,1])$. Now we define functions f', g' from Ω to $[0,1]$ by $f'(t,i) = f(t)$ and $g'(t,i) = i$ for all $(t,i) \in \Omega = [0,1] \times \{0,1\}$. Clearly, they are both in $L^1(\mu, [0,1])$. Note that $\mathrm{dist}(f') = \mathrm{dist}(f)$, which implies $\mathrm{tp}(f') = \mathrm{tp}(f)$ by [2, Theorem 2.17]. Let $p_0(x,y)$ denote the type $\mathrm{tp}(f', g')$. Since $\mathrm{tp}(f') = \mathrm{tp}(f)$, we get that $p_0(f,y)$ is consistent. If $p_0(f,y)$ is realized in M by g, then because $g'^2 = g'$, $I(g') = \frac{1}{2}$ and $\mathrm{tp}(g) = \mathrm{tp}(g')$, we have that $g = \chi_D$ for some $D \in \mathscr{B}$ and $\lambda(D) = \frac{1}{2}$. For all $n \in \mathbb{N}$, let D_n be the set $\{x \in \{0,1\}^{\mathbb{N}} \mid x(n) = 1\}$. Then $\alpha(D_n) \in \mathscr{B}$. Now we have

$$\mu\big((t,i) \in \Omega \mid f'(t,i) \in \alpha(D_n) \text{ and } g'(t,i) = 1\big)$$
$$= \mu\big((t,i) \in \Omega \mid t \in C_n, i = 1\big)$$
$$= \frac{1}{2}\lambda(C_n) = \lambda(D)\lambda(C_n),$$

and

$$\lambda\big(t \in [0,1] \mid f(t) \in \alpha(D_n) \text{ and } g(t) = 1\big) = \lambda(D \cap C_n).$$

It follows from $\mathrm{tp}(f,g) = \mathrm{tp}(f', g')$ that

$$\mu\big((t,i) \in \Omega \mid f'(t,i) \in \alpha(D_n) \text{ and } g'(t,i) = 1\big)$$
$$= \lambda\big(t \in [0,1] \mid f(t) \in \alpha(D_n) \text{ and } g(t) = 1\big),$$

and thus $\lambda(D \cap C_n) = \lambda(D)\lambda(C_n)$. Therefore D is independent from C_n for all $n \in \mathbb{N}$. Similarly, we get that for all finite $J \subset \mathbb{N}$, the set D is independent from $\bigcap_{j \in J} C_j$. Then by [9, page 26], D is independent from $\sigma(\{C_n\}_{n \in \mathbb{N}}) = \mathscr{B}$. Hence, $\lambda(D \cap D) = \lambda(D)^2$. Consequently, $\lambda(D) = 0$ or 1, which contradicts $\lambda(D) = \frac{1}{2}$. $\qquad\square$

Remark 5.2. This proof borrows ideas from [10, Theorem 3B.1]. In [18], an alternative proof using d-finite tuples is given.

Before introducing Theorem 5.3, let us explain how to build an L_{RV}-formula with parameters from an L_{Pr}-formula with parameters. Let $(\Omega, \mathscr{B}, \mu)$ be a probability space, and let $(\hat{\mathscr{B}}, \mu)$ be the probability algebra associated to $(\Omega, \mathscr{B}, \mu)$. Let $M = L^1\big(\mu, [0,1]\big)$. Then $(\hat{\mathscr{B}}, \mu)$ is an L_{Pr}-structure, and (M, d) is an L_{RV}-structure. Let $A \subseteq \hat{\mathscr{B}}$, and let $\tilde{A} := \{\chi_a \mid a \in A\} \subseteq M$. Let $\varphi(x, a)$ be an L_{Pr}-formula, where $a = (a_1, \ldots, a_n) \in A^n$. We will build an L_{RV}-formula $\tilde{\varphi}(x, \tilde{a})$ from $\varphi(x, a)$, where $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n)$. Note that $L_{\mathsf{Pr}} = \{\mathbf{0}, \mathbf{1}, \cdot^c, \cap, \cup, \mu\}$ and $L_{\mathsf{RV}} = \{\mathbf{0}, \neg, \dot{-}, \frac{1}{2}, I\}$. In the syntax of $\varphi(x, a)$, we convert $\mathbf{0}, \mathbf{1}, \cdot^c, \mu$ into $\mathbf{0}, \neg\mathbf{0}, \neg, I$ respectively. For symbols $\cap$ and $\cup$, we convert $a \cap b$ into $\tilde{a} \wedge \tilde{b} := \tilde{a} \dot{-} (\tilde{a} \dot{-} \tilde{b})$, and we convert $a \cup b$ into $\tilde{a} \vee \tilde{b} := \neg(\neg\tilde{a} \wedge \neg\tilde{b})$. Inductively, we get an L_{RV}-formula $\tilde{\varphi}(x, \tilde{a})$ from an L_{Pr}-formula $\varphi(x, a)$.

Theorem 5.3. *Let $\mathcal{M}$ be a model of ARV. Suppose $M = L^1\big(\mu, [0,1]\big)$ for the atomless probability space $(\Omega, \mathscr{B}, \mu)$. For every uncountable cardinal κ, the structure $\mathcal{M} \models \mathsf{ARV}$ is κ-saturated if and only if $(\Omega, \mathscr{B}, \mu)$ is κ-saturated.*

Proof. $\Longrightarrow$: Suppose $\mathcal{M}$ is κ-saturated. We want to show that $\hat{\mathscr{B}}$ is also κ-saturated. For all $A \subseteq \hat{\mathscr{B}}$ with $|A| < \kappa$, and all $p(x) = \mathrm{tp}(x/A) \in S_1^{\mathsf{APr}}(A)$, let $\tilde{A} := \{\chi_a \mid a \in A\} \subseteq M$. Then $|\tilde{A}| < \kappa$. Now, let $\tilde{p}(x)$ be a type in $S_1^{\mathsf{ARV}}(\tilde{A})$ including $x \wedge \neg x = \mathbf{0}$ and all $\tilde{\varphi}(x, \tilde{a})$, where $\varphi(x, a) \in p(x)$. The way how to build $\tilde{\varphi}(x, \tilde{a})$ from $\varphi(x, a)$ is just explained before this theorem. Suppose that $p(x)$ is realized by c in an elementary extension of $\hat{\mathscr{B}}$, then it is easy to notice that χ_c is in an elementary extension of $\mathcal{M}$ and χ_c realizes $\tilde{p}(x)$. Thus, $\tilde{p}(x)$ is consistent. Since $\mathcal{M}$ is κ-saturated, we know that $\tilde{p}(x) \in S_1^{\mathsf{ARV}}(\tilde{A})$ is realized in M by an element, say f. As $f \wedge \neg f = \mathbf{0}$, we know that f is a characteristic function. Thus there is $b \in \hat{\mathscr{B}}$ such that $\chi_b = f$. Then it is easy to verify that $b \models p(x) = \mathrm{tp}(x/A)$, and thus, $\hat{\mathscr{B}}$ is κ-saturated.

$\Longleftarrow$: Let $A \subseteq M$ with $|A| < \kappa$ and $\mathrm{tp}(x/A) \in S_1^{\mathsf{ARV}}(A)$. Suppose $\mathrm{tp}(x/A)$ is realized by f in $\mathcal{N}$, which is an elementary extension of $\mathcal{M}$. For every $t \in [0,1]$, the conditional probability $\mathbb{P}\big(f > t \mid \sigma(A)\big)$ is a $\sigma(A)$-measurable function. Since κ is uncountable, $\sigma(A)$ is σ-generated by $\aleph_0 + |A| < \kappa$ many elements. By Theorem 4.1, $\mathscr{B}$ is κ-atomless, therefore it is atomless over $\sigma(A)$. Thus by [14, Lemma 4.4], there exists a $\mathscr{B}$-measurable random variable $g \colon \Omega \to [0,1]$ such that for every $t \in [0,1]$,

$$\mathbb{P}\big(g > t \mid \sigma(A)\big) = \mathbb{P}\big(f > t \mid \sigma(A)\big) \text{ almost surely.}$$

By [2, Theorem 2.17], $\mathrm{tp}(f/A) = \mathrm{tp}(g/A)$. Therefore, $\mathrm{tp}(x/A) = \mathrm{tp}(f/A)$ is realized by g in M. $\square$

Theorem 5.4. *Let $\mathcal{M} = L^1\big(\mu, [0,1]\big) \models \mathsf{ARV}$, where $(\Omega, \mathscr{B}, \mu)$ is an atomless probability space. Then the following are equivalent*:

(i) Ω *is Hoover-Keisler saturated.*
(ii) Ω *is $\aleph_1$-saturated.*
(iii) *Every cardinal in the Maharam spectrum of Ω is uncountable.*
(iv) $\mathcal{M}$ *is $\aleph_1$-saturated.*

  (v) $\mathcal{M}$ *is $\aleph_0$-saturated.*
  (vi) *For all elements $a, b, c \in M$ with* $\text{tp}(a) = \text{tp}(b)$, *there exists $d \in M$ such that* $\text{tp}(a, c) = \text{tp}(b, d)$.

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is Corollary 4.2.
   The equivalence (ii) $\Longleftrightarrow$ (iii) is Theorem 4.1.
   The equivalence (ii) $\Longleftrightarrow$ (iv) is Theorem 5.3.
   The implication (iv) $\Rightarrow$ (v) is trivial.
   (v) $\Rightarrow$ (vi): Let $p(x, y)$ be $\text{tp}(a, c)$; it is in $S_2(ARV)$. Then $\text{tp}(a) = \text{tp}(b)$ implies that $p(b, y)$ is consistent. Since $\mathcal{M}$ is $\aleph_0$-saturated, $p(b, y)$ is realized in M by some element, say d. Then $\text{tp}(b, d) = \text{tp}(a, c) = p(x, y)$.
   (vi) $\Rightarrow$ (iii): Suppose not, then by Theorem 2.4 (Maharam's Theorem), Ω has a decomposition

$$(\Omega, \mathscr{B}, \mu) \cong r \cdot ([0, 1], \mathcal{L}, \lambda) \oplus (1 - r) \cdot (\Omega', \mathscr{B}', \mu') \text{ for some } r \in (0, 1],$$

where $([0, 1], \mathcal{L}, \lambda)$ is the standard Lebesgue space, $(\Omega', \mathscr{B}', \mu')$ is a probability space, and every cardinal in the Maharam spectrum of the probability space Ω' is uncountable. Take a countable open basis for $[0, 1]$, denoted by $\{C_n \mid n \in \mathbb{N}\}$, and let α be a measure-preserving isomorphism between probability spaces

$$\prod_{n \in \mathbb{N}} (\{0, 1\}, P(\{0, 1\}), \mu_0)$$

and $([0, 1], \mathscr{B}, \lambda)$. Define $f \colon \Omega = [0, 1] \sqcup \Omega' \to [0, 1]$ by $f(\omega') = 0$ for all $\omega' \in \Omega'$ and

$$f(t) = \alpha((\chi_{C_1}(t), \chi_{C_2}(t), \ldots, \chi_{C_n}(t), \ldots)) \text{ for all } t \in [0, 1].$$

Then we know that $f \in L^1(\mu, [0, 1])$. Now define $(\overline{\Omega}, \bar{\mu})$ as $(\Omega, \mu) \times (\{0, 1\}, \mu_0)$ where $\mu_0(\{0\}) = \mu_0(\{1\}) = \frac{1}{2}$. Then define functions f', g' from $\overline{\Omega} = \Omega \times \{0, 1\}$ to $[0, 1]$ by $f'(\omega, i) = f(\omega)$ and $g'(\omega, i) = i$ for all $(\omega, i) \in \Omega \times \{0, 1\}$. They are in $L^1(\bar{\mu}, [0, 1])$. Note that $\text{dist}(f') = \text{dist}(f)$, which implies $\text{tp}(f') = \text{tp}(f)$ by [2, Theorem 2.17]. Let $p(x, y)$ denote the type $\text{tp}(f', g')$. By [2, Theorem 2.17], the theory ARV is separably categorical. Therefore $p(x, y)$ is realized in M, say by (f'', g''). Then $\text{tp}(f'') = \text{tp}(f)$. By the assumption of (vi), we know that $p(f, y)$ is realized in M by an element g, whence $\text{tp}(f, g) = \text{tp}(f', g')$. Because $g'^2 = g'$, $I(g') = \frac{1}{2}$, and $\text{tp}(g) = \text{tp}(g')$, we have that $g = \chi_D$ for some $D \in \mathscr{B}$ and $\mu(D) = \frac{1}{2}$. Say $D = D_0 \sqcup D'$ with $D_0 \subseteq [0, 1]$ and $D' \subseteq \Omega'$. For all $n \in \mathbb{N}$, let D_n be the set $\{x \in \{0, 1\}^{\mathbb{N}} \mid x(n) = 1\}$. Then $\alpha(D_n)$ is a measurable subset of $[0, 1]$. Thus we have

$$\bar{\mu}\big((\omega, i) \in \overline{\Omega} \mid f'(\omega, i) \in \alpha(D_n) \text{ and } g'(\omega, i) = 1\big)$$
$$= \bar{\mu}\big((t, i) \in \bar{\Omega} \mid t \in C_n, i = 1\big) = \frac{1}{2}\mu(C_n) = \mu(D)\mu(C_n),$$

and

$$\mu\big(\omega \in \Omega \mid f(\omega) \in \alpha(D_n) \text{ and } g(\omega) = 1\big) = \mu(D \cap C_n).$$

If follows from $\mathrm{tp}(f,g) = \mathrm{tp}(f',g')$ that

$$\bar{\mu}\big((\omega,i) \in \overline{\Omega} \mid f'(\omega,i) \in \alpha(D_n) \text{ and } g'(\omega,i) = 1\big)$$
$$= \mu\big(\omega \in \Omega \mid f(\omega) \in \alpha(D_n) \text{ and } g(\omega) = 1\big),$$

and thus $\mu(D \cap C_n) = \mu(D)\mu(C_n)$. Therefore D is independent from C_n for all $n \in \mathbb{N}$. Similarly, we get that for all finite $J \subset \mathbb{N}$, the set D is independent from $\bigcap_{j \in J} C_j$. Then by [9, page 26], the set D is independent from $\sigma(\{C_n\}_{n \in \mathbb{N}})$. Therefore D is independent from D_0. Hence, $\mu(D \cap D_0) = \mu(D_0) = \mu(D)\mu(D_0)$. Since $\mu(D) \neq 0$, we have $\mu(D_0) = 0$. Thus we have $f \cdot g = 0$, but $f' \cdot g' \neq 0$, which is a contradiction to $\mathrm{tp}(f,g) = \mathrm{tp}(f',g')$. $\square$

Remark 5.5. [18, Theorem 1.3] gives another equivalent condition; *i.e.*, let f and g be two random variables valued on Polish spaces X and Y respectively, where f is not a discrete random variable. If the probability space $(\Omega, \mathscr{B}, \mu)$ has the saturation property for $\mathrm{dist}(f,g)$ while the standard Lebesgue unit interval $([0,1], \mathcal{L}, \lambda)$ does not, then Ω is Hoover-Keisler saturated.

Theorem 5.6. *For every uncountable cardinal κ, we have that $L^1\big(([0,1]^\kappa, A_\kappa, \mu_\kappa), [0,1]\big) \models \mathsf{ARV}$ is κ-saturated and strongly κ-homogeneous. Moreover, it is the unique model of ARV of density character κ with these properties.*

Proof. The first part follows from Theorem 4.3 and Theorem 5.3. The uniqueness follows from the following argument: for a complete theory T and $\kappa \geq \mathrm{Card}(T)$, by the standard back-and-forth argument any two κ-saturated models of T of density character κ are isomorphic to each other. $\square$

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