DIGITAL HOMOLOGY GROUPS OF DIGITAL WEDGE SUMS

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Abstract. The present paper investigates some properties of the digital homology in [1, 4, 5] and points out some unclearness of the definition of a digital homology and further, suggests a suitable form of a digital homology. Finally, we calculate a digital homology group and a relative digital homology group of some digital wedge sums. Finally, the paper corrects some errors in [6]. In the present paper all digital images (X, k) are assumed to be non-empty and k-connected.

1. Introduction

Digital topology has a focus on studying digital topological properties of nD digital images, which has contributed to the study of some areas of computer science such as computer graphics, image processing, approximation theory, mathematical morphology, optimization theory and so forth [7, 8, 10, 18, 19, 20, 21]. Hence for a digital image $X \subset \mathbf{Z}^n$ with a k-adjacency, denoted by (X, k) [20, 21], using the digital homology proposed in the paper [1, 3], the authors of [3] formulated a digital version of the Euler characteristic denoted by $\chi(X, k)$ and further, they studied its various properties. However, in Sections 3 and 4 we will discuss about some limitations of this digital homology. More precisely, the paper [6] asserted that if $A \subset X$ is a deformation k-retract, then $H_q^k(X, A) = 0$ for any $q \geq 0$. But we correct this approach with Proposition 5.2.

The rest of the paper is organized as follows: Section 2 provides basic notions from digital topology. Section 3 points out some unclearness of the definition of a finite digital simplical complex given in [1, 3] and

further, it refers some differences between a simplicial complex in algebraic topology and a digital simplicial complex(see Remarks 3.1 and 3.2 and Proposition 3.3). Section 4 calculates a homology group of a digital wedge sum. Section 5 corrects some errors in the paper [6]. Section 6 concludes the paper with some remarks.

2. Preliminaries

To study various properties of a digital version of a homology group of a digital image and to make the paper self-contained, we need to recall some basic notions from digital topology and digital homology such as k-adjacency relations of nD integer grids, a digital k-neighborhood, digital continuity, a digital m-simplex and so forth [7, 19, 20, 21]. Let $\mathbf{N}$ and $\mathbf{R}$ represent the sets of natural numbers and real numbers, respectively. Let $\mathbf{Z}^n$ be the set of points in the Euclidean nD space with integer coordinates, $n \in \mathbf{N}$.

To study nD digital images, we will say that two distinct points $p, q \in \mathbf{Z}^n$ are k-(or $k(m,n)$-)adjacent if they satisfy the following property [7] (see also [13, 14]):
For a natural number m, $1 \le m \le n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n,$$

are $k(m,n)$-(k-, for brevity)adjacent if

at most m of their coordinates differs by ± 1, and all others coincide.
$$(2.1)$$

Concretely, these $k(m,n)$-adjacency relations of $\mathbf{Z}^n$ are determined according to the number $m \in \mathbf{N}$ [7] (see also [13]).

According to the operator of (2.1), the k-adjacency relations of $\mathbf{Z}^n$ are obtained [7] (see also [11, 12]) as follows:

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\, i!}. \qquad (2.2)$$

A. Rosenfeld [20] called a set $X \subset \mathbf{Z}^n$ with a k-adjacency a digital image, denoted by (X, k). Indeed, to follow a graph theoretical approach of studying nD digital images [16, 21], both the k-adjacency relations of $\mathbf{Z}^n$ of (2.2) and a digital k-neighborhood has been used to study digital images. More precisely, we say that a digital k-neighborhood of p in $\mathbf{Z}^n$ is the set [20] $N_k(p) := \{q \,|\, p \text{ is } k\text{-adjacent to } q\} \cup \{p\}$.

For $a, b \in \mathbf{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \,|\, a \leq n \leq b\}$ with 2-adjacency is called a digital interval [19]. Besides, for a k-adjacency relation of $\mathbf{Z}^n$, a simple k-path with $l + 1$ elements in $\mathbf{Z}^n$ is assumed to be a subset $(x_i)_{i \in [0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|i - j| = 1$ [19]. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x, y)$, is the number l. We say that a digital image (X, k) is k-connected if for any two points in X there is a k-path in X connecting these two points. A simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [7, 19] (see Figure 1(b)), is the simple k-path $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $|i - j| = 1 (mod\, l)$ [19].

For a digital image (X, k), as a generalization of $N_k(p)$, the digital k-neighborhood of $x_0 \in X$ with radius ε is defined to be the following subset [7] of X

$$N_k(x_0, \varepsilon) := \{x \in X \,|\, l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \qquad (2.3)$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$. Concretely, for $X \subset \mathbf{Z}^n$ we obtain [10]

$$N_k(x, 1) = N_k(x) \cap X. \qquad (2.4)$$

In Section 3, in relation to the study of digital homology, the works [1, 3, 8, 11] used the notion of a digital simplicial complex in a digital image (X, k), as follows:

Definition 1. [1, 3] *Let S be a set of nonempty subsets of a digital image (X, k). Then the members of S are called simplices of (X, k) if the following hold:*
(a) If p and q are distinct points of $s \in S$, then p and q are k-adjacent.
(b) If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

We say that an m-simplex is a simplex S such that $|S| = m + 1$ [1]. In this case we call the digital m-simplex a *digital (k, m)-complex* because it is inherited from the k-adjacency of (X, k). Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' is called a face of P (for more details, see [1, 3]). Besides, a digital k-subcomplex A of a digital k-complex is a digital simplicial complex contained in a digital simplicial complex X with $Vert(A) \subset Vert(X)$ [1].

Definition 2. [3] *Let (X, k) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some nonnegative integer d. If the following statements hold, then (X, k) is called a finite digital simplicial complex [1]:*
(1) If P belongs to X, then every face of P also belongs to X.

(2) If $P, Q \in X$, then $P \cap Q$ is either empty or a common proper face of P and Q.

3. Improvement of the digital homology group in [1, 3]

Since Definition 2 is very unclear, this section improves it.

Remark 3.1. *Let us consider the set $X := \{x_0 := (0,0), x_1 := (0,1), x_2 := (1,0), x_3 := (1,1)\}$ (see Figure 1(a)). Let us consider two cases such as $(X, 4)$ or $(X, 8)$. In case of $(X, 4)$, we have only four 0-simplices and four 1-simplices. In case of $(X, 8)$, we obtain four 0-simplices, six 1-simplices, four 2-simplices and one 3-simplex. The distinctive feature is that if we take 4-adjacency for X, then the maximum dimension of any simplex is $1 \lneq 2$ (the dimension of the grid), if we adopt 8-adjacency into X, then we obtain simplices with dimension $3 \gneq 2$. In view of this observation, the digital m-simplices and the digital simplicial complex of Definitions 1 and 2 are quite different from those in algebraic topology [22].*

In view of Remark 3.1, we can see some big differences between ordinary simplical complexes and their digital versions as follows:

Remark 3.2. *(1) Let us consider the case $(X, 8)$ of Remark 3.1. Then, owing to the properties (a) and (b) of Definition 1, we have "one 3-simplex" such as $< x_0, x_1, x_2, x_3 >$ in a 2D digital image, which is quite distinctive feature compared to the ordinary simplices in [22].*

(2) Under the property (2) of Definition 2, consider the digital image $(Y, 8)$ in Figure 1(a), where $Y := \{c_0 := (0,0), c_1 := (0,1), c_2 := (1,0)\}$. Let us now consider the following 1-simplex and 0-simplex: $P_1 := < c_1, c_2 >$ and $Q_1 := < c_1 >$. Then it is clear that $P_1 \cap Q_1$ is neither empty nor a common face of P_1 and Q_1, which does not support the criterion for a finite digital simplicial complex. More precisely, the intersection of P_1 and Q_1 is exactly the singleton $\{c_1\}$ which is not a face of both $P_1 := < c_1, c_2 >$ and $Q_1 := < c_1 >$.

In view of Remark 3.2, we have the following:

Proposition 3.3. *In view of Remark 3.2, the statement of Definition 2(2) should be written as follows:*
Let (X, k) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some nonnegative integer d. If the following hold, then (X, k) is called a finite digital simplicial complex:

(1) If P belongs to X, then every face of P also belongs to X.
(2) If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q (in case either of P and Q is not a singleton as a subset of the other).

The part(2) of Proposition 3.3 implies that not every mutually k-adjacent set of $m + 1$ points has to be included as an m-simplex. And this means that given a digital image (X, k) it is possible to derive many distinct simplicial complexes.

Let us now move into the notion of a digital wedge sum which was firstly introduced in [7] and has been used to study digital wedge sums in the field of digital homotopy and digital homology, as follows:

Definition 3. [7] *(see also [13]) For pointed digital images $((X, x_0), k_0)$ in $\mathbf{Z}^{n_0}$ and $((Y, y_0), k_1)$ in $\mathbf{Z}^{n_1}$, the wedge sum of (X, k_0) and (Y, k_1), written $(X \vee Y, (x_0, y_0))$, is the digital image in $\mathbf{Z}^n$*

$$\{(x, y) \in X \times Y \mid \quad x = x_0 \quad or \quad y = y_0\} \tag{3.1}$$

with the following compatible $k(m, n)$(or k)-adjacency relative to both (X, k_0) and (Y, k_1), and the only one point (x_0, y_0) in common such that

(W1) the $k(m, n)$ (or k)-adjacency is determined by the numbers m and n with $n = \max\{n_0, n_1\}$, $m = \max\{m_0, m_1\}$ satisfying $(W1-1)$ below, where the numbers m_i are taken from the k_i(or $k(m_i, n_i)$)-adjacency relations of the given digital images $((X, x_0), k_0)$ and $((Y, y_0), k_1)$, $i \in \{0, 1\}$.

(W 1-1) In view of (3.1), induced from the projection maps, we can consider the natural projection maps

$$W_X : (X \vee Y, (x_0, y_0)) \to (X, x_0) \quad and \quad W_Y : (X \vee Y, (x_0, y_0)) \to (Y, y_0).$$

In relation to the establishment of a compatible k-adjacency of the digital wedge sum $(X \vee Y, (x_0, y_0))$, the following restriction maps of W_X and W_Y on $(X \times \{y_0\}, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ and $(\{x_0\} \times Y, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ satisfy the following properties, respectively:

$$\begin{cases} (1)\, W_X|_{X \times \{y_0\}} : (X \times \{y_0\}, k) \to (X, k_0) \text{ is a } (k, k_0)\text{-isomorphism; and} \\ (2)\, W_Y|_{\{x_0\} \times Y} : (\{x_0\} \times Y, k) \to (Y, k_1) \text{ is a } (k, k_1)\text{-isomorphism.} \end{cases}$$

(W2) Any two distinct elements $x(\neq x_0) \in X \subset X \vee Y$ and $y(\neq y_0) \in Y \subset X \vee Y$ are not $k(m, n)$ (or k)-adjacent to each other.

When establishing a digital wedge sum, we strongly need to examine if there exists a compatible k-adjacency of a digital wedge sum. Hereafter, for convenience we may use a digital wedge sum $(X \vee Y, (x_0, y_0))$ without any notation of the common point, *i.e.* $X \vee Y$ instead of $(X \vee Y, (x_0, y_0))$ if there is no danger of ambiguity.

Example 3.4. *Given the set $X \vee Y$ in Figure 1(c) we have only the compatible 4-adjacency such as $(X \vee Y, 4)$. According to Definition 3, we cannot have $(X \vee Y, 8)$.*
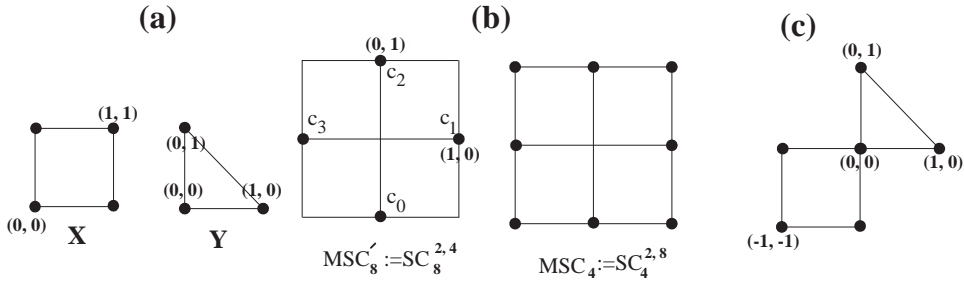

FIGURE 1. (a) A simple closed 4-curve X and another digital image $(Y, k), k \in \{8, 4\}$; (b) simple closed k-curve, $k \in \{4, 8\}$; (c) a digital wedge sum $(X \vee Y, 4)$.

The digital homology group of digital image (X, k) in [1, 3] is certainly motivated by the homology in [22]. Some distinctive differences are that in digital homology theory we use digital simplicial complexes of X instead of ordinary simplicial complexes in [22]. More precisely, the dimension of a digital simplicial complex X is the largest integer m such that X has an m-simplex [1, 3]. It turns out that a digital continuous map induces a simplicial map [1]. We say that $C_q^k(X)$ is a free abelian group [1] with a base of all digital (k, q)-complex in X. Let $(X, k) \subset \mathbf{Z}^n$ be a digital simplicial complex of dimension m. Then, for all $q \gneq m$, $C_q^k(X)$ is a trivial group [1]. The homomorphism $\partial_q : C_q^k(X) \to C_{q-1}^k(X)$ defined with the same method in [22]. Finally, in [1, 3]
(1) $Z_q^k(X) := Ker\partial_q$ is called the group of digital simplicial q-cycles.
(2) $B_q^k(X) := Im\partial_{q+1}$ is called the group of digital simplicial q-boundaries. Based on this approach, the papers [1, 3] have the following q-th digital simplicial group.

$$H_q^k(X) := Z_q^k(X)/B_q^k(X) \qquad (3.2)$$

In view of Definition 2 and the property (3.2), we have $H_q^2([0, l]_\mathbf{z}) = \mathbf{Z}$, $q = 0$ and it is trivial if $q \neq 0$ [1, 3]. Besides, for a singleton $\{x_0\}$ it is obvious that $H_q^k(\{x_0\})$ is isomorphic to $\mathbf{Z}$ if $q = 0$, and it is trivial if $q \neq 0$ [3]. In addition, $H_q^k(SC_k^{n,l}) = \mathbf{Z}$ if $q \in \{0, 1\}$ [1, 3], and it is trivial if $q \notin \{0, 1\}$.

4. Some properties of digital homology groups of some digital wedge sums

To study some properties of digital homology groups, we need to recall the digital continuity of a map $f : (X, k_0) \to (Y, k_1)$ by saying that f maps every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1) [21]. Motivated by this approach, since the digital k-neighborhood of (2.4) has been often used in digital topology, the digital continuity of a map between digital images was represented in terms of the digital k-neighborhood of (2.4), as follows:

Proposition 4.1. [7, 10] *Let (X, k_0) and (Y, k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : (X, k_0) \to (Y, k_1)$ is digitally (k_0, k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

Hereafter, we will use the term "(k_0, k_1)-continuous" for short instead of "digitally (k_0, k_1)-continuous". In Proposition 4.1 in case $n_0 = n_1$ and $k_0 = k_1 := k$, the map f is called a "k-continuous" map instead of a "(k, k)-continuous" map. A k-path $(x_i)_{i \in [0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ is considered to be the image of an injective $(2, k)$-continuous map $f : [0, l]_{\mathbf{Z}} \to \mathbf{Z}^n$ given by $f(i) = x_i$.

Since an nD digital image (X, k) is considered to be a set $X \subset \mathbf{Z}^n$ with one of the k-adjacency relations of (2.2) (or a digital k-graph in [8, 11]), in relation to the classification of nD digital images, we use the term a (k_0, k_1)-isomorphism as in [8] (see also [16]) rather than a (k_0, k_1)-homeomorphism as in [2].

Definition 4. [2] *(see also [8, 16]) Consider two digital images (X, k_0) and (Y, k_1) in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Then a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous.*

In Definition 4, in case $n_0 = n_1$ and $k_0 = k_1 := k$, we call it a k-*isomorphism* [8]. Furthermore, we denote by $X \approx_k Y$ a k-isomorphism from (X, k) to (Y, k) [8, 16].

Lemma 4.2. [3] *For the digital image $(X, 4)$ in Figure 1, we obtain*

$$H_q^4(X) = \begin{cases} \mathbf{Z}, & q = 0; \\ \mathbf{Z}, & q = 1; \\ 0, & q \notin \{0, 1\}. \end{cases}$$

The paper [6] studied a digital homology group of a digital wedge sum in $\mathbf{Z}^2$ with coefficients as a commutative ring $\mathbf{Z}_2$. However, we can studied digital homology groups of any digital wedge sums in $\mathbf{Z}^n$ as follows:

Theorem 4.3. *For a digital wedge sum $(X \vee Y, 4)$ in $\mathbf{Z}^2$ from two digital images $(X, 4)$ on $\mathbf{Z}^2$ and $(Y, 4)$ on $\mathbf{Z}^2$ (see Figure 1(c)). Then we obtain*

$$H_q^4(X \vee Y) = \begin{cases} \mathbf{Z}, & q \in \{0, 1\} \\ 0, & q \notin \{0, 1\}. \end{cases}$$

Proof: It is obvious that the digital wedge sum $(X \vee Y, 4)$ has the common point of $X \vee Y$, as a 0-dimensional digital complex such as $(0, 0) := c_0$. Besides, all free abelian groups

$C_q^4(X \vee Y)$ are isomorphic to $C_q^4(X) \oplus C_q^4(Y), q \in \mathbf{N}$.   (4.2)

Furthermore, if $z_{X \vee Y} \in Z_q^4(X \vee Y)$, depending on the given digital images $(X, 4)$ and $(Y, 4)$, then we have

either $z_{X \vee Y} = z_X, z_Y$ or $z_X + z_Y$ (4.3)

because of the property (W1) and (W2) of Definition 3, where $z_X \in Z_q^4(X)$ and $z_Y \in Z_q^4(Y)$. Hence we see that $Z_q^4(X \vee Y)$ is isomorphic to $Z_q^4(X) \oplus Z_q^4(Y)$ and further, $B_q^4(X \vee Y)$ are also isomorphic to $B_q^4(X) \oplus B_q^4(Y)$ for all $q \in \mathbf{N}$. As a result, due to the properties (4.2) and (4.3), we obtain that $H_q^4(X \vee Y)$ is isomorphic to $H_q^4(X) \oplus H_q^4(Y)$. Thus we obtain

$$H_q^4(X \vee Y) = \begin{cases} \mathbf{Z}, & q \in \{0, 1\} \\ 0, & q \notin \{0, 1\}. \end{cases}$$

□

5. Relative digital homology group of a digital wedge sum

Based on the pointed digital homotopy in [17], the following notion of a k-homotopy relative to a subset $A \subset X$ used to study a k-homotopic thinning and a strong k-deformation retract of a digital image (X, k) in $\mathbf{Z}^n$ [7, 10].

Definition 5. [7] *(see also [10]) Let $((X, A), k_0)$ and (Y, k_1) be a digital image pair and a digital image, respectively. Let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function*

$F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that
- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is $(2, k_1)$-continuous;
- for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is (k_0, k_1)-continuous.
Then we say that F is a (k_0, k_1)-homotopy between f and g [2].
- Furthermore, for all $t \in [0, m]_{\mathbf{Z}}$, $F_t(x) = f(x) = g(x)$ for all $x \in A$.
Then we call F a (k_0, k_1)-homotopy relative to A between f and g, and we say that f and g are (k_0, k_1)-homotopic relative to A in Y, $f \simeq_{(k_0, k_1)relA} g$ in symbols.

If, for some $x_0 \in X$, 1_X is k-homotopic to the constant map in the space $\{x_0\}$ relative to $\{x_0\}$, then we say that (X, x_0) is pointed k-contractible [2].

Definition 6. [10] (see also [14]) For a digital space pair $((X, A), k)$, we say that A is a strong k-deformation retract of X if there is a digital k-continuous map r from X onto A such that $i \circ r \simeq_{k \cdot rel.A} 1_X$ and $r \circ i = 1_A$.

Using the digital homology groups of digital wedge sums in Section 4, let us now correct some errors in the paper [6]. Motivated by the relative homology [22], the recent paper [6] introduced the relative homology for digital images, and studied its properties. More precisely, let (A, k) be a digital subcomplex of the digital simplicial complex (X, k). Then the chain group $C_k^q(A)$ is a subgroup of the chain group $C_q^k(X)$ (see Section 2). The quotient group $C_q^k(X, A) = C_q^k(X)/C_q^k(A)$ is called the group of relative chains of X modulo A. The boundary operator $\partial_q : C_q^k(A) \to C_{q-1}^k(A)$ is the restriction of the boundary operator on $C_q^k(X), q \in \mathbf{N}$. It induces a homomorphism $C_q^k(X, A) \to C_{q-1}^k(X, A)$ of the relative chain groups and this is also denoted by ∂_q.

Definition 7. [22, 6] Let (A, k) be a digital subcomplex of the digital simplicial complex (X, k).
- $Z_q^k(X, A) = Ker\partial_q$ is called the group of digital relative simplicial q-cycles.
- $B_q^k(X, A) = Im\partial_{q+1}$ is called the group of digital relative simplicial q-boundaries.
- $H_q^k(X, A) = Z_q^k(X, A)/B_q^k(X, A)$ is called the qth digital relative simplicial homology group.

Based on this approach, the resent paper [6] asserted the following:

Theorem 5.1. [6] *(see Proposition 3.8 of [6]) If $A \subset X$ is a deformation k-retract, then $H_q^k(X, A) = 0$ for any $q \geq 0$.*

Unfortunately, this assertion is incorrect. Indeed, under the hypothesis, the proof of this theorem in [6] is based on the following assertion of (5.1) (for more detail, see [6])

$$i_* : H_q^k(A) \to H_q^k(X) \text{ is an isomorphism} \qquad (5.1)$$

for $q \geq 0$.

However, we shall prove that this property (5.1) is incorrect (see Proposition 5.2). Due to this wrong approach, Theorem 5.1 cannot be valid. Let us now prove the invalidity of Theorem 5.1 by proposing a counterexample in $\mathbf{Z}^2$ and $\mathbf{Z}^3$.
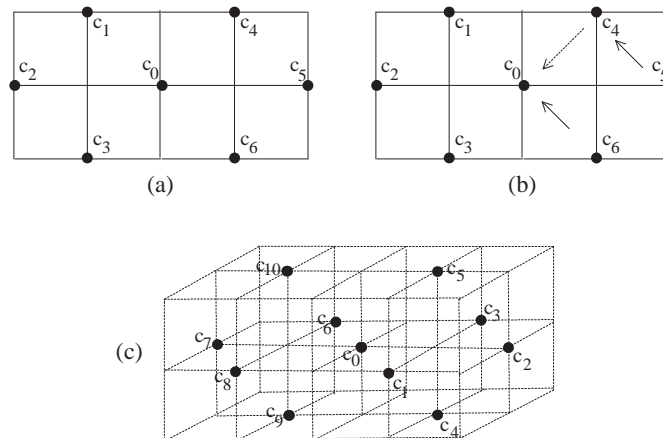

FIGURE 2. (a) $(SC_8^{2,4} \vee SC_8^{2,4}, 8)$, $A := \{c_0, c_1, c_2, c_3\} \subset SC_8^{2,4} \vee SC_8^{2,4}$; (b) An 8-homotopy relative to $A := SC_8^{2,4}$ between $1_{SC_8^{2,4} \vee SC_8^{2,4}}$ and $1_{SC_8^{2,4}}$; (c) $(MSS_{18}' \vee MSS_{18}', 18)$

The paper [9] developed the minimal simple closed 18-surface, denoted by MSS_{18}', in $\mathbf{Z}^3$ and it turns out that MSS_{18}' is 18-contractible [9].

Proposition 5.2. *(1) Although $SC_8^{2,4}$ is a strong 8-deformation retract of $(SC_8^{2,4} \vee SC_8^{2,4}, 8)$, $H_q^8(SC_8^{2,4} \vee SC_8^{2,4}, SC_8^{2,4})$ is not trivial.*
(2) Although MSS_{18}' is a strong 18-deformation retract of $(MSS_{18}' \vee$

$MSS'_{18}, 18)$, $H_q^{18}(MSS'_{18} \vee MSS'_{18}, MSS'_{18})$ is not trivial.

Proof: (1) Consider an 8-homotopy between $1_{SC_8^{2,4} \vee SC_8^{2,4}}$ and $1_{SC_8^{2,4}}$ relative to $SC_8^{2,4} := A = \{c_0, c_1, c_2, c_3\}$ as a subset of $(SC_8^{2,4} \vee SC_8^{2,4}, 8)$ (see Figure 2(a)). More precisely, according to the 8-contractibility of $SC_8^{2,4}$, there exist $[0, 2]_{\mathbf{Z}}$ and a function

$$F : SC_8^{2,4} \vee SC_8^{2,4} \times [0, 2]_{\mathbf{Z}} \to SC_8^{2,4}$$

such that
• for all $x \in SC_8^{2,4} \vee SC_8^{2,4}, F(x, 0) = 1_{SC_8^{2,4} \vee SC_8^{2,4}}$ and $F(x, 2) = 1_{SC_8^{2,4}}$;
• for all $x \in SC_8^{2,4} \vee SC_8^{2,4}$, the induced function $F_x : [0, 2]_{\mathbf{Z}} \to SC_8^{2,4} := A$ given by $F_x(t) = F(x, t)$ for all $t \in [0, 2]_{\mathbf{Z}}$ is $(2, 8)$-continuous;
• for all $t \in [0, 2]_{\mathbf{Z}}$, the induced function $F_t : SC_8^{2,4} \vee SC_8^{2,4} \to SC_8^{2,4} := A$ given by $F_t(x) = F(x, t)$ for all $x \in SC_8^{2,4} \vee SC_8^{2,4}$ is 8-continuous.
Furthermore,
• for all $t \in [0, 2]_{\mathbf{Z}}, F_t(x) = 1_{SC_8^{2,4} \vee SC_8^{2,4}}(x) = 1_{SC_8^{2,4}}(x)$ for all $x \in SC_8^{2,4}$.
However, we see that $H_1^8(SC_8^{2,4} \vee SC_8^{2,4}, SC_8^{2,4}) \simeq \mathbf{Z}$, which is not trivial.

(2) Consider an 18-homotopy between $1_{MSS'_{18} \vee MSS'_{18}}$ and $1_{MSS'_{18}}$ relative to $MSS'_{18} := B = \{c_i \mid i \in [0, 5]_{\mathbf{Z}}\}$ as a subset of $(MSS'_{18} \vee MSS'_{18}, 18)$ (see Figure 2(c)). More precisely, according to the 18-contractibility of MSS'_{18} [9], there exist $[0, 2]_{\mathbf{Z}}$ and a function

$$H : MSS'_{18} \vee MSS'_{18} \times [0, 2]_{\mathbf{Z}} \to MSS'_{18}$$

such that
• for all $x \in MSS'_{18} \vee MSS'_{18}, H(x, 0) = 1_{MSS'_{18} \vee MSS'_{18}}$ and $H(x, 2) = 1_{MSS'_{18}}$;
• for all $x \in MSS'_{18} \vee MSS'_{18}$, the induced function $H_x : [0, 2]_{\mathbf{Z}} \to MSS'_{18} := B$ given by $H_x(t) = H(x, t)$ for all $t \in [0, 2]_{\mathbf{Z}}$ is $(2, 18)$-continuous;
• for all $t \in [0, 2]_{\mathbf{Z}}$, the induced function $H_t : MSS'_{18} \vee MSS'_{18} \to MSS'_{18} := B$ given by $H_t(x) = H(x, t)$ for all $x \in MSS'_{18} \vee MSS'_{18}$ is 18-continuous.
Furthermore,
• for all $t \in [0, 2]_{\mathbf{Z}}, H_t(x) = 1_{MSS'_{18} \vee MSS'_{18}}(x) = 1_{MSS'_{18}}(x)$ for all $x \in MSS'_{18}$.
However, we see that $H_2^{18}(MSS'_{18} \vee MSS'_{18}, MSS'_{18}) \simeq \mathbf{Z}$, which is not trivial.
□

Remark 5.3. *In views of Proposition 5.2, the assertion of Proposition 3.8 of* [6] *is invalid.*

6. Further remarks

Although in algebraic topology we have used the notion of a barycentric subdivision for studying simplical complexes, in digital topology we cannot do this work (see Definitions 1 and 2). Besides, although many works have tried to study fixed point theory of digital images in terms of Lefschetz number induced by digital homology, such a kind of approach cannot be suitable in digital topology because digital homology is not invariant under the digital homotopy equivalence (for more details, see [14, 15]).

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