J. Korean Math. Soc. **53** (2016), No. 2, pp. 263–286
http://dx.doi.org/10.4134/JKMS.2016.53.2.263

GALOIS COVERINGS AND JACOBI VARIETIES OF COMPACT RIEMANN SURFACES

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Dedicated to Professor Changho Keem on his 60th birthday

Abstract. We discuss relations between Galois coverings of compact Riemann surfaces and their Jacobi varieties. We prove a theorem of a kind of Galois correspondence for Abelian subvarieties of Jacobi varieties. We also prove a theorem on the sets of points in Jacobi varieties fixed by Galois group actions.

1. Introduction

Let $\pi : X \to Y$ be a holomorphic map of a compact Riemann surface X of genus $g = g(X)$ onto a compact Riemann surface Y of genus $g_0 = g(Y)$. π is called a *Galois covering* if there is a biholomorphic map $\hat{\pi} : X/G \to Y$ such that $\pi = \hat{\pi} \circ pr$, where G is a finite subgroup of the automorphism group $\mathrm{Aut}(X)$ of X and $pr : X \to X/G$ is the canonical projection. G is called the *Galois group of* π. (We sometimes identify Y with X/G through $\hat{\pi}$.)

The purpose of this paper is to discuss relations between the Galois covering π and the Jacobi varieties $J(X)$ and $J(Y)$ of X and Y, respectively. After discussing properties of an Abelian subvariety $A(\pi)$ of $J(X)$ which is isogeneous to $J(Y)$ and each of whose points is fixed by the action of G, we prove a theorem of a kind of Galois correspondence (Theorem 3) for Abelian subvarieties of $J(X)$.

Next, we discuss existence or non-existence of invariant linear systems on X under the action of G, using Abel-Jacobi maps $\Phi_m : S^m(X) \to J(X)$, where $S^m(X)$ is the m-th symmetric product of X. For a fixed positive integer m, the action of G on $J(X)$ must be regarded as not linear action but affine action in order to be equivariant with the action on $S^m(X)$ with respect to Φ_m. So, we denote this action as (G, m)-action. We show that there is a positive integer m_0 such that the set $\mathrm{Fix}(J(X), G, m)$ of (G, m)-fixed points in $J(X)$ is non-empty if and only if m is divisible by m_0. Moreover, in this case, the number of

Received November 17, 2014; Revised May 4, 2015.
2010 *Mathematics Subject Classification.* Primary 14H7; Secondary 14H30.
Key words and phrases. compact Riemann surface, Galois covering, Jacobi variety.

the points of $\mathrm{Fix}_0(J(X), G, m)$ (the set of points of $\mathrm{Fix}(J(X), G, m)$, which is *orthogonal* to $A(\pi)$) is finite and constant for all m divisible by m_0 (Theorem 4). Finally, we give some examples for determinations of existence or non-existence of invariant linear systems using Theorem 4.

2. Some linear projections

For a compact Riemann surface X of genus g, let $H^0(X, K_X)$ be the complex vector space of dimension g of holomorphic differentials on X. Let $\{\omega_1, \ldots, \omega_g\}$ be a basis of $H^0(X, K_X)$. Then the Jacobi variety $J(X)$ of X is defined to be the complex torus $\mathbb{C}^g/\Gamma_X$, where Γ_X is the additive group of period vectors:

$$\Gamma_X = \{(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g) \mid \gamma \in H_1(X, \mathbb{Z})\}.$$

$J(X)$ is then an Abelian variety with the principal polarization given by its symplectic basis of the integral 1-st homology group $H_1(X, \mathbb{Z})$. $J(X)$ is isomorphic to the Picard variety $\mathrm{Pic}^0(X)$ of X, where

$$\mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\{\text{principal divisors}\},$$
$$\mathrm{Div}^0(X) = \{D \mid D \text{ is a divisor of } X \text{ with } \deg(D) = 0\}.$$

The isomorphism $\mathrm{Pic}^0(X) \cong J(X)$ is given by

$$D \,(\text{mod(principal divisors)}) \longmapsto \sum_{i=1}^{m} \int_{q_i}^{p_i} \Omega \,(\text{mod}\,\Gamma_X),$$

via Abel's theorem, where $D = p_1 + \cdots + p_m - q_1 - \cdots - q_m$ and $\Omega = (\omega_1, \ldots, \omega_g)$.

Now let $\pi : X \to Y = X/G$ be a Galois covering of a compact Riemann surface X of genus $g = g(X)$ onto a compact Riemann surface $Y = X/G$ of genus $g_0 = g(Y)$ with the Galois group G, which is a finite subgroup of $\mathrm{Aut}(X)$. Henceforth, we assume

$$g \geq 2.$$

We put

$$G = \{\varphi_1 = e, \ldots, \varphi_d\} \quad (d \text{ is the order of } G).$$

For any $\varphi \in G$, the pull-back φ^* is a linear isomorphism of the g-dimensional complex vector space $H^0(X, K_X)$ of holomorphic differentials on X. We define a linear map M of $H^0(X, K_X)$ into itself by

$$M = \frac{1}{d}(\varphi_1^* + \cdots + \varphi_d^*).$$

Then we can easily show the following properties:

$$M \circ \varphi^* = \varphi^* \circ M = M \text{ for all } \varphi \in G, \tag{1}$$

$$M \circ M = M. \tag{2}$$

The last equality means that M is a linear projection, so $M' = I - M$ (I: the identity map) is also a linear projection. Thus $H^0(X, K_X)$ is decomposed into the direct sum:

$$H^0(X, K_X) = M(H^0(X, K_X)) \oplus M'(H^0(X, K_X)).$$

Lemma 1. (i)

$$\begin{aligned} M(H^0(X, K_X)) &= \{\omega \in H^0(X, K_X) \mid M(\omega) = \omega\} \\ &= \{\omega \in H^0(X, K_X) \mid \varphi^*(\omega) = \omega \;\; \textit{for all} \;\; \varphi \in G\} \\ &= \{\pi^*\eta \mid \eta \in H^0(Y, K_Y)\}. \end{aligned}$$

(ii) $M'(H^0(X, K_X)) = \mathrm{Ker}(M)$, *and for every* $\varphi \in G$, φ^* *acts linearly on* $M'(H^0(X, K_X))$.

Proof. The assertion in (ii) is trivial. We prove (i). The first and the second equalities in (i) follow from the properties of M in (1) and (2). We prove the third equality.

It is clear that the pull-back over π of a holomorphic differential on Y is G-invariant. Conversely, we show that a G-invariant holomorphic differential ω can be written as the pull-back over π of a holomorphic differential on Y.

For any point p in X, there is a local coordinate z around p with $z(p) = 0$ and a local coordinate w around $q = \pi(p)$ with $w(q) = 0$ such that π is locally written as

$$\pi : z \longmapsto w = z^e, \tag{3}$$

where e is a positive integer. ($e \geq 2$ if and only if p is a ramification point of π.) ω can be locally written as

$$\omega = f(z)dz,$$

where $f(z)$ is a holomorphic function around p.

If e in (3) is 1, then the pull-back over π of the holomorphic differential $f(w)dw$ defined locally around q is ω.

Suppose e in (3) is greater than or equal to 2. We expand $f(z)$ into the power series of z as follows:

$$f(z) = c_0 + c_1 z + \cdots .$$

Let $\zeta = \exp(2\pi\sqrt{-1}/e)$ be a primitive root of 1. Then there is an element φ in G such that φ is locally written as

$$\varphi(z) = \zeta z,$$

because z and ζz are in the same fiber of π, while G acts transitively on every fiber of π. Now, by the assumption on ω, we have $\varphi^*\omega = \omega$, that is, locally,

$$f(\zeta z)d(\zeta z) = f(z)dz. \tag{4}$$

The left hand side of (4) can be written as

$$(\zeta c_0 + (\zeta)^2 c_1 z + \cdots + (\zeta)^e c_{e-1} z^{e-1} + (\zeta)^{e+1} c_e z^e + \cdots)dz.$$

Hence (4) implies

$$c_0 = 0, \dots, c_{e-2} = 0, c_e = 0, \dots .$$

Hence

$$\omega = f(z)dz = (c_{e-1}z^{e-1} + c_{2e-1}z^{2e-1} + \cdots)dz = \pi^*(g(w)dw),$$

where $g(w)$ is a holomorphic function around q whose power series expansion with respect to w is

$$g(w) = \frac{1}{e}c_{e-1} + \frac{1}{e}c_{2e-1}w + \cdots .$$

The locally defined holomorphic differentials $f(w)dw$ (for $e = 1$) and $g(w)dw$ (for $e \geq 2$) can be patched up and define a global holomorphic differential η on Y such that $\pi^*\eta = \omega$. □

From Lemma 1, we have:

Theorem 1. $g_0 = g(Y)$ *vanishes if and only if* $\varphi_1^*\omega + \cdots + \varphi_d^*\omega = 0$ *for all* $\omega \in H^0(X, K_X)$*, where* $G = \{\varphi_1, \dots, \varphi_d\}$.

Let $\eta_1, \dots, \eta_{g_0}$ be a basis of $H^0(Y, K_Y)$ and let $\omega_1, \dots, \omega_l$ be a basis of $M'(H^0(X, K_X))$. $(l = g - g_0)$. In the sequel, we use the following basis of $H^0(X, K_X)$:

$$\{\pi^*\eta_1, \dots, \pi^*\eta_{g_0}, \omega_1, \dots, \omega_l\}.$$

Then every point in the Jacobi variety $J(X)$ can be written as

$$(z, w) \pmod{\Gamma_X}, \tag{5}$$

where

$$z = (z_1, \dots, z_{g_0}) \in \mathbb{C}^{g_0},$$
$$w = (w_1, \dots, w_l) \in \mathbb{C}^l.$$

We sometimes use the 'coordinate' in (5) for a point of $J(X)$. We put

$$\Omega = (\pi^*\eta_1, \dots, \pi^*\eta_{g_0}, \omega_1, \dots, \omega_l).$$

Every $\varphi \in G$ induces an automorphism φ_* of $J(X) = \mathbb{C}^g/\Gamma_X$:

$$\sum_{j=1}^m \int_{q_j}^{p_j} \Omega \pmod{\Gamma_X} \longmapsto \sum_{j=1}^m \int_{\varphi(q_j)}^{\varphi(p_j)} \Omega \pmod{\Gamma_X} = \sum_{j=1}^m \int_{q_j}^{p_j} \varphi^*(\Omega) \pmod{\Gamma_X}.$$

Hence we may regard φ_* as a linear transformation on $\mathbb{C}^g$, which induces the above $\varphi_* : J(X) \to J(X)$, as follows:

$$\varphi_* : \sum_{j=1}^m \int_{q_j}^{p_j} \Omega \longmapsto \sum_{j=1}^m \int_{q_j}^{p_j} \varphi^*\Omega. \tag{6}$$

(We remark that φ_* maps Γ_X onto itself, for

$$\varphi_* : \int_\gamma \Omega \longmapsto \int_{\varphi_*(\gamma)} \Omega \quad (= \int_\gamma \varphi^* \Omega),$$

where $\gamma \in H_1(X, \mathbb{Z})$.)

We may thus regard φ_* as a linear transformation on $\mathbb{C}^g$, using the coordinates (z, w) as follows:

$$\varphi_* : (z, w) \longmapsto (z, wB_\varphi), \tag{7}$$

where B_φ is an $(l \times l)$-non-singular matrix defined by

$$\varphi^*(\omega_1, \ldots, \omega_l) = (\omega_1, \ldots, \omega_l) B_\varphi$$

(see (ii) of Lemma 1).

We define linear maps L and L' of $\mathbb{C}^g$ into itself as follows:

$$L = \frac{1}{d}((\varphi_1)_* + \cdots + (\varphi_d)_*),$$
$$L' = I - L.$$

Then L satisfies similar properties to M in (1) and (2):

$$L \circ \varphi_* = \varphi_* \circ L = L \text{ for all } \varphi \in G,$$
$$L \circ L = L.$$

Thus L and L' are linear projections of $\mathbb{C}^g$. In fact, they are linear projections as in (i) of the following lemma:

Lemma 2. (i) *$L : (z, w) \longmapsto (z, 0)$, $L' : (z, w) \longmapsto (0, w)$.*
(ii) *$\varphi_*(z, w) = (z, w)$ for all $\varphi \in G$ if and only if $w = 0$.*
(iii) *$B_{\varphi_1} + \cdots + B_{\varphi_d} = 0$.*

Proof. (i) By (6), we have

$$L = \frac{1}{d} \sum_{k=1}^{d} (\varphi_k)_* : \sum_{j=1}^{m} \int_{q_j}^{p_j} \Omega \longmapsto \quad \sum_{j=1}^{m} \int_{q_j}^{p_j} M(\Omega) = \sum_{j=1}^{m} \int_{q_j}^{p_j} (\pi^* \Omega_0, 0)$$
$$= (\sum_{j=1}^{m} \int_{q_j}^{p_j} \pi^* \Omega_0, 0).$$

Hence $L : (z, w) \longmapsto (z, 0)$ and so $L' : (z, w) \longmapsto (0, w)$.

(ii) $\varphi_*(z, 0) = (z, 0B_\varphi) = (z, 0)$. Conversely, assume that $\varphi_*(z, w) = (z, w)$ for all $\varphi \in G$. Then $L(z, w) = (z, w)$. The left hand side is equal to $(z, 0)$ by (i). Hence $w = 0$.

(iii) follows from (i) and (7). $\square$

The linear projections L and L' can be regarded as 'dual' operators to M and M' in the following sense: The real 1-st homology group $H_1(X, \mathbb{R})$ of X

can be considered as the dual vector space over $\mathbb{R}$ to $H^0(X, K_X)$ (which is regarded as a real vector space in this time), by the pairing

$$(\gamma, \omega) \in H_1(X, \mathbb{R}) \times H^0(X, K_X) \longmapsto \mathrm{Re}(\int_\gamma \omega) = \int_\gamma \mathrm{Re}(\omega) \in \mathbb{R}, \tag{8}$$

where $\mathrm{Re}(\int_\gamma \omega)$ (resp. $\mathrm{Re}(\omega)$) is the real part of $\int_\gamma \omega$ (resp. ω). This is because the imaginary part can be written as

$$\mathrm{Im}(\int_\gamma \omega) = \int_\gamma \mathrm{Im}(\omega) = \int_\gamma \mathrm{Re}(-\sqrt{-1}\omega) = \mathrm{Re}(\int_\gamma (-\sqrt{-1}\omega)). \tag{9}$$

The group G acts on $H_1(X, \mathbb{R})$ as follows:

$$\gamma \longmapsto \varphi_*(\gamma).$$

The equality

$$\int_{\varphi_*(\gamma)} \omega = \int_\gamma \varphi^*(\omega)$$

for $\gamma \in H_1(X, \mathbb{R})$ and $\omega \in H^0(X, K_X)$, and (9) imply

$$(\varphi_*(\gamma), \omega) = (\gamma, \varphi^*(\omega))$$

for the pairing in (8). Hence φ_* is the dual linear operator to φ^*, so

$$M^* = \frac{1}{d}\sum_{j=1}^{d} (\varphi_j)_* \tag{10}$$

is the dual linear operator to M:

$$(M^*(\gamma), \omega) = (\gamma, M(\omega)).$$

M^* and $M'^* = I - M^*$ are linear projections in $H_1(X, \mathbb{R})$.

Next, let $\mathbb{A}$ be a linear isomorphism of $H_1(X, \mathbb{R})$ onto $\mathbb{C}^g$ (over $\mathbb{R}$) defined by

$$\mathbb{A}(\gamma) = \int_\gamma \Omega \ \text{ for } \ \gamma \in H_1(X, \mathbb{R}).$$

Then we have easily the following equalities:

$$\mathbb{A}M^*\mathbb{A}^{-1} = L,$$
$$\mathbb{A}M'^*\mathbb{A}^{-1} = L'.$$

In this sense, L and L' are 'dual' operators to M and M', respectively.

We also note

$$\mathbb{A}(H_1(X, \mathbb{Z})) = \Gamma_X.$$

Thus the Jacobi variety $J(X)$ is isomorphic (as a real torus) to

$$H_1(X, \mathbb{R})/H_1(X, \mathbb{Z}).$$

3. Some Abelian subvarieties

Using the linear isomorphism $\mathbb{A}$ in the previous section, we can use $H_1(X,\mathbb{R})$ instead of $\mathbb{C}^g$ for the discussion on discrete subgroups in it: By the expression of M^* in (10), we have

$$M^*(H_1(X,\mathbb{Z})) \subset \frac{1}{d}H_1(X,\mathbb{Z}) \tag{11}$$

and so

$$M'^*(H_1(X,\mathbb{Z})) \subset \frac{1}{d}H_1(X,\mathbb{Z}). \tag{12}$$

Hence $M^*(H_1(X,\mathbb{Z}))$ and $M'^*(H_1(X,\mathbb{Z}))$ are discrete subgroups in $H_1(X,\mathbb{R})$ and

$$M^*(H_1(X,\mathbb{Z})) \oplus M'^*(H_1(X,\mathbb{Z})) \subset \frac{1}{d}H_1(X,\mathbb{Z}). \tag{13}$$

Moreover, from (11) and (12), we have

$$M^*(dH_1(X,\mathbb{Z})) \subset M^*(H_1(X,\mathbb{Z})) \cap H_1(X,\mathbb{Z}),$$

and

$$M'^*(dH_1(X,\mathbb{Z})) \subset M'^*(H_1(X,\mathbb{Z})) \cap H_1(X,\mathbb{Z}).$$

Hence we have

$$dH_1(X,\mathbb{Z}) \subset (M^*(H_1(X,\mathbb{Z})) \cap H_1(X,\mathbb{Z})) \oplus (M'^*(H_1(X,\mathbb{Z})) \cap H_1(X,\mathbb{Z})).$$

From this, together with (13), we see the ranks of these discrete subgroups are

$$\operatorname{rank}(M^*(H_1(X,\mathbb{Z})) \cap H_1(X,\mathbb{Z})) = \operatorname{rank}(M^*(H_1(X,\mathbb{Z}))) = 2g_0,$$

and

$$\operatorname{rank}(M'^*(H_1(X,\mathbb{Z})) \cap H_1(X,\mathbb{Z})) = \operatorname{rank}(M'^*(H_1(X,\mathbb{Z}))) = 2l.$$

Also, we see the following equalities:

$$M^*(H_1(X,\mathbb{Z})) \cap H_1(X,\mathbb{Z}) = M^*(H_1(X,\mathbb{R})) \cap H_1(X,\mathbb{Z}) \tag{14}$$

and

$$M'^*(H_1(X,\mathbb{Z})) \cap H_1(X,\mathbb{Z}) = M'^*(H_1(X,\mathbb{R})) \cap H_1(X,\mathbb{Z}) \tag{15}$$

for $M^*M^* = M^*$ and $M'^*M'^* = M'^*$.

Thus, correspondingly, we have discrete subgroups

$$\Gamma_X \cap \{(z,0)\}, L(\Gamma_X)$$

of $\{(z,0)\} = L(\mathbb{C}^g)$ of rank $2g_0$, and discrete subgroups

$$\Gamma_X \cap \{(0,w)\}, L'(\Gamma_X)$$

of $\{(0,w)\} = L'(\mathbb{C}^g)$ of rank $2l$.

Consider the complex tori

$$A(\pi) = \{(z,0)\}/\Gamma_X \cap \{(z,0)\},$$

$$
\begin{aligned}
B(\pi) &= \{(z,0)\}/L(\Gamma_X), \\
A'(\pi) &= \{(0,w)\}/\Gamma_X \cap \{(0,w)\}, \\
B'(\pi) &= \{(0,w)\}/L'(\Gamma_X).
\end{aligned}
$$

Since

$$
\begin{aligned}
\{(z,0)\}/\Gamma_X \cap \{(z,0)\} &\cong (\{(z,0)\} + \Gamma_X)/\Gamma_X, \\
\{(0,w)\}/\Gamma_X \cap \{(0,w)\} &\cong (\{(0,w)\} + \Gamma_X)/\Gamma_X,
\end{aligned}
$$

$A(\pi)$ and $A'(\pi)$ are regarded as complex subtori, so regarded as Abelian subvarieties of $J(X)$. Later, we show that $B(\pi)$ and $B'(\pi)$ are Abelian varieties dual to $A(\pi)$ and $A'(\pi)$, respectively.

Note first the inclusion relation

$$
\begin{aligned}
\Gamma_X \cap \{(z,0)\} &= \Gamma_X \cap L(\Gamma_X) \subset L(\Gamma_X), \\
\Gamma_X \cap \{(0,w)\} &= \Gamma_X \cap L'(\Gamma_X) \subset L'(\Gamma_X),
\end{aligned}
$$

(see (14) and (15)), of discrete subgroups of the same ranks. Hence $B(\pi)$ and $B'(\pi)$ are isogeneous to $A(\pi)$ and $A'(\pi)$, respectively. Moreover,

Lemma 3. *There are following exact sequences*:
(i) $0 \to A'(\pi) \to J(X) \to B(\pi) \to 0$;
(ii) $0 \to A(\pi) \to J(X) \to B'(\pi) \to 0$.

Proof. $L : \mathbb{C}^g \to \{(z,0)\} = L(\mathbb{C}^g)$ induces the homomorphism

$$L : J(X) = \mathbb{C}^g/\Gamma_X \to B(\pi) = (z,0)/L(\Gamma_X)$$

whose kernel is clearly

$$(\Gamma_X + \{(0,w)\})/\Gamma_X \cong \{(0,w)\}/\Gamma_X \cap \{(0,w)\} = A'(\pi).$$

Hence we have the exact sequence (i). The exact sequence (ii) can be shown in a similar way. □

Next, note that $H_1(X,\mathbb{R})$ and $H^0(X,K_X)$ are dual locally compact abelian groups in the sense of Pontryagin [4, Chapter 6] with respect to the pairing

$$\langle \gamma, \omega \rangle = \exp(2\pi\sqrt{-1}\ \mathrm{Re}(\int_\gamma \omega))$$

for $\gamma \in H_1(X,\mathbb{R})$ and $\omega \in H^0(X,K_X)$:

$$
\begin{aligned}
H_1(X,\mathbb{R})^* &= H^0(X,K_X), \\
H^0(X,K_X)^* &= H_1(X,\mathbb{R}).
\end{aligned}
$$

We recall here some results on the duality in Pontryagin [4, Chapter 6]:

(i) In general, for a locally compact abelian group B, B^{**} is canonically isomorphic to B. We identify B and B^{**} through the canonical isomorphism.

(ii) For a (locally compact) subgroup A of a locally compact abelian group B, we put

$$A^{\perp} = \{\beta \in B^* \mid \langle \beta, a \rangle = 1 \text{ for all } a \in A\}$$

and call it the *annihilator* of A. (Here $\langle , \rangle$ is the pairing of B and its dual group B^*.) Then we have

$$A^{\perp\perp} = A.$$

(iii) For an exact sequence of (locally compact) abelian groups:

$$0 \to A \to B \to C \to 0$$

we have the exact sequence of dual groups:

$$0 \longleftarrow A^* \longleftarrow B^* \longleftarrow C^* \longleftarrow 0.$$

Moreover we have

$$C^* = A^{\perp}.$$

(iv) In particular, if $B = A \times C$ (direct product), then $B^* = A^* \times C^*$, and $A^* = C^{\perp}$ and $C^* = A^{\perp}$.

Now, returning to our case,

Lemma 4. (i) $M(H^0(X, K_X))^{\perp} = M'^*(H_1(X, \mathbb{R}))$.

(ii) $M'(H^0(X, K_X))^{\perp} = M^*(H_1(X, \mathbb{R}))$ *with respect to the pairing of* $H^0(X, K_X)$ *and* $H_1(X, \mathbb{R})$.

Proof. (i) For $\gamma \in H_1(X, \mathbb{R})$, assume that

$$\langle \gamma, M(\omega) \rangle = 1 \quad \text{for all } \omega \in H^0(X, K_X).$$

Then

$$\begin{aligned} 1 = \langle \gamma, M(\omega) \rangle &= \exp 2\pi\sqrt{-1}\, \mathrm{Re}(\int_{\gamma} M(\omega)) \\ &= \exp 2\pi\sqrt{-1}\, \mathrm{Re}(\int_{M^*(\gamma)} \omega) \quad \text{for all } \omega \in H^0(X, K_X). \end{aligned}$$

This implies

$$\mathrm{Re}(\int_{M^*(\gamma)} \omega) \in \mathbb{Z} \quad \text{for all } \omega \in H^0(X, K_X).$$

Since $H^0(X, K_X)$ is a complex vector space, we have

$$\int_{M^*(\gamma)} \omega = 0 \quad \text{for all } \omega \in H^0(X, K_X).$$

Hence $M^*(\gamma) = 0$, that is, $\gamma \in M'^*(H_1(X, \mathbb{R}))$. This argument is reversible. Hence (i) is proved.

(ii) can be shown in a similar way. □

Lemma 5. (i) $(M^*(H_1(X,\mathbb{R})))^* = M(H^0(X,K_X))$.
(ii) $(M'^*(H_1(X,\mathbb{R})))^* = M'(H^0(X,K_X))$.

Proof. By the decompositions into the direct sums:

$$H^0(X,K_X) = M(H^0(X,K_X)) \oplus M'(H^0(X,K_X)),$$
$$H_1(X,\mathbb{R}) = M^*(H_1(X,\mathbb{R})) \oplus M'^*(H_1(X,\mathbb{R}))$$

and by Lemma 4, the equalities (i) and (ii) are obtained. □

Using the intersection number $\alpha \cdot \beta$ in $H_1(X,\mathbb{R})$, we define a linear isomorphism

$$\mathbb{B} : H_1(X,\mathbb{R}) \to H^0(X,K_X)$$

by

$$\mathrm{Re}(\int_\beta \mathbb{B}(\alpha)) = \alpha \cdot \beta \quad \text{for } \alpha, \beta \in H_1(X,\mathbb{R}).$$

Hence $H^0(X,K_X)$ is isomorphic to its dual group, that is, $H^0(X,K_X)$ is self-dual. Thus we may say that $H^0(X,K_X)$, $H_1(X,\mathbb{R})$ and (through the linear isomorphisms $\mathbb{A}$ and $\mathbb{A}\mathbb{B}^{-1}$) $\mathbb{C}^g$ are self-dual.

Lemma 6. $M(\mathbb{B}(H_1(X,\mathbb{Z})))^\perp \cap M^*(H_1(X,\mathbb{R})) = H_1(X,\mathbb{Z}) \cap M^*(H_1(X,\mathbb{Z}))$.

Proof. For $\alpha \in H_1(X,\mathbb{R})$, assume

$$\mathrm{Re}(\int_{M^*(\alpha)} M(\mathbb{B}(\beta))) \in \mathbb{Z}$$

for all $\beta \in H_1(X,\mathbb{Z})$. Then

$$\begin{aligned}\mathrm{Re}(\int_{M^*(\alpha)} M(\mathbb{B}(\beta))) &= \mathrm{Re}(\int_{M^*M^*(\alpha)} \mathbb{B}(\beta)) \\ &= \mathrm{Re}(\int_{M^*(\alpha)} \mathbb{B}(\beta)) = \beta \cdot M^*(\alpha) \in \mathbb{Z}\end{aligned}$$

for all $\beta \in H_1(X,\mathbb{Z})$. Hence

$$\gamma = M^*(\alpha) \in H_1(X,\mathbb{Z}).$$

Since

$$\gamma = M^*(\alpha) = M^*M^*(\alpha) = M^*(\gamma),$$

we conclude

$$\gamma \in M^*(H_1(X,\mathbb{Z})) \cap H_1(X,\mathbb{Z}).$$

This argument can be reversible. Hence

$$M(\mathbb{B}(H_1(X,\mathbb{Z})))^\perp \cap M^*(H_1(X,\mathbb{R})) = H_1(X,\mathbb{Z}) \cap M^*(H_1(X,\mathbb{Z})).$$ □

By a similar argument, we have:

Lemma 7. $M'(\mathbb{B}(H_1(X,\mathbb{Z})))^\perp \cap M'^*(H_1(X,\mathbb{R})) = H_1(X,\mathbb{Z}) \cap M'^*(H_1(X,\mathbb{Z}))$.

Now, consider the following exact sequence:

$$0 \to L(\Gamma_X) \cap \Gamma_X \to \{(z,0)\} \to A(\pi) \to 0.$$

Using the self-duality of $\mathbb{C}^g$ and Lemma 6, the dual exact sequence of this exact sequence is given as follows:

$$0 \longleftarrow (L(\Gamma_X) \cap \Gamma_X)^* \longleftarrow \{(z,0)\} \longleftarrow L(\Gamma_X) \longleftarrow 0.$$

Hence we have

$$A(\pi)^* = (\{(z,0)\}/L(\Gamma_X) \cap \Gamma_X)^* = L(\Gamma_X), \tag{16}$$

$$B(\pi)^* = (\{(z,0)\}/L(\Gamma_X))^* = L(\Gamma_X) \cap \Gamma_X. \tag{17}$$

Thus $A(\pi)$ and $B(\pi)$ are dual Abelian varieties.

In a similar way, (using Lemma 7), we see that $A'(\pi)$ and $B'(\pi)$ are dual Abelian varieties.

4. Accola's theorem

We now define two homomorphisms

$$\pi_* : J(X) \to J(Y), \tag{18}$$

$$\pi^* : J(Y) \to J(X) \tag{19}$$

as follows:

$$\pi_* : \sum_{j=1}^{m} \int_{q_j}^{p_j} \Omega \pmod{\Gamma_X} \longmapsto \sum_{j=1}^{m} \int_{\pi(q_j)}^{\pi(p_j)} \Omega_0 \pmod{\Gamma_Y} \tag{20}$$

and

$$\pi^* : \sum_{j=1}^{m} \int_{Q_j}^{P_j} \Omega_0 \pmod{\Gamma_Y} \longmapsto \sum_{j=1}^{m} \sum_{k=1}^{d} \int_{q_{jk}}^{p_{jk}} \Omega \pmod{\Gamma_X}, \tag{21}$$

where

$$\begin{aligned} \Omega &= (\pi^*\eta_1, \ldots, \pi^*\eta_{g_0}, \omega_1, \ldots, \omega_l), \\ \Omega_0 &= (\eta_1, \ldots, \eta_{g_0}), \\ \pi^{-1}(P_j) &= p_{j1} + \cdots + p_{jd} \quad \text{and} \\ \pi^{-1}(Q_j) &= q_{j1} + \cdots + q_{jd}. \end{aligned}$$

(Note that these homomorphisms are well defined, for $\pi_*(\gamma) \in H_1(Y,\mathbb{Z})$ for $\gamma \in H_1(X,\mathbb{Z})$, and $\pi^{-1}(\delta) \in H_1(X,\mathbb{Z})$ for $\delta \in H_1(Y,\mathbb{Z})$.) Then, using the 'coordinates' (z,w) in (5), these homomorphisms can be written as follows:

$$\pi_* : (z,w) \pmod{\Gamma_X} \longmapsto z \pmod{\Gamma_Y}, \tag{22}$$

$$\pi^* : z \pmod{\Gamma_Y} \longmapsto (dz, 0) \pmod{\Gamma_X}. \tag{23}$$

(22) follows from the following property of integration:

$$\int_{\pi(q)}^{\pi(p)} \Omega_0 = \int_q^p \pi^* \Omega_0.$$

(23) follows from the property of integration and the G-invariance of the image in (21) (see (ii) of Lemma 2).

We first discuss the homomorphism π_* in (18). The homomorphism π_* can be decomposed as follows:

$$\pi_* = \hat{\pi}_* \circ L,$$

where

$$L : J(X) \to B(\pi) = \{(z, 0)\}/L(\Gamma_X),$$
$$(z, w) \ (\mathrm{mod}\, \Gamma_X) \longmapsto (z, 0) \ (\mathrm{mod}\, L(\Gamma_X))$$

and

$$\hat{\pi}_* : B(\pi) \to J(Y),$$
$$(z, 0) \ (\mathrm{mod}\, L(\Gamma_X)) \longmapsto z \ (\mathrm{mod}\, \Gamma_Y).$$

Here L and $\hat{\pi}_*$ are surjective homomorphism. Moreover, since $\mathrm{dim} B(\pi) = g_0 = \mathrm{dim} J(Y)$, the kernel of $\hat{\pi}_*$ is a finite subgroup of $B(\pi)$. Hence,

Lemma 8. (i) *The decomposition* $\pi_* = \hat{\pi}_* \circ L$ *gives the Stein factorization of the map* π_*.

(ii) $B(\pi)$ (*and so* $A(\pi)$) *is isogeneous to* $J(Y)$.

Moreover we have:

Lemma 9. *The kernel* $\mathrm{Ker}(\hat{\pi}_*)$ *of* $\hat{\pi}_*$ *is isomorphic to* $H_1(Y, \mathbb{Z})/\pi_*(H_1(X, \mathbb{Z}))$, *which is isomorphic to the Galois group of the maximal unbranched abelian covering of* Y *in* X.

Proof. Note that

$$\hat{\pi}_* : L(\Gamma_X) \to \Gamma_Y$$

is injective. Hence $L(\Gamma_X)$ and $\hat{\pi}_*(L(\Gamma_X)) = \pi_*(\Gamma_X)$ is isomorphic via $\hat{\pi}_*$. Hence $\hat{\pi}_* : B(\pi) \to J(Y)$ can be decomposed as follows:

$$\hat{\pi}_* : B(\pi) = \{(z, 0)\}/L(\Gamma_X) \cong \mathbb{C}^{g_0}/\pi_*(\Gamma_X) \twoheadrightarrow \mathbb{C}^{g_0}/\Gamma_Y = J(Y).$$

Hence $\mathrm{Ker}(\hat{\pi}_*)$ is isomorphic to $\Gamma_Y/\pi_*(\Gamma_X)$, which is isomorphic to

$$H_1(Y, \mathbb{Z})/\pi_*(H_1(X, \mathbb{Z})).$$

In order to show the last assertion, take a point q_0 in Y, which is not contained in the branch locus B_π of π. Take a point $p_0 \in \pi^{-1}(q_0)$. Put

$$U = \pi_1(Y - B_\pi, q_0),$$
$$V = \pi_1(X - \pi^{-1}(B_\pi), p_0)$$

(the fundamental groups). We may consider V as a normal subgroup of U such that $U/V \cong G$, through the injective homomorphism π_*.

Consider the following commutative diagram of injective homomorphisms:

$$\begin{array}{ccc} [V,V] & \longrightarrow & V \\ {\scriptstyle \pi_*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_*} \\ [U,U] & \longrightarrow & U \end{array}$$

This diagram induces the homomorphism

$$\pi_* : H_1(X - \pi^{-1}(B_\pi), \mathbb{Z}) \cong V/[V,V] \to H_1(Y - B_\pi, \mathbb{Z}) \cong U/[U,U].$$

Moreover, note that π_* maps a small circle γ_p around a point p in $\pi^{-1}(B_\pi)$ to e-times of a circle δ_q around $q = \pi(p)$, where e is the ramification index of π at p. Hence π_* induces the homomorphism

$$\begin{aligned} \pi_* : H_1(X, \mathbb{Z}) \cong & H_1(X - \pi^{-1}(B_\pi), \mathbb{Z})/\langle \gamma_p \mid p \in \pi^{-1}(B_\pi)\rangle \\ \to & H_1(Y, \mathbb{Z}) \cong H_1(Y - B_\pi, \mathbb{Z})/\langle \delta_q \mid q \in B_\pi \rangle. \end{aligned}$$

Note that U/V and $[U,U]V/V$ are isomorphic to G and $[G,G]$, respectively. Hence we conclude that $H_1(Y,\mathbb{Z})/\pi_*(H_1(X,\mathbb{Z}))$ is isomorphic to the Galois group of the maximal unbranched abelian covering of Y in X. □

Next, we discuss the homomorphism π^* in (19).

Lemma 10. (i) *If δ and δ' are homologous 1-cycles on Y, then $\pi^{-1}(\delta)$ and $\pi^{-1}(\delta')$ are homologous 1-cycles on X.*

(ii) *For $\delta \in H_1(Y, \mathbb{Z})$,*

$$\int_{\pi^{-1}(\delta)} \Omega = (\int_{\pi^{-1}(\delta)} \pi^*\Omega_0, 0) = (d \int_\delta \Omega_0, 0) \in L(\Gamma_X) \cap \Gamma_X.$$

Proof. (i) is obvious. In the assertion (ii), the equality

$$\int_{\pi^{-1}(\delta)} \pi^*\Omega_0 = d \int_\delta \Omega_0$$

follows from a property of integration. Next, note that $\pi^{-1}(\delta)$ is G-invariant:

$$\varphi_*(\pi^{-1}(\delta)) = \pi^{-1}(\delta)$$

for all $\varphi \in G$. Hence

$$\varphi_*(\int_{\pi^{-1}(\delta)} \Omega) = \int_{\varphi_*(\pi^{-1}(\delta))} \Omega = \int_{\pi^{-1}(\delta)} \Omega$$

for all $\varphi \in G$. Hence the first equality in (ii) follows from (ii) of Lemma 2. The left hand side of the first equality of (ii) belongs to Γ_X, while the right hand side belongs to $L(\Gamma_X)$. □

Lemma 10 implies that

$$\pi^*(\Gamma_Y) \subset L(\Gamma_X) \cap \Gamma_X.$$

Hence the image of the homomorphism π^* in (19) is $A(\pi) = \{(z, 0)\}/L(\Gamma_X) \cap \Gamma_X$:

$$\pi^* : J(Y) \twoheadrightarrow A(\pi) \hookrightarrow J(X).$$

Moreover, note that

$$\begin{aligned} \pi^* : \mathbb{C}^{g_0} &\to \{(z, 0)\} \\ z &\longmapsto (dz, 0) \end{aligned}$$

is bijective. Hence the kernel of π^* in (19) is isomorphic to

$$(L(\Gamma_X) \cap \Gamma_X)/\pi^*(\Gamma_Y),$$

which is isomorphic (via $\mathbb{A}^{-1}$) to

$$(M^*(H_1(X, \mathbb{Z})) \cap H_1(X, \mathbb{Z}))/\pi^*(H_1(Y, \mathbb{Z})).$$

Thus we conclude:

Lemma 11. *The kernel of the homomorphism* $\pi^* : J(Y) \to J(X)$ *is isomorphic to* $(M^*(H_1(X, \mathbb{Z})) \cap H_1(X, \mathbb{Z}))/\pi^*(H_1(Y, \mathbb{Z}))$.

Now we are ready to prove the following theorem of Accola [1, p. 5]:

Theorem 2 (Accola)**.** *The kernel of the homomorphism* $\pi^* : J(Y) \to J(X)$ *is a finite group isomorphic to the dual group of the Galois group of the maximal unbranched abelian covering of* Y *in* X.

Proof. By Lemma 9 and Lemma 11, it suffices to show that $(M^*(H_1(X, \mathbb{Z})) \cap H_1(X, \mathbb{Z}))/\pi^*(H_1(Y, \mathbb{Z}))$ is isomorphic to the dual group of $\mathrm{Ker}(\hat{\pi}_*)$. From the exact sequence of abelian groups:

$$0 \to \mathrm{Ker}(\hat{\pi}_*) \to B(\pi) \to J(\pi) \to 0,$$

we have the exact sequence of the dual abelian groups:

$$0 \longleftarrow (\mathrm{Ker}(\hat{\pi}_*))^* \longleftarrow B(\pi)^* \longleftarrow J(\pi)^* \longleftarrow 0.$$

Note that $J(Y)^* \cong H_1(Y, \mathbb{Z})$ and $B(\pi)^* \cong (M^*(H_1(X, \mathbb{Z})) \cap H_1(X, \mathbb{Z}))$ (see (17)). Also note that the dual homomorphism of $\hat{\pi}_*$ coincides with the injective homomorphism $\pi^* : H_1(Y, \mathbb{Z}) \to (M^*(H_1(X, \mathbb{Z})) \cap H_1(X, \mathbb{Z}))$. Hence $(\mathrm{Ker}(\hat{\pi}_*))^*$ is isomorphic to $(M^*(H_1(X, \mathbb{Z})) \cap H_1(X, \mathbb{Z}))/\pi^*(H_1(Y, \mathbb{Z}))$. $\square$

Remark 1. [1, p. 5] asserts more generally that Theorem 2 holds for (not necessarily Galois) finite covering $\pi : X \to Y$. (Note that the homomorphisms π_* and π^* in (18) and (19) can be defined as in (20) and (21) for (not necessarily Galois) finite coverings.)

5. Galois correspondence

In our point of view, $A(\pi)$ looks most important among Abelian subvarieties of $J(X)$. Every point of $A(\pi)$ is fixed by every element of G (see (7)).

Lemma 12. *If A is an Abelian subvariety of $J(X)$, each of whose points is fixed by every element of G, then $A \subset A(\pi)$.*

Proof. Any Abelian subvariety A of $J(X)$ can be expressed as

$$A = \frac{S}{S \cap \Gamma_X} \cong \frac{S + \Gamma_X}{\Gamma_X},$$

where S is a linear subspace of $\mathbb{C}^g$. Suppose that every point of A is fixed by every element of G. For every vector $v \in S$ and every $\varphi \in G$, there exists a vector $a(v, \varphi) \in \Gamma_X$ such that

$$\varphi_*(v) = v + a(v, \varphi), \text{that is}, a(v, \varphi) = \varphi_*(v) - v.$$

Then, for a complex parameter t,

$$a(tv, \varphi) = \varphi_*(tv) - tv = t\varphi_*(v) - tv = ta(v, \varphi).$$

Since Γ_X is a discrete subgroup of $\mathbb{C}^g$, this relation implies $a(v, \varphi) = 0$. Hence

$$\varphi_*(v) = v \quad \text{for all } \ v \in S, \ \varphi \in G.$$

By (ii) of Lemma 2, we have $v \in \{(z, 0)\}$. That is, $S \subset \{(z, 0)\}$. Hence $A \subset A(\pi)$. □

For a subgroup H of G, the set of all points in $J(X)$ fixed by every element of H forms a subgroup of $J(X)$. Let $A(H)$ be the connected component of 0 of the subgroup. This is the largest Abelian subvariety each of whose points is fixed by every element of H. By Lemma 12,

$$A(G) = A(\pi).$$

In general, we have

$$A(H) = A(\pi_H),$$

where

$$\pi_H : X \to X/H$$

is the canonical projection.

An Abelian subvariety A of $J(X)$ is said to be *maximal* if $A = A(H)$ for a subgroup H of G.

Theorem 3 (Galois correspondence)**.** *Assume $g_0 = g(X/G) \geq 2$. Then the correspondence*

$$H \longmapsto A(H)$$

is bijective between the set $\{H\}$ of subgroups of G and the set $\{A\}$ of maximal Abelian subvarieties of $J(X)$. The correspondence reverses the inclusion relations.

For the proof of Theorem 3, we first note the following relation:

$$A(G) \subset A(H) \subset J(X).$$

Next, we prove:

Lemma 13. *For subgroups H and H' of G with $H \subsetneqq H'$,*
(i) $A(G) \subset A(H') \subset A(H) \subset J(X)$.
(ii) *If the genus of X/H' satisfies $g(X/H') \geq 2$, then $A(H') \subsetneqq A(H)$.*

Proof. (i) is clear from the definition of $A(H)$. We prove (ii). Let

$$\pi_{H,H'} : X/H \to X/H'$$

be the canonical projection. This is a finite covering. The Riemann-Hurwitz formula for $\pi_{H,H'}$ can be written as

$$2g(X/H) - 2 = d_1(2g(X/H') - 2) + \sum_p (e_p - 1),$$

where $d_1 = [H' : H]$ is the mapping degree of $\pi_{H,H'}$ and $e_p(\geq 2)$ is the ramification index at the ramification point $p \in X/H$.

Since $d_1 \geq 2$ and $g(X/H') \geq 2$, we have

$$2g(X/H) - 2 \geq d_1(2g(X/H') - 2) > 2g(X/H') - 2.$$

Hence $g(X/H) > g(X/H')$. Hence

$$\dim A(H) = g(X/H) > g(X/H') = \dim A(H'). \qquad \square$$

Now for an Abelian subvariety A of $J(X)$, we put

$$H(A) = \{\varphi \in G \mid \text{every point of } A \text{ is fixed by } \varphi\}.$$

Then $H(A)$ is a subgroup of G.

Lemma 14. *If $g_0 = g(X/G) \geq 2$, then $H(A(H)) = H$.*

Proof. Put $H' = H(A(H))$. Then by the definition, we have $H \subset H'$. Hence $A(H') \subset A(H)$. But note that every point of $A(H)$ is fixed by every element of H'. Hence, by the maximality of $A(H')$, we have $A(H) \subset A(H')$. Hence $A(H) = A(H')$. Now, by Lemma 13, we have $H = H'$. $\square$

Now Theorem 3 follows from Lemma 13 and Lemma 14.

6. G-invariant linear systems

We want to look for G-invariant linear systems on X, making use of $J(X)$ and its Abelian subvariety $A(\pi)$. A linear system Λ on X is said to be G-invariant if every element of G maps every divisor in Λ to a divisor in Λ.

Lemma 15. (i) *For a* (*not necessarily positive*) *divisor D on X, assume that $D \sim \varphi(D)$* (*linearly equivalent*) *for all $\varphi \in G$. Then for any* (*not necessarily positive*) *divisor E with $E \sim D$, we have $E \sim \varphi(E)$ for all $\varphi \in G$.*

(ii) *If a linear system Λ is G-invariant, then the complete linear system $|D|$ containing Λ is G-invariant.*

(iii) *If a divisor D on X is G-invariant, then the complete linear system $|D|$ is G-invariant.*

(iv) *If $|D|$ and $|D'|$ are G-invariant, then $|D + D'|$ is G-invariant.*

Proof. (i) There is a meromorphic function f on X such that

$$D - E = (f) = D_f(0) - D_f(\infty),$$

where (f), $D_f(0)$, $D_f(\infty)$ are the principal divisor of f, the zero-divisor of f and the polar-divisor of f, respectively. Then, for any element φ in G, we have

$$\varphi(D) - \varphi(E) = \varphi(D_f(0)) - \varphi(D_f(\infty)) = (f \circ \varphi^{-1}).$$

Hence $\varphi(D)$ and $\varphi(E)$ are linearly equivalent. By the assumption, D and $\varphi(D)$ are linearly equivalent. Hence, $\varphi(E)$ is linearly equivalent to D and so, linearly equivalent to E.

(ii), (iii) and (iv) follow from (i). □

Hence, our first task is to look for G-invariant complete linear systems on X. Complete linear systems appear as the inverse images of points in $J(X)$ of the Abel-Jacobi map $\Phi_m \quad (m \in \mathbb{Z}_{>0})$. Here the Abel-Jacobi map

$$\Phi_m : S^m(X) \to J(X)$$

($S^m(X)$: the m-th symmetric product of X) is defined as follows: Take a point p_0 in X and fix it. For a divisor $D = p_1 + \cdots + p_m$ in $S^m(X)$, Φ_m maps D to

$$\sum_{j=1}^{m} \int_{p_0}^{p_j} \Omega \,(\mathrm{mod}\, \Gamma_X).$$

The group G acts on $S^m(X)$. So we must modify the action of G on $J(X)$ so that Φ_m becomes equivariant under the actions:

$$\sum_{j=1}^{m} \int_{p_0}^{p_j} \Omega \,(\mathrm{mod}\, \Gamma_X) \longmapsto \sum_{j=1}^{m} \int_{p_0}^{\varphi(p_j)} \Omega \,(\mathrm{mod}\, \Gamma_X)$$

$$= \sum_{j=1}^{m} \int_{\varphi(p_0)}^{\varphi(p_j)} \Omega + m \int_{p_0}^{\varphi(p_0)} \Omega \,(\mathrm{mod}\, \Gamma_X)$$

$$=\sum_{j=1}^{m}\int_{p_0}^{p_j}\varphi^*(\Omega)+m\int_{p_0}^{\varphi(p_0)}\Omega\ (\mathrm{mod}\,\Gamma_X),$$

where $\varphi \in G$. Thus we define the modified action of G on $J(X)$, which we denote (G,m)-action, as follows: (Writing $\varphi^{(m)}$, instead of φ),

$$\varphi^{(m)}((z,w)\ (\mathrm{mod}\,\Gamma_X)) = \varphi_*(z,w)+mv_\varphi\ (\mathrm{mod}\,\Gamma_X) \tag{24}$$

$$= (z,wB_\varphi)+mv_\varphi\ (\mathrm{mod}\,\Gamma_X), \tag{25}$$

where $v_\varphi = \int_{p_0}^{\varphi(p_0)}\Omega$.

Hence we may define an affine transformation $\varphi^{(m)}$ on $\mathbb{C}^g$, which induces the action $\varphi^{(m)}$ on $J(X)$, as follows:

$$\varphi^{(m)}(z,w) = \varphi_*(z,w)+mv_\varphi = (z,wB_\varphi)+mv_\varphi, \tag{26}$$

where $v_\varphi = \int_{p_0}^{\varphi(p_0)}\Omega$.

Now, for any (not necessarily positive) integer m, we *define* the (G,m)-action on $J(X)$ and on $\mathbb{C}^g$ by (24),(25) and (26), respectively.

By (24) and (25), a point $(z_0,w_0)\ (\mathrm{mod}\,\Gamma_X)$ in $J(X)$ is a fixed point of $\varphi^{(m)}$ $(m\in\mathbb{Z})$ if and only if

$$(z_0,w_0B_\varphi)+mv_\varphi\ (\mathrm{mod}\,\Gamma_X),=(z_0,w_0)\ (\mathrm{mod}\,\Gamma_X).$$

This occurs if and only if

$$(0,w_0B_\varphi)+mv_\varphi\ (\mathrm{mod}\,\Gamma_X)=(0,w_0)\ (\mathrm{mod}\,\Gamma_X).$$

That is, this occurs if and only if

$$\varphi^{(m)}((0,w_0)\ (\mathrm{mod}\,\Gamma_X)) = \varphi_*(0,w_0)+mv_\varphi\ (\mathrm{mod}\,\Gamma_X) = (0,w_0)\ (\mathrm{mod}\,\Gamma_X).$$

That is, $(z_0,w_0)(\mathrm{mod}\Gamma_X)$ is a fixed point of $\varphi^{(m)}$ if and only if $(0,w_0)(\mathrm{mod}\Gamma_X)$ is a fixed point of $\varphi^{(m)}$.

We put, for $m\in\mathbb{Z}$,

$$\mathrm{Fix}(J(X),G,m)=\{(z_0,w_0)\ (\mathrm{mod}\,\Gamma_X)\in J(X)\mid (z_0,w_0)\ (\mathrm{mod}\,\Gamma_X)\ \text{is a fixed point of}\ \varphi^{(m)}\ \text{for all}\ \varphi\in G\},$$

and

$$\mathrm{Fix}_0(J(X),G,m)=\{(0,w_0)\ (\mathrm{mod}\,\Gamma_X)\in J(X)\mid (0,w_0)\ (\mathrm{mod}\,\Gamma_X)\ \text{is a fixed point of}\ \varphi^{(m)}\ \text{for all}\ \varphi\in G\}.$$

We may express $\mathrm{Fix}_0(J(X),G,m)$ the set of points of $\mathrm{Fix}(J(X),G,m)$, which is *orthogonal to* $A(\pi)$. Then the above discussion implies:

Lemma 16. (i) $\mathrm{Fix}(J(X),G,m)$ *is non-empty if and only if* $\mathrm{Fix}_0(J(X),G,m)$ *is non-empty.*

(ii) *If* $\mathrm{Fix}(J(X),G,m)$ *is non-empty, then*

$$\mathrm{Fix}(J(X),G,m)=\mathrm{Fix}_0(J(X),G,m)+A(\pi),$$

where the addition is the addition in $J(X)$.

Thus we may restrict our discussion mainly to $\mathrm{Fix}_0(J(X), G, m)$.

Lemma 17. $\mathrm{Fix}_0(J(X), G, 0)$ *is non-empty and is contained in the set of* d-*division points of* 0 *of* $J(X)$.

Proof. $0 = (0, 0) (\mathrm{mod}\, \Gamma_X)$ is a point of $\mathrm{Fix}_0(J(X), G, 0)$.

For a point $(0, w_0)(\mathrm{mod}\, \Gamma_X)$ of $\mathrm{Fix}_0(J(X), G, 0)$, we have $(0, w_0 B_\varphi) = (0, w_0) \; (\mathrm{mod}\, \Gamma_X)$ for all $\varphi \in G$. We add this equality for $\varphi = \varphi_1, \ldots, \varphi_d$ and get

$$(0, w_0(B_{\varphi_1} + \cdots + B_{\varphi_d})) = (0, dw_0) = d(0, w_0) \; (\mathrm{mod}\, \Gamma_X).$$

The left hand side is equal to $(0, 0)$ by (iii) of Lemma 2. Hence $(0, w_0)$ is a d-division point of 0 of $J(X)$. □

The set $\Delta_0(0, d)$ of d-division points of 0 of $J(X)$ of the form $(0, w)$ forms a finite subgroup of $J(X)$ of order d^{2l}, where $l = g - g_0$. Since the sum of two points of $\mathrm{Fix}_0(J(X), G, 0)$ is again a point of $\mathrm{Fix}_0(J(X), G, 0)$, $\mathrm{Fix}_0(J(X), G, 0)$ is a subgroup of the finite subgroup $\Delta_0(0, d)$.

Lemma 18. *For a point* $(0, w_0)(\mathrm{mod}\, \Gamma_X) \in \mathrm{Fix}_0(J(X), G, m)$,

$$\mathrm{Fix}_0(J(X), G, m) = \mathrm{Fix}_0(J(X), G, 0) + (0, w_0) \; (\mathrm{mod}\, \Gamma_X),$$

where the addition is the addition in $J(X)$.

Proof. Take a point $(0, w)(\mathrm{mod}\, \Gamma_X) \in \mathrm{Fix}_0(J(X), G, 0)$. Then

$$\varphi_*(0, w) \; (\mathrm{mod}\, \Gamma_X) = (0, w) \; (\mathrm{mod}\, \Gamma_X)$$

for all $\varphi \in G$. On the other hand, we have

$$\varphi_*(0, w_0) + m v_\varphi \; (\mathrm{mod}\, \Gamma_X) = (0, w_0) \; (\mathrm{mod}\, \Gamma_X) \tag{27}$$

for all $\varphi \in G$. Adding these equality, we have

$$\varphi_*(0, w + w_0) + m v_\varphi \; (\mathrm{mod}\, \Gamma_X) = (0, w + w_0) \; (\mathrm{mod}\, \Gamma_X)$$

for all $\varphi \in G$. Hence

$$\mathrm{Fix}_0(J(X), G, m) \supset \mathrm{Fix}_0(J(X), G, 0) + d(0, w_0) \; (\mathrm{mod}\, \Gamma_X).$$

Conversely, for a point $(0, w_0')(\mathrm{mod}\, \Gamma_X)$, we have,

$$\varphi_*(0, w_0') + m v_\varphi \; (\mathrm{mod}\, \Gamma_X) = (0, w_0') \; (\mathrm{mod}\, \Gamma_X)$$

for all $\varphi \in G$. Subtracting (27) from this equality, we have

$$\varphi_*(0, w_0' - w_0) \; (\mathrm{mod}\, \Gamma_X) = (0, w_0' - w_0) \; (\mathrm{mod}\, \Gamma_X)$$

for all $\varphi \in G$. Hence $((0, w_0') - (0, w_0))(\mathrm{mod}\, \Gamma_X) \in \mathrm{Fix}_0(J(X), G, 0)$. □

A similar argument to the proof of Lemma 18 shows the following lemma:

Lemma 19. (i) *For m and m' in $\mathbb{Z}$, if* $\mathrm{Fix}_0(J(X), G, m)$ *and* $\mathrm{Fix}_0(J(X), G, m')$ *are non-empty, then* $\mathrm{Fix}_0(J(X), G, m+m')$ *is non-empty and* $\mathrm{Fix}_0(J(X), G, m)$ $+\mathrm{Fix}_0(J(X), G, m') = \mathrm{Fix}_0(J(X), G, m+m')$*, where the addition is the addition in $J(X)$.*

(ii) $\mathrm{Fix}_0(J(X), G, m) + \cdots + \mathrm{Fix}_0(J(X), G, m) = \mathrm{Fix}_0(J(X), G, km)$ *for $m \in \mathbb{Z}$, where the left hand side is the k-times summation in $J(X)$.*

(iii) $\mathrm{Fix}_0(J(X), G, -m) = -\mathrm{Fix}_0(J(X), G, m)$ *for $m \in \mathbb{Z}$.*

Lemma 20. $\mathrm{Fix}_0(J(X), G, d)$ *is non-empty.*

Proof. For any point $P \in Y$, the divisor

$$\pi^{-1}(P) = p_1 + \cdots + p_d \in S^d(X)$$

is fixed by every element of G. Hence, by (iii) of Lemma 15,

$$\Phi_d(\pi^{-1}(P)) \in \mathrm{Fix}(J(X), G, d).$$

Hence $\mathrm{Fix}_0(J(X), G, d)$ is non-empty. □

Now, Lemma 18, Lemma 19 and Lemma 20 imply:

Theorem 4. (i) *There is a positive integer m_0 with $m_0 \mid d$ such that, for $m \in \mathbb{Z}$, $Fix_0(J(X), G, m)$ is non-empty if and only if m is divisible by m_0.*

(ii) *The number of points of non-empty* $\mathrm{Fix}_0(J(X), G, m)$ *is constant for m.*

Let

$$B_\pi = \{Q_1, \ldots, Q_s\} \quad (\subset Y)$$

be the set of all branch points of π. Put

$$\pi^{-1}(Q_j) = q_{j1} + \cdots + q_{jl_j},$$

for $j = 1, \ldots, s$, where $l_j = d/e_j$ and $e_j (\geq 2)$ is the ramification index at q_{jk} for $k = 1, \ldots, l_j$.

The divisor $\pi^{-1}(Q_j)$ is fixed by every element of G. Hence, by (iii) of Lemma 15,

$$\Phi_{d/e_j}(\pi^{-1}(Q_j)) \in \mathrm{Fix}(J(X), G, d/e_j).$$

Proposition 1. (i) $\mathrm{Fix}_0(J(X), G, d/e_j)$ *is non-empty for $j = 1, \ldots, s$.*

(ii) $\mathrm{Fix}_0(J(X), G, d/e_0)$ *is non-empty, where $e_0 = LCM(e_1, \ldots, e_s)$.*

(iii) *d/e_0 is divisible by the positive integer m_0 in Theorem 4.*

Proof. (i) is clear from the above argument.

(ii) Note that

$$GCD(d/e_1, \ldots, d/e_s) = d/LCM(e_1, \ldots, e_s).$$

Now (ii) follows from Lemma 19.

(iii) follows from (ii) and Theorem 4. □

Put

$$\mathrm{Fix}(J(X),G) = \bigcup_{m\in\mathbb{Z}} \mathrm{Fix}(J(X),G,m),$$
$$\mathrm{Fix}_0(J(X),G) = \bigcup_{m\in\mathbb{Z}} \mathrm{Fix}_0(J(X),G,m).$$

Then, by Lemma 19, we have:

Proposition 2. (i) $\mathrm{Fix}_0(J(X),G) \subset \mathrm{Fix}(J(X),G)$.
(ii) *They are subgroups of* $J(X)$.

A point in $\mathrm{Fix}(J(X),G)$ may not be in the image W_m of $\Phi_m : S^m(X) \to J(X)$, if $m < g$. So, we put

$$\mathrm{Fix}^*(J(X),G) = \bigcup_{m\in\mathbb{Z}_{>0}} \mathrm{Fix}(J(X),G,m) \cap W_m,$$
$$\mathrm{Fix}_0^*(J(X),G) = \bigcup_{m\in\mathbb{Z}_{>0}} \mathrm{Fix}_0(J(X),G,m) \cap W_m.$$

(Here, we put $W_m = \Phi_m(S^m(X))$, and so $W_m = J(X)$ if $m \geq g$.) Then, by (iv) of Lemma 15, we have:

Proposition 3. (i) $\mathrm{Fix}_0^*(J(X),G) \subset \mathrm{Fix}^*(J(X),G)$.
(ii) *They are subsemigroups of* $\mathrm{Fix}_0(J(X),G)$ *and of* $\mathrm{Fix}(J(X),G)$*, respectively.*

Finally, we give some examples.

Example 1.

Let X be a hyperelliptic Riemann surface of genus g, that is, the normalization of the algebraic curve defined by the equation

$$X : y^2 = (x - a_1)(x - a_2)\cdots(x - a_{2g+2}),$$

where $a_1, a_2, \ldots, a_{2g+2}$ are mutually distinct complex numbers and $g \geq 2$. Let

$$\pi : X \to \mathbb{P}^1, \quad (x,y) \longmapsto x$$

be the Galois covering with $G = \{1, \varphi\}$, where $\mathbb{P}^1$ is the complex projective line and

$$\varphi : (x,y) \longmapsto (x,-y)$$

is the involution of X.

In this case, $g_0 = 0$ and so $l = g$. Hence $\mathrm{Fix}_0(J(X),G,m) = \mathrm{Fix}(J(X),G,m)$. Moreover, $\mathrm{Fix}(J(X),G,m)$ is non-empty for any integer m, (hence $m_0 = d/e_0 = 1$, in this case) and consists of 2^{2g}-points. Hence $\mathrm{Fix}(J(X),G,0)$ coincides with the set $\Delta(0,2)$ of 2-division points of 0 in $J(X)$.

In fact, put $p_j = (a_j, 0)$ for $j = 1, 2, \ldots, 2g+2$. Then it is easy to see that

$$\mathrm{Fix}(J(X),G,1) \cap W_1 = \{\Phi_1(p_j) \mid j = 1, 2, \ldots, 2g+2\}.$$

In order to count the number of points in $\mathrm{Fix}(J(X), G, g)$, we divide our discussion into 2-cases:

Case 1. g : odd.

$\mathrm{Fix}(J(X), G, g)$ consists of the following points:

$\Phi_g((g-1)p_1 + p_j)$ — $\binom{2g+2}{1}$-points,

$\Phi_g((g-3)p_1 + p_j + p_k + p_l)$ —$\binom{2g+2}{3}$-points, where p_j, p_k and p_l are mutually distinct points,

...

$\Phi_g(p_{j_1} + p_{j_2} + \cdots + p_{j_g})$ — $\binom{2g+2}{g}$-points.

Case 2. g : even.

$\mathrm{Fix}(J(X), G, g)$ consists of the following points:

$\Phi_g(gp_1)$ — $\binom{2g+2}{0}$-points,

$\Phi_g((g-2)p_1 + p_j + p_k)$ —$\binom{2g+2}{2}$-points, where p_j and p_k are distinct points,

...

$\Phi_g(p_{j_1} + p_{j_2} + \cdots + p_{j_g})$ —$\binom{2g+2}{g}$-points.

Thus our assertion follows from the following lemma:

Lemma 21. *The following formula holds*:

(i) *If n is odd, then*

$$\binom{2n+2}{1} + \binom{2n+2}{3} + \cdots + \binom{2n+2}{n} = 2^{2n}.$$

(ii) *If n is even, then*

$$\binom{2n+2}{0} + \binom{2n+2}{2} + \cdots + \binom{2n+2}{n} = 2^{2n}.$$

Proof. This formula follows from the relation between the $(2n+1)$-th row and the $(2n+2)$-th row in Pascal's triangle. □

In a similar way, it is possible to count the number of points in

$$\mathrm{Fix}(J(X), G, m) \cap W_m$$

for $2 \leq m < g$.

Example 2.

Let X be the hyperelliptic Riemann surface of genus $g = 2$ defined by

$$X : y^2 = x^6 + 1.$$

Let $G = \mathrm{Aut}(X)$. It is well-known (see e.g. Namba [2]) that there is an exact sequence

$$1 \to Z \to G \to U \to 1,$$

where Z is the center of G:

$$Z = \{1, \varphi\}, \quad \varphi : (x, y) \longmapsto (x, -y),$$

and U is the subgroup of $\mathrm{Aut}(\mathbb{P}^1)$, isomorphic to the dihedral group D_6 of order 12:

$$U = \langle \hat{\psi}, \hat{\eta} \rangle,$$
$$\hat{\psi} : x \longmapsto \zeta^2 x, \quad \hat{\eta} : x \longmapsto 1/x,$$

where $\zeta = \exp(2\pi\sqrt{-1}/12)$. In fact, G is generated by φ, ψ and η, where

$$\psi : (x, y) \longmapsto (\zeta^2 x, y),$$
$$\eta : (x, y) \longmapsto (1/x, y/x^3).$$

In this case, $g_0 = 0$ and so $l = g = 2$. Moreover $\pi : X \to X/G = \mathbb{P}^1$ is given by

$$\pi : (x, y) \longmapsto x^6 + 1/x^6.$$

The branch locus of π is

$$B_\pi = \{-2, 2, \infty\},$$

with the ramification index $4, 2, 6$, respectively. Hence

$$e_0 = 12, \quad d/e_0 = 2.$$

We show that $\mathrm{Fix}(J(X), G, 2)$ consists of one point. Note that $H^0(X, K_X)$ is 2-dimensional and has the basis $dx/y, xdx/y$. Hence the canonical linear system $|K_X|$ is 1-dimensional and gives the meromorphic function $f : (x, y) \longmapsto x$ on X. $|K_X|$ is G-invariant. For a positive divisor D of degree 2, which is not a canonical divisor, we have $|D| = \{D\}$, by Riemann-Roch theorem. Hence $|D|$ is not G-invariant. Hence $\mathrm{Fix}(J(X), G, 2)$ consists of one point.

It is clear that $\mathrm{Fix}(J(X), G, 1) \cap W_1$ is empty. But we show that $\mathrm{Fix}(J(X), G, 1)$ is non-empty, (and so, $m_0 = 1$ in this case). For this purpose, it is enough to show that $\mathrm{Fix}(J(X), G, 3)$ is non-empty, for 2 and 3 are coprime.

Consider the meromorphic function

$$h : (x, y) \longmapsto y/(x - \zeta)(x - \zeta^5)(x - \zeta^9).$$

Then

$$D_0(h) = \{(\zeta^3, 0), (\zeta^7, 0), (\zeta^{11}, 0)\},$$
$$D_\infty(h) = \{(\zeta, 0), (\zeta^5, 0), (\zeta^9, 0)\}.$$

Moreover, we have

$$h \circ \varphi = h,$$
$$h \circ \psi^{-1} = 1/h,$$
$$h \circ \eta = \zeta^3/h = \sqrt{-1}/h.$$

Hence the linear pencil determined by h is G-invariant.

Thus we conclude that $\mathrm{Fix}(J(X), G, m)$ consists of one point for all integer m in this case.

Example 3.

Let X be a non-singular plane quartic curve, called *Klein curve* defined by the equation:

$$X : X_0X_1^3 + X_1X_2^3 + X_2X_0^3 = 0,$$

where $(X_0 : X_1 : X_2)$ is a homogeneous coordinate in the complex projective plane $\mathbb{P}^2$. Then $g = g(X) = 3$. Put $G = \mathrm{Aut}(X)$, which is known to be the simple group of order 168. It is known that $\pi : X \to \mathbb{P}^1 = X/G$ is a Galois covering whose branch locus is Q_1, Q_2, Q_3 with the ramification indices $2, 3, 7$, respectively. Hence $g_0 = 0$, $l = g = 3$, $e_0 = 42$ and $d/e_0 = 4$.

The canonical linear system $|K_X|$ is degree 4, consists of line sections of the Klein curve and is G-invariant. Other complete linear systems of degree 4 are linear pencils (by Riemann-Roch theorem) and are not G-invariant. This is because there is no non-trivial homomorphism of G into $\mathrm{Aut}(\mathbb{P}^1)$. Hence $\mathrm{Fix}(J(X), G, 4)$ and so every non-empty $\mathrm{Fix}(J(X), G, m)$ consists of one point.

We show that $\mathrm{Fix}(J(X), G, 3)$ is empty. In fact, every linear pencil of degree 3 can be obtained by the linear projection of the Klein curve with the center a point on the curve (see, e.g. Namba [3, p. 372, Theorem 5.3.17]). Such a linear pencil can not be G-invariant by a geometric reason, or by the same reason as in the case of $m = 4$. Other linear systems of degree 3 have dimension 0 and can not be G-invariant. Hence $\mathrm{Fix}(J(X), G, 3)$ is empty, so $\mathrm{Fix}(J(X), G, 1)$ is empty.

It is clear that $\mathrm{Fix}(J(X), G, 2) \cap W_2$ is empty. But we do not know that $\mathrm{Fix}(J(X), G, 2)$ itself is empty or not. Hence we do not know which is correct: $m_0 = 2$ or $m_0 = 4$ in this case.

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