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Information measures for generalized hesitant fuzzy information

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Abstract

In this paper, we present the entropy and similarity measure for generalized hesitant fuzzy information, and discuss their desirable properties. Some measure formulas are developed, and the relationships among them are investigated. We show that the similarity measure and entropy for generalized hesitant fuzzy information can be transformed by each other based on their axiomatic definitions. Furthermore, an approach of multiple attribute decision making problems where attribute weights are unknown and the evaluation values of attributes for each alternative are given in the form of GHFEs is investigated.

Key Words: Generalized hesitant fuzzy sets, Entropy, Similarity measure

1. Introduction

During the evaluating process to get a more reasonable decision result, a decision organization, which contains a lot of experts, is authorized to provide the preference information about a set of alternatives. In practice, they may have several possible membership degrees take the forms of both crisp values and interval values in $[0, 1]$ when discussing the membership degree of an alternative with respect to a criterion. To deal with such cases, Qjan et al. [1] introduced the concept of generalized hesitant fuzzy sets (GHFSs) considered as a generalization of both IFSs and HFSs. GHFS can reflect the human's hesitance more objectively than other extensions of fuzzy set (IFS, IVIFS and HFS). Since hesitation among several possible membership degrees with uncertainties in evaluating process is a very common problem in practical decision making, it is necessary to develop some entropy and similarity measures for GHFSs. In this paper, we present axiomatic definitions of entropy and similarity measure for GHFEs, and show that the entropy and the similarity measure for GHFEs can be transformed by each other based on their axiomatic definitions.

2. Entropy and similarity measures for generalized hesitant fuzzy elements

Definition 2.1. [1] Let $([0,1]\times[0,1])^* = \{(x,y)|x,y\in[0,1], x+y\le 1\}$. Given a fixed set X, the generalized hesitant fuzzy set (GHFS) on X is in terms of a function $\tilde{\alpha}$ that when applied to X returns a subset of $([0,1]\times[0,1])^*$, which can be represented as the following mathematical symbol:

$$A = \{\langle x, \tilde{\alpha}(x)\rangle | x \in X\},$$

where $\tilde{\alpha}(x) = \{(\mu_{\tilde{\alpha}}(x), \nu_{\tilde{\alpha}}(x))\}$ is a set of some values in $([0,1]\times[0,1])^*$ (i.e, a set of some IFVs in $[0,1]$), denoting the possible membership degrees of the element $x \in X$

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to the set A. For convenience, Qjan et al. [1] called $\tilde{\alpha}(x)$ a generalized hesitant fuzzy element (GHFE) and the set of all GHFEs is denoted by GH.

In particular, if there is only one value in $\tilde{\alpha}(x)$, then the GHFS reduces to the IFS; if $\mu_{\tilde{\gamma}} + \nu_{\tilde{\gamma}} = 1$ for each $\tilde{\gamma} = (\mu_{\tilde{\gamma}}, \nu_{\tilde{\gamma}}) \in \tilde{\alpha}(x)$, then the GHFS reduces to the HFS; if $\tilde{\alpha}(x)$ contains only one value $\tilde{\gamma}$ and $\mu_{\tilde{\gamma}} + \nu_{\tilde{\gamma}} = 1$, then the GHFS reduces to the FS. Thus, it indicates that FSs, IFSs and HFSs are special types of GHFSs.

It is noted that the number of IFVs in different GHFEs may be different, let $l_{\tilde{\alpha}}$ be the number of IFVs in $\tilde{\alpha}$. By comparison method [3] of IFVs, we arrange the elements in $\tilde{\alpha}$ in increasing order, let $\tilde{\alpha}^{\sigma(i)} = (\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)})$ $(i = 1, 2, \ldots, l_{\tilde{\alpha}})$ be the ith smallest IFV in $\tilde{\alpha}$. To operate correctly, we assume that the GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$ should have the same length l when we compare them.

Now, we give the generalized hesitant fuzzy entropy and generalized hesitant fuzzy similarity measure defined as follows:

Definition 2.2. An entropy on GHFE $\tilde{\alpha}$ is a real-valued function $E : GH \to [0, 1]$, satisfying the following axiomatic requirements:

(1) $E(\tilde{\alpha}) = 0$ if and only if $\tilde{\alpha} = (0, 1)$ or $\tilde{\alpha} = (1, 0)$;

(2) $E(\tilde{\alpha}) = 1$ if and only if $\mu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}}^{\sigma(i)}$ for $i = 1, 2, \ldots, l_{\tilde{\alpha}}$;

(3) $E(\tilde{\alpha}) \le E(\tilde{\beta})$ if $\mu_{\tilde{\alpha}}^{\sigma(i)} \le \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \ge \nu_{\tilde{\beta}}^{\sigma(i)}$ for $\mu_{\tilde{\beta}}^{\sigma(i)} \le \nu_{\tilde{\beta}}^{\sigma(i)}$, or if $\mu_{\tilde{\alpha}}^{\sigma(i)} \ge \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \le \nu_{\tilde{\beta}}^{\sigma(i)}$ for $\mu_{\tilde{\beta}}^{\sigma(i)} \ge \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$;

(4) $E(\tilde{\alpha}) = E(\tilde{\alpha}^c)$.

Definition 2.3. For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, the similarity measure between $\tilde{\alpha}$ and $\tilde{\beta}$, denoted as $S(\tilde{\alpha}, \tilde{\beta})$, should satisfy the following properties:

(1) $S(\tilde{\alpha}, \tilde{\beta}) = 0$ if and only if $\tilde{\alpha} = (0, 1)$, $\tilde{\beta} = (1, 0)$ or $\tilde{\alpha} = (1, 0)$, $\tilde{\beta} = (0, 1)$;

(2) $S(\tilde{\alpha}, \tilde{\beta}) = 1$ if and only if $\tilde{\alpha} = \tilde{\beta}$, i.e. $\mu_{\tilde{\alpha}}^{\sigma(i)} = \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$;

(3) $S(\tilde{\alpha}, \tilde{\gamma}) \le S(\tilde{\alpha}, \tilde{\beta})$, $S(\tilde{\alpha}, \tilde{\gamma}) \le S(\tilde{\beta}, \tilde{\gamma})$, if $\mu_{\tilde{\alpha}}^{\sigma(i)} \le \mu_{\tilde{\beta}}^{\sigma(i)} \le \mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)} \ge \nu_{\tilde{\beta}}^{\sigma(i)} \ge \nu_{\tilde{\gamma}}^{\sigma(i)}$ or if $\mu_{\tilde{\alpha}}^{\sigma(i)} \ge \mu_{\tilde{\beta}}^{\sigma(i)} \ge \mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)} \le \nu_{\tilde{\beta}}^{\sigma(i)} \le \nu_{\tilde{\gamma}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$;

(4) $S(\tilde{\alpha}, \tilde{\beta}) = S(\tilde{\beta}, \tilde{\alpha})$.

3. Relationships between entropy and similarity measures

Theorem 3.1. Let $\tilde{\alpha}$ be a GHFE, then $S(\tilde{\alpha}, \tilde{\alpha}^c)$ is an entropy for $\tilde{\alpha}$.

Proof. (1) $S(\tilde{\alpha}, \tilde{\alpha}^c) = 0 \Leftrightarrow \tilde{\alpha} = (0, 1)$ and $\tilde{\alpha}^c = (1, 0)$ or $\tilde{\alpha} = (1, 0)$ and $\tilde{\alpha}^c = (0, 1)$;

(2) $S(\tilde{\alpha}, \tilde{\alpha}^c) = 1 \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} = \mu_{\tilde{\alpha}^c}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}^c}^{\sigma(i)}$, $i = 1, 2, \ldots, l \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$;

(3) Suppose that $\mu_{\tilde{\alpha}}^{\sigma(i)} \le \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \ge \nu_{\tilde{\beta}}^{\sigma(i)}$, for $\mu_{\tilde{\beta}}^{\sigma(i)} \le \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, then $\mu_{\tilde{\alpha}}^{\sigma(i)} \le \mu_{\tilde{\beta}}^{\sigma(i)} \le \nu_{\tilde{\beta}}^{\sigma(i)} \le \nu_{\tilde{\alpha}}^{\sigma(i)}$. Therefore, by the definition of similarity measure of GHFE, $S(\tilde{\alpha}, \tilde{\alpha}^c) \le S(\tilde{\beta}, \tilde{\alpha}^c) \le S(\tilde{\beta}, \tilde{\beta}^c)$. With the same reason, when $\mu_{\tilde{\alpha}}^{\sigma(i)} \ge \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \le \nu_{\tilde{\beta}}^{\sigma(i)}$, for $\mu_{\tilde{\beta}}^{\sigma(i)} \ge \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, we can prove $S(\tilde{\alpha}, \tilde{\alpha}^c) \le S(\tilde{\beta}, \tilde{\beta}^c)$.

(4) $S(\tilde{\alpha}, \tilde{\alpha}^c) = S(\tilde{\alpha}^c, \tilde{\alpha})$. □

Example 3.2. For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, two generalized hesitant fuzzy similarity measures can be constructed as:

$$S_1(\tilde{\alpha}, \tilde{\beta}) = 1 - \frac{1}{2l} \sum_{i=1}^{l} \left(|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}| + |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}| \right)$$

$$S_2(\tilde{\alpha}, \tilde{\beta}) = 1 - \sqrt{\frac{1}{2l} \sum_{i=1}^{l} \left((\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)})^2 + (\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)})^2 \right)}.$$

Then we can construct the following entropy formulas based on the similarity measures S_1 and S_2:

$$S_1(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \frac{1}{l} \sum_{i=1}^{l} |\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|$$

$$S_2(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \sqrt{\frac{1}{l} \sum_{i=1}^{l} (\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^2}$$

Theorem 3.3. For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, let $|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}| \le |\mu_{\tilde{\alpha}}^{\sigma(i+1)} - \mu_{\tilde{\beta}}^{\sigma(i+1)}|$, $|\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}| \ge |\nu_{\tilde{\alpha}}^{\sigma(i+1)} - \nu_{\tilde{\beta}}^{\sigma(i+1)}|$, $i = 1, 2, \ldots, l-1$, and we define a GHFE $f(\tilde{\alpha}, \tilde{\beta})$ as follows:

$$f(\tilde{\alpha}, \tilde{\beta}) = \left\{ \left(\frac{1 + |\mu_{\tilde{\alpha}}^{\sigma(1)} - \mu_{\tilde{\beta}}^{\sigma(1)}|}{2}, \frac{1 - |\nu_{\tilde{\alpha}}^{\sigma(1)} - \nu_{\tilde{\beta}}^{\sigma(1)}|}{2} \right), \left(\frac{1 + |\mu_{\tilde{\alpha}}^{\sigma(2)} - \mu_{\tilde{\beta}}^{\sigma(2)}|}{2}, \frac{1 - |\nu_{\tilde{\alpha}}^{\sigma(2)} - \nu_{\tilde{\beta}}^{\sigma(2)}|}{2} \right), \ldots, \left(\frac{1 + |\mu_{\tilde{\alpha}}^{\sigma(l)} - \mu_{\tilde{\beta}}^{\sigma(l)}|}{2}, \frac{1 - |\nu_{\tilde{\alpha}}^{\sigma(l)} - \nu_{\tilde{\beta}}^{\sigma(l)}|}{2} \right) \right\},$$

then $E(f(\tilde{\alpha},\tilde{\beta}))$ is the similarity measure of $\tilde{\alpha}$ and $\tilde{\beta}$.

Proof. (1) $E(f(\tilde{\alpha},\tilde{\beta})) = 0 \Leftrightarrow f(\tilde{\alpha},\tilde{\beta}) = (1,0)$ or $f(\tilde{\alpha},\tilde{\beta}) = (0,1) \Leftrightarrow \tilde{\alpha} = (0,1), \tilde{\beta} = (1,0)$ or $\tilde{\alpha} = (1,0)$, $\tilde{\beta} = (0,1)$;

(2) $E(f(\tilde{\alpha},\tilde{\beta})) = 1 \Leftrightarrow \frac{1+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|}{2} = \frac{1-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|}{2}$, $i = 1,2,\ldots,l \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} = \mu_{\tilde{\beta}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1,2,\ldots,l$;

(3) Since $\mu_{\tilde{\alpha}}^{\sigma(i)} \le \mu_{\tilde{\beta}}^{\sigma(i)} \le \mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)} \ge \nu_{\tilde{\beta}}^{\sigma(i)} \ge \nu_{\tilde{\gamma}}^{\sigma(i)}$, $i = 1,2,\ldots,l$, then we obtain $\frac{1+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|}{2} \le \frac{1+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\gamma}}^{\sigma(i)}|}{2}$ and $\frac{1-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|}{2} \ge \frac{1-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\gamma}}^{\sigma(i)}|}{2}$, $i = 1,2,\ldots,l$. Hence $\mu_{f(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)} \le \mu_{f(\tilde{\alpha},\tilde{\gamma})}^{\sigma(i)}$ and $\nu_{f(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)} \ge \nu_{f(\tilde{\alpha},\tilde{\gamma})}^{\sigma(i)}$, $i = 1,2,\ldots,l$. From the definition of $f(\tilde{\alpha},\tilde{\beta})$, we know that $\mu_{f(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)} \ge \nu_{f(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)}$, $i = 1,2,\ldots,l$, and thus $E(f(\tilde{\alpha},\tilde{\gamma})) \le E(f(\tilde{\alpha},\tilde{\beta}))$. With the same reason, we can prove that it is also true for $\mu_{\tilde{\alpha}}^{\sigma(i)} \ge \mu_{\tilde{\beta}}^{\sigma(i)} \ge \mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)} \le \nu_{\tilde{\beta}}^{\sigma(i)} \le \nu_{\tilde{\gamma}}^{\sigma(i)}$, $i = 1,2,\ldots,l$;

(4) $E(f(\tilde{\alpha},\tilde{\beta})) = E(f(\tilde{\beta},\tilde{\alpha}))$. □

Example 3.4. From Definition 2.2, we can construct two entropy formulas as follows:

$$E_1(\tilde{\alpha}) = \frac{1}{l_{\tilde{\alpha}}(\sqrt{2}-1)} \sum_{i=1}^{l_{\tilde{\alpha}}} \left(\sin \frac{\pi(1+\mu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\alpha}}^{(i)})}{4} + \sin \frac{\pi(1-\mu_{\tilde{\alpha}}^{\sigma(i)}+\nu_{\tilde{\alpha}}^{(i)})}{4} - 1 \right)$$

$$E_2(\tilde{\alpha}) = \frac{1}{l_{\tilde{\alpha}}(\sqrt{2}-1)} \sum_{i=1}^{l_{\tilde{\alpha}}} \left(\cos \frac{\pi(1+\mu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\alpha}}^{(i)})}{4} + \cos \frac{\pi(1-\mu_{\tilde{\alpha}}^{\sigma(i)}+\nu_{\tilde{\alpha}}^{(i)})}{4} - 1 \right).$$

Then can construct the following entropy formulas based on the similarity measures E_1 and E_2:

$$E_1(f(\tilde{\alpha},\tilde{\beta})) = \frac{1}{l(\sqrt{2}-1)} \tag{1}$$

$$\times \sum_{i=1}^{l} \left(\sin \frac{\pi(2+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|+|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|)}{8} + \sin \frac{\pi(2-|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|)}{8} - 1 \right) \tag{2}$$

$$E_2(f(\tilde{\alpha},\tilde{\beta})) = \frac{1}{l(\sqrt{2}-1)} \times \sum_{i=1}^{l} \left(\cos \frac{\pi(2+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|+|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|)}{8} + \cos \frac{\pi(2-|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|)}{8} - 1 \right).$$

4. Method based on information measures for multiple attribute decision making with generalized hesitant fuzzy information

Suppose that there are m alternatives y_i $(i = 1,2,\ldots,m)$ and n attributes x_j $(j = 1,2,\ldots,n)$ with the attribute weight vector $w = (w_,1 w_2,\ldots,w_n)^T$ such that $w_j \in [0,1]$, $j = 1,2,\ldots,n$, and $\sum_{j=1}^{n} w_j = 1$. Suppose that a decision organization is authorized to provide all the possible degrees that the alternative y_i satisfies the attribute x_j, denoted by a GHFE $\tilde{\alpha}_{ij}$.

In following, we develop an approach to multiple attribute decision making with generalized hesitant fuzzy information.

Approach.

Step 1. The decision organization provides all possible evaluations the alternative y_i under the attribute x_j, denoted by the GHFE $\tilde{\alpha}_{ij}$ $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$.

Step 2. Utilize the maximizing deviation method to calculate the attribute weight w_j of the attribute x_j:

$$w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})},\ j = 1,2,\ldots,n, \tag{3}$$

where $d(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})$ is distance between $\tilde{\alpha}_{ij}$ and $\tilde{\alpha}_{kj}$ such that for two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, the distance between $\tilde{\alpha}$ and $\tilde{\beta}$, denoted as $d(\tilde{\alpha},\tilde{\beta})$, defined by

$$d(\tilde{\alpha},\tilde{\beta}) = \frac{1}{2l} \sum_{i=1}^{l} \left(|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}| + |\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}| \right). \tag{4}$$

Step 3. Calculate the distance between the alternative y_i and the positive-ideal solution $\tilde{\alpha}^+ = (\tilde{\alpha}_1^+,\tilde{\alpha}_2^+,\ldots,\tilde{\alpha}_n^+)$ and the negative-ideal solution $\tilde{\alpha}^- = (\tilde{\alpha}_1^-,\tilde{\alpha}_2^-,\ldots,\tilde{\alpha}_n^-)$:

$$d^+(y_i) = \sum_{j=1}^{n} \left(w_j d(\tilde{\alpha}_{ij},\tilde{\alpha}_j^+) \right),\ i = 1,2,\ldots,m, \tag{5}$$

$$d^-(y_i) = \sum_{j=1}^{n} \left(w_j d(\tilde{\alpha}_{ij}, \tilde{\alpha}_j^-)\right), \ i = 1, 2, \ldots, m. \quad (6)$$

Step 4. Calculate the closeness degree of the alternative y_i to the positive-ideal solution $\tilde{\alpha}^+$ by using the following

$$D(y_i) = \frac{d^-(y_i)}{d^-(y_i) + d^+(y_i)}, \quad i = 1, 2, \ldots, m. \quad (7)$$

Step 5. Rank the alternatives y_i $(i = 1, 2, \ldots, m)$ according the values of $D(y_i)$ $(i = 1, 2, \ldots, m)$ in descending order, and the larger the value of $D(y_i)$, the better the alternative y_i.

In the following, we use a multiple attribute decision making problem of determining what kind of air-conditioning systems should be installed in a library (adapted from [2, 3]) to illustrate the proposed approaches.

Example 4.1. A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers four feasible alternatives y_i $(i = 1, 2, 3, 4)$, which might be adapted to the physical structure of the library. The offered air-conditioning system must take a decision according to the following five attributes: (1) performance (x_1), (2) maintainability (x_2), (3) flexibility (x_3), (4) cost (x_4), (5) safety (x_5). Let $J = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of five attributes, and assume that x_1, x_2, x_3 and x_5 are benefit attributes and x_4 is cost attribute. That is, $J_1 = \{x_1, x_2, x_3, x_5\}$ and $J_2 = \{x_4\}$.

To get the optimal alternative, the following steps are given if proposed Approach is used:

Step 1. The decision organization provides all possible evaluations of the alternative y_i, by a GHFE $\tilde{\alpha}_{ij}$, with respect to the attribute x_j, listed in Table 1 (i.e. generalized hesitant fuzzy decision matrix $D = (\tilde{\alpha}_{ij})_{4\times5}$).

Step 2. Calculate the attribute weight w_j of the attribute x_j by Eqs. (3) and (4):

$$w = (0.1375, 0.1818, 0.2040, 0.1973, 0.2794)^T.$$

Step 3. Utilize Eqs. (5) and (6) to calculate the distance between the alternative y_i and the positive-ideal solution $\tilde{\alpha}^+$ or the negative-ideal solution $\tilde{\alpha}^-$:

$$d^+(y_1) = 0.4219, d^+(y_2) = 0.2893,$$
$$d^+(y_3) = 0.4087, d^+(y_4) = 0.3653,$$

Table 1. Generalized hesitant fuzzy decision matrix

	x_1
y_1	$\{(0.3, 0.2), (0.3, 0.4)\}$
y_2	$\{(0.7, 0.2), (0.5, 0.2)\}$
y_3	$\{(0.6, 0.3), (0.5, 0.2)\}$
y_4	$\{(0.5, 0.3), (0.5, 0.4)\}$

	x_2	x_3
y_1	$\{(0.6, 0.2), (0.5, 0.2), (0.4, 0.3)\}$	$\{(0.4, 0.5), (0.3, 0.4)\}$
y_2	$\{(0.5, 0.1), (0.4, 0.2), (0.3, 0.1)\}$	$\{(0.8, 0.1), (0.7, 0.2)\}$
y_3	$\{(0.9, 0.05), (0.8, 0.1), (0.7, 0.1)\}$	$\{(0.4, 0.3), (0.4, 0.4)\}$
y_4	$\{(0.8, 0.1), (0.8, 0.3), (0.6, 0.3)\}$	$\{(0.7, 0.3), (0.5, 0.4)\}$

	x_4	x_5
y_1	$\{(0.4, 0.2), (0.3, 0.4), ((0.2, 0.6), (0.2, 0.7)\}$	$\{(0.8, 0.1), (0.7, 0.2)\}$
y_2	$\{(0.8, 0.1), (0.7, 0.2), ((0.6, 0.3), (0.5, 0.3)\}$	$\{(0.7, 0.2), (0.6, 0.3)\}$
y_3	$\{(0.8, 0.1), (0.7, 0.2), ((0.6, 0.1), (0.4, 0, 1)\}$	$\{(0.2, 0.5), (0.2, 0.6)\}$
y_4	$\{(0.8, 0.1), (0.7, 0.3), ((0.6, 0.3), (0.4, 0, 2)\}$	$\{(0.6, 0.3), (0.4, 0.5)\}$

$$d^-(y_1) = 0.5781, d^-(y_2) = 0.7107,$$
$$d^-(y_3) = 0.5913, d^-(y_4) = 0.6347.$$

Step 4. Calculate the closeness degree of the alternative y_i to negative-ideal solution $\tilde{\alpha}^+$ by Eq. (7):

$$D(y_1) = 0.5781, D(y_2) = 0.7107,$$
$$D(y_3) = 0.5913, D(y_4) = 0.6347.$$

Step 5. Rank the alternatives y_i $(i = 1, 2, 3, 4)$ according to the values of $D(y_i)$ $(i = 1, 2, 3, 4)$ in descending order:

$$y_2 \succ y_4 \succ y_3 \succ y_1.$$

5. Conclusions

In this paper, the entropy and similarity measures for GHFEs were proposed, and several theorems that the entropy and similarity measures for GHFEs can be transformed by each other were proved. Besides, an approach of multiple attribute decision making problems where attribute weights are unknown and the evaluation values of attributes for each alternative are given in the form of GHFEs was investigated. To get optimal weight vector of attributes, the proposed approach utilized the maximizing deviation method which focuses on the deviations among the decision information. This approach utilized the weights of attributes to calculate closeness degrees of alternatives and to get their ranking. Furthermore, the illustrative example demonstrated the practicality and effectiveness of the developed approaches. The prominent feature of two approaches is that they can provide a flexible way to facilitate the decision process under generalized hesitant fuzzy environment and be more applicable than existing ones, because our approaches can avoid complex computations.

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