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ON THE PRESCRIBED MEAN CURVATURE PROBLEM ON THE STANDARD n-DIMENSIONAL BALL

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Abstract. In this paper, we consider the problem of existence of conformal metrics with prescribed mean curvature on the unit ball of $\mathbb{R}^n$, $n \geq 3$. Under the assumption that the order of flatness at critical points of prescribed mean curvature function $H(x)$ is $\beta \in]1, n-2]$, we give precise estimates on the losses of the compactness and we prove new existence result through an Euler-Hopf type formula.

1. Introduction and main result

In this article, we consider the problem of existence of conformal scalar flat metric with prescribed boundary mean curvature on the standard n-dimensional ball. Let B^n be the unit ball in $\mathbb{R}^n$, $n \geq 3$, with Euclidean metric g_0. Its boundary will be denoted by S^{n-1} and will be endowed with the standard metric still denoted by g_0. Let $H : S^{n-1} \to \mathbb{R}$ be a given function, we study the problem of finding a conformal metric $g = u^{\frac{4}{n-2}} g_0$ such that $R_g = 0$ in B^n and $h_g = H$ on S^{n-1}. Here R_g is the scalar curvature of the metric g in B^n and h_g is the mean curvature of g on S^{n-1}. This problem is equivalent to solving the following nonlinear boundary value equation:

$$(P) \qquad \begin{cases} \Delta u = 0 & \text{in } B^n \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} u = \dfrac{n-2}{2} H u^{\frac{n}{n-2}} & \text{on } S^{n-1}, \end{cases}$$

where ν is the outward unit vector with respect to the metric g_0.

In general, there are several difficulties in facing this problem by means of variational methods. Indeed, in virtue of the non-compactness of the embedding $H^1(B^n) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\partial B^n)$, the Euler-Lagrange functional J associated to the problem does not satisfy the Palais-Smale condition, and that leads to the failure of the standard critical point theory. Moreover, besides the obvious necessary condition that H must be positive somewhere, there are topological obstructions of Kazdan-Warner type to solving (P).

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In this paper, we study the case where the prescribed function H satisfies some kind of flatness near its critical points. Roughly, it is assumed that there exists a real number β such that in some geodesic normal coordinate system centered at y, we have

$$(f)_\beta \qquad H(x) = H(0) + \sum_{i=1}^{n-1} b_i |(x_i)|^\beta + R(x),$$

where $b_i = b_i(y) \in \mathbb{R} \setminus \{0\}, \forall i = 1, \ldots, n-1$, $\sum_{i=1}^{n-1} b_i \neq 0$ and

$$\sum_{s=0}^{[\beta]} |\nabla^s R(x)||x|^{s-\beta} = o(1) \text{ as } x \text{ tends to zero.}$$

Here ∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β.

Let

$$\mathcal{K} = \big\{y \in S^{n-1},\ \nabla H(y) = 0\big\},\ \mathcal{K}^+ = \big\{y \in \mathcal{K},\ \sum_{i=1}^{n-1} b_i < 0\big\}$$

and

$$\widetilde{i}(y) = \sharp\Big\{b_i,\ i = 1, \ldots, n-1, \text{ such that } b_i < 0\Big\}.$$

In [1], the authors proved that under condition $(f)_\beta$, with $n-2 < \beta < n-1$, (P) has a solution provided

$$(1.1) \qquad \sum_{y \in \mathcal{K}^+} (-1)^{n-1-\widetilde{i}(y)} \neq 1.$$

We denote by Ξ the operator which associates to H the solution v of (P) and we extend the definition of Ξ to the case of weak solutions of (P). Let

$$\mathcal{K}_{n-2} = \big\{y \in \mathcal{K},\ \beta = \beta(y) = n-2\big\}.$$

For each p-tuple $(y_{l_1}, \ldots, y_{l_p}) \in \Big(\mathcal{K}^+ \cap \mathcal{K}_{n-2}\Big)^p$ such that $y_{l_i} \neq y_{l_j}$ if $i \neq j$, we associate the matrix $M = (M_{ij})$ defined by

$$\begin{cases} M_{ii} = \dfrac{n-2}{n-1}\widetilde{c}_1 \dfrac{-\sum_{k=1}^{n-1} b_k\big(y_{l_i}\big)}{H\big(y_{l_i}\big)^{n-1}}, & i \in \{1, \ldots, p\}, \\ M_{ij} = -2^{\frac{n-2}{2}} c_1 \dfrac{G\big(y_{l_i}, y_{l_j}\big)}{\big[H\big(y_{l_i}\big)H\big(y_{l_j}\big)\big]^{\frac{n-2}{2}}}, & i, j \in \{1, \ldots, p\},\ i \neq j, \end{cases}$$

where

$$c_1 = c_0^{\frac{2(n-1)}{n-2}} \int_{\mathbb{R}^{n-1}} \frac{dx}{\big(1+|x|^2\big)^{\frac{n}{2}}}, \text{ and } \widetilde{c}_1 = c_0^{\frac{2(n-1)}{n-2}} \int_{\mathbb{R}^{n-1}} \frac{|x_1|^\beta}{\big(1+|x|^2\big)^{n-1}} dx.$$

Here $G(q, \cdot)$ denotes the Green's function for the operator Ξ with point q and x_1 is the component of x in some geodesic normal coordinate system.

Let $\rho = \rho(y_{l_1}, \dots, y_{l_p})$ be the least eigenvalue of M. It was the first pointed out by Bahri [5] that when the interaction between the different bubbles is of the same order as the self interaction. ρ plays a fundamental role in the existence of solutions to problem like (P). Please see [3], such kind of phenomenon appears under $(f)_\beta$ condition when $\beta = n-2$. We assume the following

(A_0) $\qquad \rho(y_{l_1}, \dots, y_{l_p}) \neq 0$ for distinct points $y_{l_1}, \dots, y_{l_p} \in \mathcal{K}^+ \cap \mathcal{K}_{n-2}$.

We now introduce the following set

$$\begin{aligned}\mathcal{C}_\infty := \Big\{ & \tau_p = (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K}^+)^p,\ p \geq 1,\ \text{s.t. } y_i \neq y_j\ \forall i \neq j, \text{ and if} \\ & \{y_{l_1}, \dots, y_{l_p}\} \cap \mathcal{K}_{n-2} \neq \emptyset, \text{ we denote by } y_{i_1}, \dots, y_{i_q} \text{ all elements of} \\ & \{y_{l_1}, \dots, y_{l_p}\} \text{ with } \beta(y_{i_j}) = n-2 \text{ for each } j = 1, \dots, q \text{ and} \\ & \rho(y_{i_1}, \dots, y_{i_q}) > 0 \Big\}. \qquad (1.2)\end{aligned}$$

Our main result is the following:

Theorem 1.1. *Assume that H is a C^1-function satisfying (A_0) and $(f)_\beta$, with*

$$1 < \beta \leq n-2.$$

If

$$\sum_{(y_{l_1}, \dots, y_{l_P}) \in \mathcal{C}_\infty} (-1)^{P-1+\sum_{j=1}^{P} n-1+\widetilde{i}(y_{l_j})} \neq 1,$$

then (P) has at least one solution.

Our argument uses a careful analysis of the lack of compactness of the Euler Lagrange functional J associated to the problem (P). Namely, we study the non-compact orbits of the gradient of J the so-called critical points at infinity following the terminology of Bahri [5]. These critical points at infinity can be treated as usual critical points once a Morse lemma at infinity is performed from which we can derive just as in the classical Morse theory the difference of topology induced by these noncompact orbits and compute their Morse index. Such a Morse lemma at infinity is obtained through the construction of suitable pseudo-gradient for which the Palais-Smale condition is satisfied along the decreasing flow lines, as long as these flow lines do not enter the neighborhood of a finite numbers $y_1, \dots, y_p$ of critical points of K such $(y_1, \dots, y_p) \in \mathcal{C}_\infty$.

Similar Morse lemma at infinity has been established for the problem (P) on the sphere S^n, $n \geq 3$, under the hypothesis that the order of flatness at critical points of H is $\beta \in [n-2, n-1[$, see [3].

The rest of this paper is organized as follows. In Section 2, we set up the variational problem and we recall the expansion of the gradient of the associated Euler-Lagrange functional near infinity. In Section 3, we characterize the critical points at infinity of the associated variational problem. Section 4 is devoted to the proof of the main result Theorem 1.1.

2. General framework and some known facts

2.1. Variational structure and lack of compactness

In this section, we recall the functional setting and the variational problem and its main features. Problem (P) has a variational structure. The Euler-Lagrange functional is

$$J(u) = \Big(\int_{S^{n-1}} H u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\Big)^{\frac{2-n}{n-1}},$$

defined on $H^1(B^n)$ equipped with the norm

$$\|u\|^2 = \int_{B^n} |\nabla u|^2 dv_{g_0} + \frac{n-2}{2}\int_{S^{n-1}} u^2 d\sigma_{g_0},$$

where dv_{g_0} and $d\sigma_{g_0}$ denote the Riemannian measure on B^n and S^{n-1} induced by the metric g_0. We denote by Σ the unit sphere of $H^1(B^n)$ and we set,

$$\Sigma^+ = \{u \in \Sigma,\, u \geq 0\}.$$

The exponent $\frac{2(n-1)}{n-2}$ is critical for the Sobolev trace embedding $H^1(B^n) \to L^q(S^{n-1})$. This embedding being not compact, the functional J does not satisfy the Palais-Smale condition. In fact, let M be a Riemannian manifold and let $f : M \to \mathbb{R}$. We say that f satisfies the *Palais-Smale condition* if every sequence (u_k) such that $f(u_k)$ is bounded and $f'(u_k) \to 0$ has a convergent subsequence (see [12]).

In order to characterize the sequences failing the Palais-Smale condition, we need to introduce some notations.

We will use the notation x for the variables belonging to the unit ball B^n or to the half space $\mathbb{R}^n_+$ defined by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n,\ x_n > 0\}$. We will also use the notation $x = (x', x_n)$ for $x \in \mathbb{R}^n_+$. It will be convenient to perform some stereographic projection in order to reduce the above problem to $\mathbb{R}^n_+$. Let $D^{1,2}(\mathbb{R}^n_+)$ denote the completion of $C^\infty_c(\overline{\mathbb{R}^n_+})$, with respect to the Dirichlet norm. The stereographic projection π_q through an appropriate point $q \in S^{n-1}$ induces an isometry $i :\, H^1(B^n) \longrightarrow D^{1,2}(\mathbb{R}^n_+)$ according to the following formula

$$iu(x) = \Big(\frac{2}{|x'|^2 + (x_n+1)^2}\Big)^{\frac{n-2}{2}} u\Big(\frac{2x'}{|x'|^2 + (x_n+1)^2}, \frac{|x'|^2 + x_n - 1}{|x'|^2 + (x_n+1)^2}\Big),$$

where $x' = (x_1, \dots, x_{n-1})$. In particular, we can check that the following relations hold true for every $u \in H^1(B^n)$,

$$\int_{B^n} |\nabla u|^2 + \frac{n-2}{2}\int_{S^{n-1}} u^2 = \int_{\mathbb{R}^n_+} |\nabla iu|^2 \qquad \text{and}$$

$$\int_{S^{n-1}} |u|^{2\frac{(n-1)}{n-2}} = \int_{\partial\mathbb{R}^n_+} |iu|^{2\frac{(n-1)}{n-2}}.$$

In the sequel, we will identify the function H and its composition with the stereographic projection π_q. We will also identify a point x of B^n and its image by π_q. These facts will be assumed as understood in the sequel.

For $a \in \partial\mathbb{R}^n_+$ and $\lambda > 0$, we define the function:

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{\frac{n-2}{2}}}{\Big((1+\lambda x_n)^2 + \lambda^2|x'-a'|^2\Big)^{\frac{n-2}{2}}}, \tag{2.1}$$

where $x \in \mathbb{R}^n_+$, and $\bar{c}$ is chosen such that $\delta_{a,\lambda}$ satisfies the following equation,

$$\begin{cases} \Delta u &= 0 \quad \text{and } u > 0 \quad \text{in } \mathbb{R}^n_+ \\ -\dfrac{\partial u}{\partial x_n} &= u^{\frac{n}{n-2}} \qquad\qquad \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

So, these bubbles $\delta_{(a,\lambda)}$ are the unique solution of the Yamabe problem on $\mathbb{R}^n_+$, see for example [13]. Set

$$\tilde{\delta}_{a,\lambda} = i^{-1}(\delta_{(a,\lambda)}). \tag{2.2}$$

For $\varepsilon > 0$, $p \in \mathbb{N}$, let us define

$$V(p,\varepsilon) = \begin{cases} u \in \Sigma \text{ s.t. } \exists a_1,\dots,a_p \in S^{n-1}, \exists \alpha_1,\dots,\alpha_p > 0, \\ \exists \lambda_1,\dots,\lambda_p > \varepsilon^{-1} \text{ with } \Big\| u - \displaystyle\sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} \Big\| < \varepsilon, \;\; \varepsilon_{ij} < \varepsilon, \; \forall\, i \neq j, \\ \text{ and } \Big| \dfrac{\alpha_i^{\frac{2}{n-2}} H(a_i)}{\alpha_j^{\frac{2}{n-2}} H(a_j)} - 1 \Big| < \varepsilon, \;\; \forall i,j = 1,\dots,p, \end{cases}$$

where

$$\varepsilon_{ij} = \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2 \Big)^{\frac{2-n}{2}}.$$

If u is a function in $V(p,\varepsilon)$, one can find an optimal representation, following the ideas introduced in Proposition 5.2 of [5] (see also pages 348–350 of [6]). Namely we have:

Proposition 2.1. *For any $p \in \mathbb{N}$, there is $\varepsilon_p > 0$ such that if $\varepsilon \le \varepsilon_p$ and $u \in V(p,\varepsilon)$, then the following minimization problem*

$$\min_{\alpha_i > 0, \lambda_i > 0, a_i \in S^{n-1}} \Big\| u - \sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} \Big\|$$

has a unique solution (α, λ, a) up to a permutation.

In particular, we can write u as follows

$$u = \sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} + v,$$

where v belongs to $H^1(B^n)$ and it satisfies the following:

$$(V_0): \quad \Big\langle v, \psi \Big\rangle = 0 \ \text{ for } \ \psi \in \{\tilde{\delta}_i, \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}, \frac{\partial \tilde{\delta}_i}{\partial a_i}, i = 1, \dots, p\}$$

here $\tilde{\delta}_i = \tilde{\delta}_{(a_i,\lambda_i)}$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product defined on $H^1(\mathbb{B}^n)$ by

$$\langle u, v \rangle = \int_{B^n} \nabla u \nabla v \, dv_{g_0} + \frac{n-2}{2} \int_{S^{n-1}} uv \, d\sigma_{g_0}.$$

The behavior of sequences failing the Palais-Smale condition can be characterized taking into account the uniqueness result of Li and Zhu [13] and following the ideas introduced in [2]. We have the following proposition:

Proposition 2.2. *Let (u_k) be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to zero. Then there exist an integer $p \in \mathbb{N}$, a sequence $(\varepsilon_k) > 0$, ε_k tends to zero, and an extracted subsequence of u_k's, again denoted (u_k) such that $u_k \in V(p, \varepsilon_k)$.*

Now arguing as in [6] (pages 326, 327 and 334), the following Morse lemma allows us to get rid of the v-contributions and shows that it can be neglected with respect to the concentration phenomenon.

Proposition 2.3. *There is a C^1-map which to each $(\alpha_i, a_i, \lambda_i)$ such that $\sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)}$ belongs to $V(p, \varepsilon)$ associates $\overline{v} = \overline{v}(\alpha, a, \lambda)$ such that $\overline{v}$ is unique and satisfies:*

$$J\Big(\sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} + \overline{v}\Big) = \min_{v \in (V_0)} \Big\{ J\Big(\sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} + v\Big)\Big\}.$$

Moreover, there exists a change of variables $v - \overline{v} \to V$ such that

$$J\Big(\sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} + v\Big) = J\Big(\sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} + \overline{v}\Big) + \| V \|^2 .$$

We notice that in the V variable, we define a pseudo-gradient by setting

$$\frac{\partial V}{\partial s} = -\mu V,$$

where μ is a very large constant. Then at $s = 1$, $V(s) = e^{-\mu s} V(0)$ will be as small as we wish. This shows that, in order to define our deformation, we can work as if V was zero. The deformation will extend immediately with the same properties to a neighborhood of zero in the V variable.

The following proposition gives precise estimates of $\overline{v}$.

Proposition 2.4 ([3])**.** *There exists $c > 0$ such that the following holds*

$$\|\overline{v}\| \leq c \sum_{i=1}^p \Big[\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla H(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{\frac{n}{2(n-1)}}}{\lambda_i^{\frac{n}{2}}} \Big]$$

$$+\ c\begin{cases}\displaystyle\sum_{k\neq r}\varepsilon_{k\,r}^{\frac{n}{2(n-2)}}\Big(\log\varepsilon_{kr}^{-1}\Big)^{\frac{n}{2(n-1)}} & \textit{if}\ \ n\geq 4\\ \displaystyle\sum_{k\neq r}\varepsilon_{k\,r}\Big(\log\varepsilon_{kr}^{-1}\Big)^{\frac{1}{2}} & \textit{if}\ \ n=3.\end{cases}$$

At the end of this section, we give the following definition extracted from ([5], definition 09; see also [6] pages 333–334).

Definition 2.1. A critical point at infinity of J on Σ^+ is a limit of a flow line $u(s)$ of the equation

$$\begin{cases}\dfrac{\partial u}{\partial s}=-\partial J(u(s))\\ u(0)=u_0\end{cases}$$

such that $u(s)$ remains in $V(p,\varepsilon(s))$ for $s\geq s_0$. Here, w is either zero or a solution of (P) and $\varepsilon(s)$ is some positive function tending to zero when $s\to+\infty$. Using Proposition 2.1, $u(s)$ can be written as:

$$u(s)=\sum_{i=1}^{p}\alpha_i(s)\delta_{(a_i(s),\,\lambda_i(s))}+v(s).$$

Denoting $\widetilde{\alpha}_i:=\lim_{s\to+\infty}\alpha_i(s)$, $\widetilde{y}_i:=\lim_{s\to+\infty}a_i(s)$, we denote by

$$\sum_{i=1}^{p}\widetilde{\alpha}_i\delta_{(\widetilde{y}_i,\,\infty)}\ \text{or}\ (\widetilde{y}_1,\dots,\widetilde{y}_p)_\infty$$

such a critical point at infinity.

2.2. Expansion of the gradient of the functional

In this subsection, we recall the expansion of the gradient of the functional J in $V(p,\varepsilon)$, $p\geq 1$. These expansions are extracted from [3].

Proposition 2.5 ([3])**.** *Let $u=\sum_{j=1}^{p}\alpha_j\tilde{\delta}_j\in V(p,\varepsilon)$. For every i, $1\leq i\leq p$, we have the following expansions*

(i)

$$\Big\langle\partial J(u),\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}\Big\rangle=-2c_2J(u)\sum_{i\neq j}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+o\Big(\sum_{i\neq j}\varepsilon_{ij}\Big)+o\Big(\frac{1}{\lambda_i}\Big),$$

where $c_2=c_0^{\frac{2(n-1)}{n-2}}\int_{\mathbb{R}^{n-1}}\frac{dy}{\left(1+|y|^2\right)^{\frac{n}{2}}}$.

(ii) *If $a_i\in B(y_{l_i},\rho)$, $y_{l_i}\in\mathcal{K}$ with $1<\beta\leq n-2$ and ρ is a positive constant small enough, we have*

$$\Big\langle\partial J(u),\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}\Big\rangle$$

$$=2J(u)\Big[-c_2\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+\frac{n-2}{2(n-1)}c_0^{\frac{2(n-1)}{n-2}}\beta\frac{\alpha_i}{H(a_i)}\,\frac{1}{\lambda_i^{\beta}}\sum_{k=1}^{n-1}b_k$$

$$\times \int_{\mathbb{R}^{n-1}} sign\Big(x_k+\lambda_i(a_i-y_{j_i})_k\Big)\Big|x_k+\lambda_i(a_i-y_{j_i})_k\Big|^{\beta-1}\frac{x_k}{(1+|x|^2)^{n-1}}dx$$

$$+\,o\Big(\sum_{j\neq i}\varepsilon_{ij}+\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta}}\Big)\Big]. \tag{2.3}$$

Furthermore, if $\lambda_i|a_i-y_{j_i}|<\delta$ *for* δ *very small, we then have*

$$\Big\langle \partial J(u),\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}\Big\rangle = 2J(u)\Big[\frac{n-2}{2(n-1)}\,\beta\,c_3\,\frac{\alpha_i}{H(a_i)}\,\frac{\sum_{k=1}^{n-1}b_k}{\lambda_i^{\beta}}-c_2\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}$$

$$+\,o\Big(\sum_{j\neq i}\varepsilon_{ij}+\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta}}\Big)\Big], \tag{2.4}$$

where $c_3=c_0^{\frac{2(n-1)}{n-2}}\int_{\mathbb{R}^{n-1}}\frac{|x_1|^{\beta}}{(1+|x|^2)^{n-1}}dx$.

Proposition 2.6 ([3]). *Let* $u=\sum_{j=1}^{p}\alpha_j\tilde{\delta}_j\in V(p,\varepsilon)$ *and let* i, $1\le i\le p$. *We have the following expansions*:

(i)

$$\Big\langle \partial J(u),\frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i}\Big\rangle = -\,c_5(J(u))^{\frac{2n-3}{n-2}}\alpha_i^{\frac{n}{n-2}}\frac{\nabla H(a_i)}{\lambda_i}+O\Big(\sum_{i\neq j}\frac{1}{\lambda_i}\Big|\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big|\Big)$$

$$+\,o\Big(\sum_{i\neq j}\varepsilon_{ij}+\frac{1}{\lambda_i}\Big),$$

(ii) *If* $a_i\in B(y_{j_i},\rho)$, $y_{j_i}\in\mathcal{K}$ *with* $1<\beta\le n-2$, *we have*

$$\Big\langle \partial J(u),\frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial(a_i)_k}\Big\rangle$$

$$=\,-2(n-2)c_0^{\frac{2(n-1)}{n-2}}\alpha_i^{\frac{n}{n-2}}(J(u))^{\frac{2n-3}{n-2}}\frac{1}{\lambda_i^{\beta}}\int_{\mathbb{R}^{n-1}}b_k\big|x_k+\lambda_i(a_i-y_{j_i})_k\big|^{\beta}\frac{x_k}{(1+|x|^2)^{n}}dx$$

$$+\,o\Big(\sum_{i\neq j}\varepsilon_{ij}\Big)+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta}}\Big)+O\Big(\sum_{i\neq j}\frac{1}{\lambda_i}\Big|\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big|\Big),$$

where $(a_i)_k$ *is the* k^{th} *component of* a_i *in some geodesic normal coordinates system and* $c_5=\int_{\mathbb{R}^{n-1}}\frac{dy}{\big(1+|y|^2\big)^{n-1}}$.

3. Characterization of the critical points at infinity

This section is devoted to the characterization of the critical points at infinity in $V(p,\varepsilon)$, $p\ge 1$, under β-flatness condition with $1<\beta\le n-2$. This characterization is obtained through the construction of a suitable pseudo-gradient at infinity for which the Palais-Smale condition is satisfied along the decreasing flow-lines as long as these flow-lines do not enter in the neighborhood of finite number of critical points $y_i, i = 1, \dots, p$ of H such that $(y_1, \dots, y_p) \in \mathcal{C}_\infty$.

Now we introduce the following main result.

Theorem 3.1. *Assume that H satisfies (A_0) and $(f)_\beta$, $1 < \beta \le n-2$.*

Let $\beta := \max\{\beta(y)/y \in \mathcal{K}\}$. For $p \ge 1$, there exists a pseudo-gradient W in $V(p,\varepsilon)$ so that the following holds:

There exists a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i \lambda_i)} \in V(p,\varepsilon)$ such that

(i) $\Big\langle \partial J(u), W(u) \Big\rangle \le -c\Big(\sum_{i=1}^p \frac{1}{\lambda_i^\beta} + \sum_{i=1}^p \frac{|\nabla H(a_i)|}{\lambda_i} + \sum_{j \ne i} \varepsilon_{ij} \Big)$.

(ii)

$$\Big\langle \partial J(u+\overline{v}), W(u) + \frac{\partial \overline{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W(u)) \Big\rangle \le -c\Big(\sum_{i=1}^p \frac{1}{\lambda_i^\beta} + \sum_{i=1}^p \frac{|\nabla H(a_i)|}{\lambda_i} + \sum_{j \ne i} \varepsilon_{ij} \Big).$$

Furthermore, $|W|$ is bounded and the only case where the maximum of the λ_i's is not bounded is when $a_i \in B(y_{l_i}, \rho)$ with $y_{l_i} \in \mathcal{K}$, $\forall i = 1, \dots, p$, $(y_{l_1}, \dots, y_{l_p}) \in \mathcal{C}_\infty$.

We will prove Theorem 3.1 at the end of the section. Now we state two results which deal with two specific cases of Theorem 3.1. Let

$$V_1(p,\varepsilon) = \Big\{ u = \sum_{i=1}^p \alpha_i \tilde{\delta}_i \in V(p,\varepsilon) \text{ s.t. } a_i \in B(y_{l_i}, \rho),\ y_{l_i} \in \mathcal{K} \setminus \mathcal{K}_{n-2}\ \forall i = 1, \dots, p \Big\}.$$

and

$$V_2(p,\varepsilon) = \Big\{ u = \sum_{i=1}^p \alpha_i \tilde{\delta}_i \in V(p,\varepsilon) \text{ s.t. } a_i \in B(y_{l_i}, \rho),\ y_{l_i} \in \mathcal{K}_{n-2}\ \forall i = 1, \dots, p \Big\}.$$

We then have:

Proposition 3.1 (See [3], Proposition 3.4)**.** *For $p \ge 1$ there exists a pseudo-gradient W_2 in $V_2(p,\varepsilon)$ such that $\forall u = \sum_{i=1}^p \alpha_i \tilde{\delta}_i \in V_2(p,\varepsilon)$, we have*

$$\Big\langle \partial J(u), W_2(u) \Big\rangle \le -c\Big(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2}} + \sum_{i \ne j} \varepsilon_{ij} + \sum_{i=1}^p \frac{|\nabla H(a_i)|}{\lambda_i} \Big),$$

where c is a positive constant independent of u. Furthermore, we have $|W_2|$ is bounded and the only case where the maximum of λ_i's is not bounded is when $a_i \in B(y_{l_i}, \rho)$, $y_{l_i} \in \mathcal{K}^+$, $\forall\, i = 1, \dots, p$, with $\rho(y_{l_1}, \dots, y_{l_p}) > 0$.

Proposition 3.2. *For $p \ge 1$, there exists a pseudo-gradient W_1 in $V_1(p,\varepsilon)$ so that the following holds:*

There exists $c>0$ independent of $u=\sum_{i=1}^p \alpha_i\tilde{\delta}_i \in V_1(p,\varepsilon)$ such that

$$\Big\langle \partial J(u), W_1(u)\Big\rangle \le -c\Big(\sum_{i=1}^p \frac{1}{\lambda_i^\beta}+\sum_{i\neq j}\varepsilon_{ij}+\sum_{i=1}^p \frac{|\nabla H(a_i)|}{\lambda_i}\Big).$$

Furthermore $|W_1|$ is bounded and the only case where the maximum of the λ_i's is not bounded is when $a_i \in B(y_{l_i},\rho)$ with $y_{l_i}\in \mathcal{K}^+$, $\forall i=1,\dots,p$, and $y_i\neq y_j\ \forall\ i\neq j$.

Observe that in $V_1(p,\varepsilon)$, the interaction of two bubbles is negligible with respect to the self interaction. Similar phenomena occur for the scalar curvature problem, see [10], so the proof of Proposition 3.2 is similar to the corresponding statement in [10]. Before giving the proof of Theorem 3.1, we state the following notations extracted from [3].

Let M_1 be a fixed positive constant large enough and let $u=\sum_{i=1}^p \alpha_i\tilde{\delta}_{(a_i,\lambda_i)} \in V(p,\varepsilon)$ such that $a_i\in B(a_i,\rho)$, $y_{l_i}\in\mathcal{K}$, $\forall\ i=1,\dots,p$. For any index i, $1\le i\le p$, we define the following vector fields

$$Z_i(u)=\alpha_i\lambda_i\frac{\partial\tilde{\delta}_{(a_i,\lambda_i)}}{\partial\lambda_i} \tag{3.1}$$

and

$$X_i=\alpha_i\sum_{k=1}^{n-1}\frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_{(a_i,\lambda_i)}}{\partial(a_i)_k}\int_{\mathbb{R}^{n-1}} b_k\frac{|x_k+\lambda_i(a_i-y_{l_i})_k|^{\beta_i}}{(1+\lambda_i|(a_i-y_{l_i})_k|)^{\beta_i-1}}\frac{x_k}{(1+|x|^2)^{n+1}}dx. \tag{3.2}$$

We claim that X_i is bounded. Indeed, the claim is trivial if $\lambda_i|a_i-y_{l_i}|\le M_1$. If $\lambda_i|a_i-y_{l_i}|>M_1$, for any k, $1\le k\le n-1$, such that $\lambda_i|(a_i-y_{l_i})_k|>\frac{M_1}{\sqrt{n-1}}$, we have the following estimate

$$\int_{\mathbb{R}^n}\frac{\big|x_k+\lambda_i(a_i-y_{l_i})_k\big|^{\beta_i}x_k}{(1+|x|^2)^{n+1}}dx = c(\text{signe}\lambda_i(a_i-y_{l_i})_k)(\lambda_i|(a_i-y_{l_i})_k|)^{\beta_i-1} \times(1+o(1)). \tag{3.3}$$

Hence, our claim follows. Next, we will say that

$$i\in L_1 \text{ if } \lambda_i|a_i-y_{l_i}|\le M_1,$$

$$i\in L_2 \text{ if } \lambda_i|a_i-y_{l_i}|> M_1,$$

and we will denote by k_i, the index satisfying

$$|(a_i-y_{l_i})_{k_i}|=\max_{1\le k\le n-1}|(a_i-y_{l_i})_k|. \tag{3.4}$$

It is easy to see that if $i\in L_2$, then $\lambda_i|(a_i-y_{l_i})_{k_i}|>\frac{M_1}{\sqrt{n-1}}$.

Now, we introduce the following two lemmas.

Lemma 3.1. *Let* $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} \in V(p,\varepsilon)$ *such that* $a_i \in B(y_{l_i}, \rho)$, $y_{l_i} \in \mathcal{K}$, $\forall\, i = 1,\dots,p$. *We then have*

$$\begin{aligned}\langle \partial J(u), Z_i(u)\rangle = & -2c_2 J(u)\sum_{j\neq i}\alpha_i\alpha_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + O\Big(\frac{1}{\lambda_i^{\beta_i}}\Big)\\ & + \Big[O\Big(\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2}\Big), \textit{if } i\in L_2\Big]\\ & + o\Big(\sum_{j\neq i}\varepsilon_{ij}\Big) + o\Big(\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta_j}}\Big),\end{aligned}$$

where k_i *is defined in* (3.4).

Proof. Using Proposition 2.5, we have

$$\begin{aligned}\langle \partial J(u), Z_i(u)\rangle = & -2c_2 J(u)\sum_{j\neq i}\alpha_i\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \frac{n-2}{2(n-1)}c_0^{\frac{2(n-1)}{n-2}}\beta\frac{\alpha_i^2}{H(a_i)}\\ & \times\int_{\mathbb{R}^{n-1}} signe\Big(x_k + \lambda_i(a_i - y_{j_i})_k\Big)\Big|x_k + \lambda_i(a_i - y_{j_i})_k\Big|^{\beta-1}\\ & \times\frac{x_k}{(1+|x|^2)^{n-1}}dx + o\Big(\sum_{j\neq i}\varepsilon_{ij}\Big) + o\Big(\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta_j}}\Big).\end{aligned}$$

Observe that for $k\in\{1,\dots,n-1\}$, if $\lambda_i|(a_i - y_{l_i})_k| > \frac{M_1}{\sqrt{n-1}}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}} signe\Big(x_k + \lambda_i(a_i - y_{j_i})_k\Big)\frac{\big|x_k + \lambda_i(a_i - y_{l_i})_k\big|^{\beta_i-1}x_k}{(1+|x|^2)^{n-1}}dx\\ &= c\; signe\Big(\lambda_i(a_i - y_{j_i})_k\Big)(\lambda_i|(a_i - y_{l_i})_k|)^{\beta_i-2}(1+o(1)),\end{aligned}\tag{3.5}$$

taking M_1 large enough. If not, we have

$$\int_{\mathbb{R}^{n-1}}\frac{\big|x_k + \lambda_i(a_i - y_{l_i})_k\big|^{\beta_i-1}|x_k|}{(1+|x|^2)^{n-1}}dx = O(1).$$

Using the fact that k_i defined in (3.4) satisfies $\lambda_i|(a_i - y_{l_i})_{k_i}| > \frac{M_1}{\sqrt{n-1}}$, if $i\in L_2$, Lemma 3.1 follows. □

Lemma 3.2. *For* $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} \in V(p,\varepsilon)$ *such that* $a_i \in B(y_{l_i}, \rho)$, $y_{l_i} \in \mathcal{K}$, $\forall\, i = 1,\dots,p$, *we have*

$$\begin{aligned}\langle \partial J(u), X_i(u)\rangle \le & O\Big(\sum_{j\neq i}\frac{1}{\lambda_i}\Big|\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big|\Big) + O\Big[\Big(\frac{1}{\lambda_i^{\beta_i}}\Big), \textit{if } i\in L_1\Big]\\ & + \Big[-c\Big(\frac{1}{\lambda_i^{\beta_i}} + \frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i-1}}{\lambda_i}\Big), \textit{if } i\in L_2\Big] + o\Big(\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta_j}}\Big),\end{aligned}$$

where k_i *is defined in* (3.4).

Proof. Using Proposition 2.6, we have

$$\begin{aligned}
&\langle \partial J(u), X_i(u)\rangle \\
= &-2(n-2)c_0^{\frac{2(n-1)}{n-2}}\alpha_i^{\frac{n}{n-2}}(J(u))^{\frac{2n-3}{n-2}}\frac{1}{\lambda_i^{\beta_i}} \\
&\times\sum_{k=1}^{n-1}\left(\int_{\mathbb{R}^{n-1}} b_k\frac{|x_k+\lambda_i(a_i-y_{l_i})_k|^{\beta_i}}{(1+\lambda_i|(a_i-y_{l_i})_k|)^{(\beta_i-1)/2}}\frac{x_k}{(1+|x|^2)^n}dx\right)^2 \\
&+O\Big(\sum_{j\neq i}\frac{1}{\lambda_i}|\frac{\partial\varepsilon_{ij}}{\partial a_i}|\Big)+o\Big(\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta_j}}\Big) \\
\leq &-c\frac{1}{\lambda_i^{\beta_i}}\left(\int_{\mathbb{R}^{n-1}} b_{k_i}\frac{|x_k+\lambda_i(a_i-y_{l_i})_{k_i}|^{\beta_i}}{(1+\lambda_i|(a_i-y_{l_i})_{k_i}|)^{(\beta_i-1)/2}}\frac{x_{k_i}}{(1+|x|^2)^n}dx\right)^2 \\
&+O\Big(\sum_{j\neq i}\frac{1}{\lambda_i}|\frac{\partial\varepsilon_{ij}}{\partial a_i}|\Big)+o\Big(\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta_j}}\Big). \qquad (3.6)
\end{aligned}$$

Using (3.3) and the fact that $\lambda_i|(a_i-y_{l_i})_{k_i}| > \frac{M_1}{\sqrt{n-1}}$, if $i\in L_2$, Lemma 3.2 follows. □

Proof of Theorem 3.1. In order to complete the construction of the pseudo-gradient W suggested in Theorem 3.1, it only remains (using Propositions 3.1 and 3.2) to focus attention at the two following subsets of $V(p,\varepsilon)$.

Subset 1. We consider here the case of $u=\sum_{i=1}^p \alpha_i\tilde{\delta}_{(a_i\lambda_i)}=\sum_{i\in I_1}\alpha_i\tilde{\delta}_{(a_i\lambda_i)} + \sum_{i\in I_2}\alpha_i\tilde{\delta}_{(a_i\lambda_i)}$ such that

$$I_1\neq\emptyset, I_2\neq\emptyset, \sum_{i\in I_1}\alpha_i\tilde{\delta}_{(a_i\lambda_i)}\in V_1(\sharp I_1,\varepsilon) \text{ and } \sum_{i\in I_2}\alpha_i\tilde{\delta}_{(a_i\lambda_i)}\in V_2(\sharp I_2,\varepsilon).$$

Without loss of generality, we can assume that

$$\lambda_1\leq\cdots\leq\lambda_p.$$

We distinguish three cases.

Case 1. $u_1 := \sum_{i\in I_1}\alpha_i\tilde{\delta}_{(a_i\lambda_i)}\notin V_1^1(\sharp I_1,\varepsilon)=\{u=\sum_{j=1}^{\sharp I_1}\alpha_j\tilde{\delta}_{(a_j,\lambda_j)}, a_j\in B(y_{l_j},\rho), y_{l_j}\in\mathcal{K}^+\ \forall\ j=1,\dots,\sharp I_1 \text{ and } y_{l_j}\neq y_{l_k}\forall j\neq k\}$.

Let $\widetilde{W_1}$ be the pseudo-gradient on $V(p,\varepsilon)$ defined by $\widetilde{W_1}(u)=W_1(u_1)$, where W_1 is the vector filed defined by Proposition 3.2 in $V_1(\sharp I_1,\varepsilon)$. Note that if $u_1\notin V_1^1(\sharp I_1,\varepsilon)$, then the pseudo-gradient $W_1(u_1)$ does not increase the maximum of the λ_i's, $i\in I_1$. Using Proposition 3.2, we have

$$\begin{aligned}
\Big\langle\partial J(u),\widetilde{W_1}(u)\Big\rangle\leq &-c\Big(\sum_{i\in I_1}\frac{1}{\lambda_i^{\beta_i}}+\sum_{j\neq i, i,j\in I_1}\varepsilon_{ij}+\sum_{i\in I_1}\frac{|\nabla H(a_i)|}{\lambda_i}\Big) \\
&+O\Big(\sum_{i\in I_1, j\in I_2}\varepsilon_{ij}\Big). \qquad (3.7)
\end{aligned}$$

An easy calculation yields

$$\varepsilon_{ij} = O\Big(\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\Big) = o\Big(\frac{1}{\lambda_i^{\beta_i}}\Big) + o\Big(\frac{1}{\lambda_j^{\beta_j}}\Big). \tag{3.8}$$

Fix $i_0 \in I_1$, we denote by

$$J_1 = \{i \in I_2 \text{ s.t. } \lambda_i^{n-2} \geq \frac{1}{2}\lambda_{i_0}^{\beta_{i_0}}\} \text{ and } J_2 = I_2 \setminus J_1.$$

Using (3.7) and (3.8), we find that

$$\begin{aligned}\Big\langle \partial J(u), \widetilde{W_1}(u)\Big\rangle \leq & -c\Big(\sum_{i\in I_1\cup J_1}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in I_1}\frac{|\nabla H(a_i)|}{\lambda_i} + \sum_{j\neq i\in I_1}\varepsilon_{ij}\Big)\\ & + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big).\end{aligned} \tag{3.9}$$

From another part, by Lemma 3.1 we have

$$\begin{aligned}\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u)\Big\rangle \leq c & \sum_{j\neq i\ ,i\in J_1} 2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + O\Big(\sum_{i\in J_1}\frac{1}{\lambda_i^{\beta_i}}\Big)\\ & + O\Big(\sum_{i\in J_1\cap L_2}\frac{|(a_i-y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2}\Big).\end{aligned} \tag{3.10}$$

Observe that using a direct calculation, we have

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \leq -c\,\varepsilon_{ij}, \text{ if } \lambda_i \geq \lambda_j \text{ or } \lambda_i\sim\lambda_j \text{ or } |a_i-a_j|\geq\delta_0>0. \tag{3.11}$$

Since for $i<j$, we have

$$2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + 2^j\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} \leq -c\,\varepsilon_{ij}, \tag{3.12}$$

and for $i\in J_1$ and $j\in J_2$, we have $\lambda_j\leq\lambda_i$. So, we obtain $\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\leq -c\,\varepsilon_{ij}$. These estimates yields

$$\begin{aligned}\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u)\Big\rangle \leq & -c\sum_{j\neq i\ ,i\in J_1,\ j\in J_1\cup J_2}\varepsilon_{ij} + O\Big(\sum_{i\in J_1}\frac{1}{\lambda_i^{\beta_i}}\Big)\\ & + O\Big(\sum_{i\in J_1\cap L_2}\frac{|(a_i-y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2}\Big)\\ & + O\Big(\sum_{i\in J_1,\ j\in I_1}\varepsilon_{ij}\Big).\end{aligned}$$

It is easy to see that for any index $i\in L_2$, we have

$$\frac{|(a_i-y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2} \leq \frac{\sqrt{n-1}}{M_1}\frac{|(a_i-y_{l_i})_{k_i}|^{\beta_i-1}}{\lambda_i},$$

where k_i is defined in (3.4) and M_1 large enough. Thus, we derive that

$$\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 2}}{\lambda_i^2} = o\Big(\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 1}}{\lambda_i}\Big) \text{ for any } i \in L_2. \tag{3.13}$$

Let $m_1 > 0$ small enough, using Lemma 3.2, (3.13) and (3.8), we get

$$\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u) + m_1 \sum_{i \in J_1 \cap L_2} X_i(u)\Big\rangle$$
$$\leq -c\Big(\sum_{j\neq i\ , i\in J_1,\ j\in J_1\cup J_2} \varepsilon_{ij} + \sum_{i\in J_1} \frac{|\nabla H(a_i)|}{\lambda_i}\Big) + O\Big(\sum_{i\in J_1}\frac{1}{\lambda_i^{\beta_i}}\Big) + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big),$$

and by (3.9) we obtain

$$\Big\langle \partial J(u), \widetilde{W_1}(u) + m_1\Big(\sum_{i\in J_1} -2^i Z_i(u) + m_1 \sum_{i\in J_1\cap L_2} X_i(u)\Big)\Big\rangle$$
$$\leq -c\Big(\sum_{i\in I_1\cup J_1}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i\neq j\in I_1}\varepsilon_{ij} + \sum_{j\neq i\ , i\in J_1,\ j\in J_1\cup J_2}\varepsilon_{ij} + \sum_{i\in I_1\cup J_1}\frac{|\nabla H(a_i)|}{\lambda_i}\Big)$$
$$+ o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big). \tag{3.14}$$

We need to add the remainder indices $i \in J_2$. Note that $\widetilde{u} := \sum_{j\in J_2} \alpha_j \tilde{\delta}_j \in V_2(\sharp J_2, \varepsilon)$. Thus, using Proposition 3.1, we apply the associated vector field which we will denote $\widetilde{W_2}$. We then have the following estimate

$$\Big\langle \partial J(u), \widetilde{W_2}(u)\Big\rangle \leq -c\Big(\sum_{j\in J_2}\frac{1}{\lambda_j^{\beta_j}} + \sum_{i\neq j,\ i,j\in J_2}\varepsilon_{ij} + \sum_{j\in J_2}\frac{|\nabla H(a_j)|}{\lambda_j}\Big)$$
$$+ O\Big(\sum_{j\in J_2,\ i\in J_1}\varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big), \tag{3.15}$$

since $|a_i - a_j| \geq \rho$ for $j \in J_2$ and $i \in I_1$.

Let in this case $W = \widetilde{W_1} + m_1\Big(\widetilde{W_2} + \sum_{i\in J_1} -2^i Z_i + m_1 \sum_{i\in J_1\cap L_2} X_i\Big)$.

From (3.14) and (3.15) we obtain

$$\Big\langle \partial J(u), W(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla H(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Case 2. $u_1 := \sum_{i\in I_1}\alpha_i\tilde{\delta}_{(a_i\lambda_i)} \in V_1^1(\sharp I_1, \varepsilon)$ and $u_2 := \sum_{i\notin I_2}\alpha_i\tilde{\delta}_{(a_i\lambda_i)} \notin V_2^1(\sharp I_2, \varepsilon) := \{u = \sum_{j=1}^{\sharp I_2}\alpha_j\tilde{\delta}_{(a_j,\lambda_j)},\ a_j \in B(y_{l_j}, \rho),\ y_{l_j} \in \mathcal{K}^+,\ \forall j = 1, \dots, \sharp I_2$ and $\rho(y_{l_1}, \dots, y_{\sharp I_2}) > 0\}$.

Since $u_2 \in V_2(\sharp I_2, \varepsilon)$, by Proposition 3.1, we can apply the associated vector field which we will denote V_1. We get

$$\begin{aligned}\Big\langle \partial J(u), V_1(u)\Big\rangle \leq & -c\Big(\sum_{i\in I_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2}\frac{|\nabla H(a_i)|}{\lambda_i}+\sum_{i\neq j,\ i,j\in I_2}\varepsilon_{ij}\Big)\\ &+O\Big(\sum_{i\in I_2,\ j\in I_1}\varepsilon_{ij}\Big).\end{aligned}\tag{3.16}$$

Observe that $V_1(u)$ does not increase the maximum of the λ_i's, $i \in I_2$, since $u_2 \notin V_2^1(\sharp I_2, \varepsilon)$. Fix $i_0 \in I_2$ and let

$$\widetilde{J_1} = \{i \in I_1 \text{ s.t. } \lambda_i^{\beta_i} \geq \frac{1}{2}\lambda_{i_0}^{n-2}\} \text{ and } \widetilde{J_2} = I_1 \setminus \widetilde{J_1}.$$

Using (3.16) and (3.8), we get

$$\begin{aligned}\Big\langle \partial J(u), V_1(u)\Big\rangle \leq & -c\Big(\sum_{i\in I_2\cup\widetilde{J_1}}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2}\frac{|\nabla H(a_i)|}{\lambda_i}+\sum_{i\neq j,\ i,j\in I_2}\varepsilon_{ij}\Big)\\ &+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big).\end{aligned}\tag{3.17}$$

We need to add the indices i, $i \in \widetilde{J_2}$. Let $\widetilde{u} := \sum_{j\in\widetilde{J}_2}\alpha_j\tilde{\delta}_{(a_j\lambda_j)}$, since $\widetilde{u} \in V_1(\sharp\widetilde{J}_2, \varepsilon)$, we can apply the associated vector field giving by Proposition 3.1. Let V_2 be this vector field. By Proposition 3.2, we have

$$\begin{aligned}\Big\langle \partial J(u), V_2(u)\Big\rangle \leq & -c\Big(\sum_{j\in\widetilde{J_2}}\frac{1}{\lambda_j^{\beta_j}}+\sum_{j\in\widetilde{J_2}}\frac{|\nabla H(a_j)|}{\lambda_j}+\sum_{i\neq j,\ i,j\in\widetilde{J_2}}\varepsilon_{ij}\Big)\\ &+O\Big(\sum_{j\in\widetilde{J_2},\ i\notin\widetilde{J_2}}\varepsilon_{ij}\Big).\end{aligned}$$

Observe that $I_1 = \widetilde{J_1} \cup \widetilde{J_2}$ and we are in the case where $\forall\, i \neq j \in I_1$, we have $|a_i - a_j| \geq \rho$. Thus by (3.8), we get

$$O\Big(\sum_{j\in\widetilde{J_2},i\notin\widetilde{J_2}}\varepsilon_{ij}\Big)=o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big),$$

and hence

$$\Big\langle \partial J(u), V_1(u)+V_2(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2\cup\widetilde{J_2}}\frac{|\nabla H(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Let in this case $W = V_1 + V_2 + m_1\sum_{i\in\widetilde{J_1}}X_i(u)$, m_1 small enough.

Using the above estimate and Lemma 3.2, we find that

$$\Big\langle \partial J(u), W(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla H(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Case 3. $u_1 \in V_1^1(\sharp I_1, \varepsilon)$ and $u_2 \in V_2^1(\sharp I_2, \varepsilon)$.

Let $\widetilde{V_1}$ (respectively $\widetilde{V_2}$) be the pseudo-gradient in $V(p, \varepsilon)$ defined by $\widetilde{V_1}(u) = W_1(u_1)$ (respectively $\widetilde{V_2}(u) = W_2(u_2)$) where W_1 (respectively W_2) is the vector field defined by Proposition 3.2 (respectively 3.1) in $V_1^1(\sharp I_1, \varepsilon)$ (respectively $V_2^1(\sharp I_2, \varepsilon)$) and let in this case

$$W = \widetilde{V_1} + \widetilde{V_2}.$$

Using Proposition 3.1, Proposition 3.2 and (3.8) we get

$$\Big\langle \partial J(u), W(u) \Big\rangle \leq -c\bigg(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p} \frac{|\nabla H(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \bigg).$$

Notice that in the first and second cases, the maximum of the λ_i's, $1 \leq i \leq p$, is a bounded function and hence the Palais-Smail condition is satisfied along the flow-lines of W. However in the third case all the λ_i's, $1 \leq i \leq p$, will increase and goes to $+\infty$ along the flow-lines generated by W.

Subset 2. We consider the case of $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \in V(p, \varepsilon)$ such that there exist a_i satisfying $a_i \notin \cup_{y \in \mathcal{K}} B(y, \rho)$.

In this region, the construction of the pseudo-gradient W proceeds exactly as the proof of (Theorem 3.2, of subset 2) of [3].

Finally, observe that our pseudo-gradient W in $V(p, \varepsilon)$ satisfies claim (i) of Theorem 3.1 and it is bounded, since $||\lambda_i \frac{\partial \tilde{\delta}_{(a_i, \lambda_i)}}{\partial \lambda_i}||$ and $||\frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_{(a_i, \lambda_i)}}{\partial a_i}||$ are bounded. From the definition of W, the λ_i's, $1 \leq i \leq p$ decrease along the flow-lines of W as long as these flow-lines do not enter in the neighborhood of finite number of critical points y_{l_i}, $i = 1, \ldots, p$, of $\mathcal{K}$ such that $(y_{l_1}, \ldots, y_{l_p}) \in \mathcal{C}_\infty$.

Now, arguing as in Appendix 2 of [6], see also Appendix B of [9], claim (ii) of Theorem 3.1 follows from (i) and Proposition 2.4. This complete the proof of Theorem 3.1. □

Corollary 3.1. *Let $p \geq 1$. The critical points at infinity of J in $V(p, \varepsilon)$ correspond to*

$$(y_{l_1}, \ldots, y_{l_p})_\infty := \sum_{i=1}^{p} \frac{1}{H(y_{l_i})^{\frac{n-2}{2}}} \, \tilde{\delta}_{(y_{l_i}, \infty)},$$

where $(y_{l_1}, \ldots, y_{l_p}) \in \mathcal{C}_\infty$. Moreover, such a critical point at infinity has an index equal to $i(y_{l_1}, \ldots, y_{l_p})_\infty = p - 1 + \sum_{i=1}^{p} n - 1 - \widetilde{i}(y_{l_i})$.

4. Proof of Theorem 1.1

We prove the existence result by contradiction. Assume that J has no critical point in Σ^+. It follows from Corollary 3.1 that the critical points at infinity of the associated variational problem are in one to one correspondence with the elements of $\mathcal{C}_\infty$ defined in (1.2).

Notice that, just like for usual critical points, it is associated to each critical point at infinity w_∞ of J stable and unstable manifolds $W_s^\infty(w_\infty)$ and

$W_u^\infty(w_\infty)$, (see [6], pages 356–357). These manifolds can be easily described once a finite dimensional reduction like the one we performed in Section 3 is established.

For any $w_\infty = (y_{i_1}, \ldots, y_{i_p}) \in \mathcal{C}_\infty$, let $c(w)_\infty = S_n\big(\sum_{j=1}^{p} \frac{1}{H(y_{l_j})^{\frac{n-2}{2}}}\big)^{\frac{2}{n}}$ denote the associated critical value. Here, we choose to consider a simplified situation, where for any $w_\infty \neq w'_\infty$, $c(w)_\infty \neq c(w')_\infty$ and thus order the $c(w)_\infty$'s, $w_\infty \in \mathcal{C}_\infty$ as

$$c(w_1)_\infty < \cdots < c(w_{k_0})_\infty.$$

By using a deformation lemma (see Proposition 7.24 and Theorem 8.2 of [8]), we know that if $c(w_{k-1})_\infty < a < c(w_k)_\infty < b < c(w_{k+1})_\infty$, then

$$J_b \simeq J_a \cup W_u^\infty(w_k)_\infty, \tag{4.1}$$

where $J_b = \{u \in \Sigma^+, J(u) \leq b\}$ and $\simeq$ denotes retracts by deformation.

We apply the Euler-Poincaré characteristic of both sides of (4.1), we find that

$$\chi(J_b) = \chi(J_a) + (-1)^{i(w_k)_\infty}, \tag{4.2}$$

where $i(w_k)_\infty$ denotes the index of the critical point at infinity $(w_k)_\infty$. Let

$$b_1 < c(w_1)_\infty = \min_{u \in V_\eta(\Sigma^+)} J(u) < b_2 < c(w_2)_\infty < \cdots < b_{k_0} < c(w_{k_0})_\infty < b_{k_0+1}.$$

Since we have assumed that (P) has no solution, $J_{b_{k_0+1}}$ is a retract by deformation of Σ^+. Therefore $\chi(J_{b_{k_0+1}}) = 1$, since Σ^+ is a contractible set. Now using (4.2), we derive after recalling that $\chi(J_{b_1}) = \chi(\emptyset) = 0$,

$$1 = \sum_{j=1}^{k_0} (-1)^{i(w_j)_\infty}. \tag{4.3}$$

Hence if (4.3) is violated, J has a critical point in Σ^+.

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