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DERIVATIVES FOR THE LINEARITY OF TERNARY NUMBER VALUED FUNCTIONS

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Abstract. The aim of this paper is to investigate the differentials of the hypercomplex valued functions in Clifford analysis. Like as the differentials defined by the naïve approach in one complex variable analysis, we define the differentials of functions with values in ternary number functions by same ways. And we survey the properties of each differential with respect to a non-commutativity of the skew field.

1. Introduction

For a hyperholomorphic function, Naser [8] has researched several definitions and theorems of the qunaternionic functions in 1971. Naser [8] has provided the corresponding Cauchy-Riemann equations and Cauchy theorem by using a differential operator $D^* = \dfrac{\partial}{\partial \overline{z_1}} + e_2 \dfrac{\partial}{\partial \overline{z_2}}$.

In 2011, Luna-Elizarrarás and Shapiro [7] have researched the differentials and the derivatives of the each function valued complex, quaternionic and the other hypercomplex numbers in some details. Luna-Elizarrarás and Shapiro [7] have surveyed the definitions of the differentials for one complex variable case in several approaches. And Luna-Elizarrarás and Shapiro [7] have investigated several properties of the differentials for the hypercomplex valued functions defined by a naïve approach.

Jung and Shon [1] have studied properties of hyperholomorphic functions in a dual ternary number field and Jung et al. [2] have provided structures for hyperholomorphic functions in a dual quaternion field.

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Kim et al. [4, 5] have investigated the regularity and properties of the functions valued ternary numbers and reduced quaternions in Clifford analysis. Lim and Shon [6] have shown the hyperholomorphy of split quaternionic functions and given the definitions of the corresponding Cauchy-Riemann equation with several examples.

2. Preliminaries

A ternary number field $\mathcal{A}$ identified with $\mathbb{R}^3$ is a non-commutative division ring (skew field) generated by $e_0,\ e_1$ and e_2. The basis of the field $\mathcal{A}$ satisfy the following:

$$\begin{aligned} &e_0 = id.,\ \ e_1^2 = e_2^2 = -1,\ \ e_1e_2 = -e_2e_1 \\ &\text{and}\ \ \overline{e_r} = -e_r\ \ (r = 1, 2). \end{aligned} \tag{1}$$

Then a ternary number z is defined by

$$z = \sum_{k=0}^{2} e_k x_k = x_0 + e_1x_1 + e_2x_2.$$

And z is denoted by (1) and using Fueter variables as

$$\begin{aligned} z =& x_0 + e_1x_1 + e_2x_2 = \left(\frac{1}{2}x_0 + e_1x_1\right) + \left(\frac{1}{2}x_0 + e_2x_2\right) \\ =& e_1\left(-\frac{1}{2}e_1x_0 + x_1\right) + e_2\left(-\frac{1}{2}e_2x_0 + x_2\right) \\ :=& e_1z_1 + e_2z_2, \end{aligned}$$

where $z_1, z_2 \in \mathbb{C}$. The conjugate z^* of z and the absolute value $|z|$ are defined by

$$\begin{aligned} z^* =& x_0 - e_1x_1 - e_2x_2 = \left(\frac{1}{2}x_0 - e_1x_1\right) + \left(\frac{1}{2}x_0 - e_2x_2\right) \\ =& - e_1\left(\frac{1}{2}e_1x_0 + x_1\right) - e_2\left(\frac{1}{2}e_2x_0 + x_2\right) \\ =& \overline{e_1z_1} + \overline{e_2z_2} \end{aligned}$$

and

$$|z|^2 = zz^* = z^*z = \sum_{k=0}^{2} {x_k}^2 = x_0^2 + x_1^2 + x_2^2.$$

The non-zero ternary number z has the unique inverse

$$z^{-1} = \frac{z^*}{|z|^2} \quad (|z| \neq 0).$$

Let Ω be a bounded open set in $\mathcal{A}$. A function $f : \Omega \to \mathcal{A}$ is defined by

$$z \mapsto f(z) = \sum_{k=0}^{2} e_k u_k(x_0, x_1, x_2),$$

where $u_k(x_0, x_1, x_2)$ are real valued functions for $j = 0, 1, 2$. We denote $u_k(x_0, x_1, x_2)$ as u_k for convenience.

Let Ω be a bounded open set and $z \in \Omega$. A function $f : \Omega \to \mathcal{A}$ is defined by $f(z) = u_0 + e_1 u_1 + e_2 u_2$. We put any ternary number $h \neq 0$ such that $z + h \in \Omega$ as the increment of an argument at the point z. And let

$$\begin{aligned}
\Delta f_z :=& f(z+h) - f(z)\\
=& \{u_0(z+h) + e_1 u_1(z+h) + e_2 u_2(z+h)\} - \{u_0(z) + e_1 u_1(z) + e_2 u_2(z)\}\\
=& \{u_0(z+h) - u_0(z)\} + e_1\{u_1(z+h) - u_1(z)\} + e_2\{u_2(z+h) - u_2(z)\}\\
:=& \Delta u_0 + e_1 \Delta u_1 + e_2 \Delta u_2.
\end{aligned}$$

3. Differentials for ternary number valued functions

By the non-commutativity of the field $\mathcal{A}$, we have two functions for the ternary number h satisfying the following:

$$h \mapsto h^{-1} \Delta f_z, \tag{2}$$

$$h \mapsto \Delta f_z h^{-1}. \tag{3}$$

So, we consider two different aspects for each functions of h to define the differentials of ternary number-valued functions: the left side is $h^{-1}\Delta f_z$ and the right side is $\Delta f_z h^{-1}$.

Remark 3.1. *Let f, g be functions defined in the field $\mathcal{A}$. We denote that $f(z) = o(g(z))$ as $z \to a$ if*

$$\lim_{z \to a} f(z) g(z)^{-1} = 0 \quad \text{and} \quad \lim_{z \to a} g(z)^{-1} f(z) = 0.$$

And $o(g(z))$ is called to the Landau notation.

Then we define the derivablity and the differentiablity of the functions.

Definition 3.2. *A function f is left (resp. right) side derivable at a point $z \in \mathcal{A}$ if (2)(resp. (3)) has a limit as $h \to 0$. And the limit is called left (resp. right) side derivative of f at z. The left (resp. right) derivative of f is denoted as*

$$'f(z) := \lim_{h\to 0} h^{-1}\Delta f_z \ \left(\text{resp. } f'(z) := \lim_{h\to 0} \Delta f_z h^{-1}\right).$$

And if there exists a constant $B_z \in \mathcal{A}$ such that

$$\Delta f_z = hB_z + o(h) \quad as \quad h \to 0$$

$$(\Delta f_z = B_z h + o(h) \quad as \quad h \to 0),$$

then we say that the function f is left (resp. right) side differentiable at z.

Lemma 3.3. *Let Ω be an open set in $\mathcal{A}$ and a function f be defined in Ω. Then*

$$\Delta f_z h^{-1} = \left(h^{*-1}\Delta f_z^*\right)^*, \tag{4}$$

$$h^{-1}\Delta f_z = \left(\Delta f_z^* h^{*-1}\right)^* \tag{5}$$

for $h \in \mathcal{A}$.

Proof. Since h is an arbitrary ternary number, h is denoted as $h = a + e_1 b + e_2 c$ for $a, b, c \in \mathbb{R}$. Then the inverses h^{-1} and h^{*-1} are

$$h^{-1} = \frac{h^*}{|h|^2} = \frac{a - e_1 b - e_2 c}{a^2+b^2+c^2} \quad \text{and} \quad h^{*-1} = \frac{h}{|h|^2} = \frac{a + e_1 b + e_2 c}{a^2+b^2+c^2}.$$

By the multiplication of the ternary number,

$$\begin{aligned}\Delta f_z h^{-1} =& (\Delta u_0 + e_1\Delta u_1 + e_2\Delta u_2)\left(\frac{a - e_1 b - e_2 c}{a^2+b^2+c^2}\right)\\ =& \frac{1}{a^2+b^2+c^2}\{(\Delta u_0 a + \Delta u_1 b + \Delta u_2 c) + e_1(-\Delta u_0 b + \Delta u_1 a)\\ &+ e_2(-\Delta u_0 c + \Delta u_2 a) + e_1 e_2(-\Delta u_1 c + \Delta u_2 b)\}.\end{aligned}$$

By the similar computation,

$$\begin{aligned}h^{*-1}\Delta f_z^* =& \left(\frac{a + e_1 b + e_2 c}{a^2+b^2+c^2}\right)(\Delta u_0 - e_1\Delta u_1 - e_2\Delta u_2)\\ =& \frac{1}{a^2+b^2+c^2}\{(a\Delta u_0 + b\Delta u_1 + c\Delta u_2) + e_1(b\Delta u_0 - a\Delta u_1)\\ &+ e_2(c\Delta u_0 - a\Delta u_2) + e_1 e_2(c\Delta u_1 - b\Delta u_2)\}.\end{aligned}$$

Thus, we obtain the result (4) as follows:

$$\{h^{*-1}\Delta f_z^*\}^* = \Delta f_z h^{-1}.$$

And (5) is also proved by the similar ways. □

We check the following theorems and examples by Definition 3.2 and Lemma 3.3.

Proposition 3.4. *Let Ω be a bounded open set in $\mathcal{A}$. The function f is left side derivable at $z \in \Omega$ if and only if f^* is right side derivable at z.*

Proof. Let the function f be left side derivable at $z \in \Omega$. Then $h^{-1}\Delta f_z$ has a limit as $h \to 0$. It is equivalent to that $\left(h^{-1}\Delta f_z\right)^*$ has a limit as $h \to 0$. By Lemma 3.3,

$$\left(h^{-1}\Delta f_z\right)^* = \Delta f_z^* h^{*-1}$$

has a limit as $h \to 0$.

Since $h \to 0$ is equivalent to $h^* \to 0$, we have

$$\exists \lim_{h^* \to 0} \Delta f_z^* h^{*-1}.$$

Thus, f^* is right side derivable at $z \in \Omega$. □

Theorem 3.5. *Let Ω be a bounded open set in $\mathcal{A}$ and $z \in \Omega$. The ternary number valued function f is left side differentiable at z if and only if f is left side derivable at z. In addition, if f satisfies this property then $'f(z) = B_z$.*

Proof. Suppose that the function f is left side differentiable at $z \in \Omega$. Then there exists a constant $B_z \in \mathcal{A}$ such that

$$\Delta f_z = hB_z + o(h) \quad \text{as} \quad h \to 0. \tag{6}$$

We know (6) is equivalent to the following:

$$h^{-1}\Delta f_z = B_z + h^{-1}o(h).$$

Now, let $h \to 0$. Then,

$$\lim_{h\to 0} h^{-1}\Delta f_z = B_z + \lim_{h\to 0} h^{-1}o(h).$$

By the definition of $o(h)$, we have

$$'f(z) = \lim_{h\to 0} h^{-1}\Delta f_z = B_z.$$

Thus, f is left side derivable at z.

Since these processes are equivalent to each other, we have

$$'f(z) = B_z.$$

The proof is done. □

We consider the linearity of ternary number-valued functions to clarify the differentiability and the derivability.

Theorem 3.6. *Let $h \in \mathcal{A}$ and a function $\rho : \mathcal{A} \to \mathcal{A}$ be defined by*

$$f : h \mapsto hB_z,$$

where B_z is the ternary number constant. Then ρ is left-linear over $\mathcal{A}$ but not right-linear.

Proof. Let $h_1, h_2 \in \mathcal{A}$. And let a, b be any ternary numbers. Then we know the following result by the rule of the multiplication:

$$\begin{aligned}\rho(ah_1 + bh_2) =&(ah_1 + bh_2)B_z = ah_1B_z + bh_2B_z \\ =&a\rho(h_1) + b\rho(h_2).\end{aligned}$$

But, we have

$$\begin{aligned}\rho(h_1a + h_2b) =&(h_1a + h_2b)B_z = h_1aB_z + h_2bB_z \\ \neq&\rho(h_1)a + \rho(h_2)b\end{aligned}$$

by the non-commutativity of $\mathcal{A}$. So we obtain the result. □

From Theorem 3.6, we know that the derivability and the differentiability of ternary number valued functions depend on the each linearity (left or right) of the increment Δf_z.

Proposition 3.7. *In the field $\mathcal{A}$, the function $f : z \mapsto az + b$ is right side differentiable for $a, b \in \mathcal{A}$. And $g : z \mapsto za + b$ is not right side differentiable but left side differentiable.*

Proof. Let the function f be defined by $f(z) = az + b$ for $a, b \in \mathcal{A}$. Then we have

$$\begin{aligned}f'(z) &= \lim_{h\to 0} \Delta f_z h^{-1} = \lim_{h\to 0} \{f(z+h) - f(z)\}\, h^{-1} \\ &= \lim_{h\to 0} \{a(z+h) + b - (az+b)\}\, h^{-1} \\ &= \lim_{h\to 0} (ah)h^{-1} = a,\end{aligned}$$

by the definition of the differentiability. Thus, f is right side differentiable.

By similar way, the linearity shows the function g is not right side differentiable but left side differentiable. □

Now, we have several examples for the derivability and the differentiability of functions.

Example 3.8. *The function f defined by $f(z) = z^2$ is not differentiable at $z \in \mathcal{A}\backslash\mathbb{R}$. So, f is not derivable at z.*

In fact, by the direct computation,

$$\begin{aligned}\Delta f_z =& f(z+h) - f(z) = (z^2 + zh + hz + h^2) - z^2 \\ =& zh + hz + h^2.\end{aligned}$$

For $z \notin \mathbb{R}$, we know the terms of $zh + hz$ are neither left linear nor right linear.

Thus, $\Delta f_z h^{-1}$ and $h^{-1}\Delta f_z$ have no limits $f'(z)$ and $'f(z)$. We know that f is neither differentiable nor derivable at non-real z.

Example 3.9. *The function f, defined by $f(z) = z^3$ is not differentiable at $z \in \mathcal{A}\backslash\mathbb{R}$.*

In fact, in similar ways with the above Example 3.8,

$$\begin{aligned}\Delta f_z =& f(z+h) - f(z) \\ =& (z^3 + zhz + hz^2 + h^2 z + z^2 h + zh^2 + hzh + h^3) - z^3 \\ =& zhz + hz^2 + h^2 z + z^2 h + zh^2 + hzh.\end{aligned}$$

The term of hzh is not effected by the linearity. But, the other terms are neither left linear nor right linear.

We obtain the result that f is not derivable at $z \in \mathcal{A}\backslash\mathbb{R}$.

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