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ON THE INDEFINITE POSITIVE QUADRIC $\mathbb{Q}_+^{n-2}$

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Abstract. The generalized Gaussian image of a spacelike surface in L^n lies in the indefinite positive quadric $\mathbb{Q}_+^{n-2}$ in the open submanifold $\mathbb{C}P_+^{n-1}$of the complex projective space $\mathbb{C}P^{n-1}$. The purpose of this paper is to find out detailed information about $\mathbb{Q}_+^{n-2} \subset \mathbb{C}P_+^{n-1}$.

1. The Generalized Gauss Map

We begin with fixing our terminology and notation. Let $L^n = (R^n, g)$ denote Lorentzian n-space with the flat Lorentzian metric g of index 1. Let M be a connected smooth orientable 2 manifold, and $X : M \longrightarrow L^n$ be a smooth imbedding of M into L^n. Throughout this paper, we assume that X is a spacelike imbedding or M is a spacelike surface in L^n, that is, the pull back X^*g of the Lorentzian metric g via X is a positive definite metric on M.

Let $M = (M, \bar{g})$ be a spacelike surface in L^n with the induced metric $\bar{g} = X^*g$ so that $X : M \longrightarrow L^n$ is an isometric imbedding. By (u_1, u_2) we always denote isothermal coordinates compatible with the orientation on M. Then the metric $\bar{g}$ is expressed locally as

$$\bar{g} = \lambda^2((du_1)^2 + (du_2)^2), \qquad \lambda > 0. \tag{1}$$

It is well known that (u_1, u_2) is defined around each point of M, and we may regard M as a Riemann surface by introducing a complex local coordinate $z = u_1 + iu_2$.

We shall define the generalized Gauss map using local coordinates. Let M be a spacelike surface in L^n, or a Riemann surface. Locally, if u_1 and u_2 are isothermal parameters in a neighborhood of p on M, then M is defined near p by a map $X(z) = (x_1(z), \ldots, x_n(z)) \in L^n$, where $z = u_1 + iu_2$. Define the generalized Gauss map Ψ by

$$\Psi(z) = \frac{\partial X}{\partial u_1} + i\frac{\partial X}{\partial u_2} \ ,$$

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where $\Psi(z) \in \mathbb{C}P_+^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}P^{n-1} \mid -z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n} > 0\}$. Let us think of the effect of choosing another isothermal parameters $\tilde{u_1}$, $\tilde{u_2}$, and $\tilde{z} = \tilde{u_1} + i\tilde{u_2}$. Since the change of coordinates on a Riemann surface is analytic, we know that

$$\frac{\partial X}{\partial \tilde{u}_1} + i\frac{\partial X}{\partial \tilde{u}_2} = (\frac{\partial X}{\partial u_1} + i\frac{\partial X}{\partial u_2})(\frac{\partial u_1}{\partial \tilde{u}_1} - i\frac{\partial u_1}{\partial \tilde{u}_2}) \ ,$$

which implies $\Psi(z) = \Psi(\tilde{z})$ in $\mathbb{C}P_+^{n-1}$. Since the pair of vectors $\frac{\partial X}{\partial u_1}$, $\frac{\partial X}{\partial u_2}$ are orthogonal and equal in length in L^n, it follows that

$$\frac{\partial X}{\partial u_1} + i\frac{\partial X}{\partial u_2} \in \mathbb{Q}_+^{n-2} \ ,$$

where $\mathbb{Q}_+^{n-2} = \{(z_1, \ldots, z_n) \in \mathbb{C}P_+^{n-1} \mid -{z_1}^2 + {z_2}^2 + \ldots + {z_n}^2 = 0\}$. Consequently, the generalized Gauss map Ψ is given locally by

$$(u_1, u_2) \longrightarrow \frac{\partial X}{\partial u_1} + i\frac{\partial X}{\partial u_2} \in \mathbb{Q}_+^{n-2} \subset \mathbb{C}P_+^{n-1} \ . \tag{2}$$

We may represent the Gauss map locally by

$$\Psi(z) = (\overline{\phi_1}(z), \ldots, \overline{\phi_n}(z)) \ ,$$

where $\phi_k = 2\frac{\partial x_k}{\partial z} = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}$. Denote $(\phi_1, \ldots, \phi_n)$ by Φ. Then Ψ is holomorphic when Φ is antiholomorphic and Ψ is antiholomorphic when Φ is holomorphic. We will consider Φ as the generalized Gauss map instead of Ψ.

2. On the Indefinite Positive Quadric $\mathbb{Q}_+^{n-2}$

Note that the complex projective space $\mathbb{C}P^{n-1} = \mathbb{C}P_+^{n-1} \cup \mathbb{C}P_0^{n-1} \cup \mathbb{C}P_-^{n-1}$, where the generalized Gaussian image of a spacelike surface in L^n lies in the indefinite positive quadric $\mathbb{Q}_+^{n-2}$ in $\mathbb{C}P_+^{n-1}$. The indefinite Fubini-Study metric on $\mathbb{C}P_+^{n-1}$ is given by

$$ds^2 = 2\frac{\sum_{j<k} \epsilon_j \mid z_j dz_k - z_k dz_j \mid^2}{(-z_1\overline{z_1} + z_2\overline{z_2} + \cdots z_n\overline{z_n})^2} \ , \tag{3}$$

where $\epsilon_1 = -1$, and $\epsilon_j = 1$ otherwise.

Proposition 2.1. *Let H be the hyperplane in $\mathbb{C}P^{n-1}(n \geq 3)$ defined by*

$$H : z_{n-1} - iz_n = 0 \tag{4}$$

Then $(\mathbb{Q}^{n-2})^* = \mathbb{Q}^{n-2} \setminus H$ is biholomorphic to $\mathbb{C}^{n-2}$ under the correspondence

$$(z_1, \cdots, z_n) = \alpha\left(2\xi_1, \cdots, 2\xi_{n-2}, 1 - 2\sum_{i=1}^{n-2}\epsilon_i(\xi_i)^2, i\left(1 - 2\sum_{i=1}^{n-2}\epsilon_i(\xi_i)^2\right)\right), \tag{5}$$

where $\alpha = \frac{z_{n-1} - iz_n}{2}$, and

$$\xi_1 = \frac{z_1}{z_{n-1} - iz_n}, \cdots, \xi_{n-2} = \frac{z_{n-2}}{z_{n-1} - iz_n} \ . \tag{6}$$

Proof. Given any point $(z_1, \cdots, z_n) \in (\mathbb{Q}^{n-2})^*$, if we define ξ_k by (6), then by the defining equation $-(z_1)^2 + (z_2)^2 + \cdots + (z_n)^2 = 0$ of $\mathbb{Q}^{n-2}$,

$$-(\xi_1)^2 + \cdots + (\xi_{n-2})^2 = -\frac{z_{n-1} + iz_n}{z_{n-1} - iz_n} \quad .$$

Hence

$$\begin{aligned} z_{n-1} &= \tfrac{(z_{n-1}-iz_n)+(z_{n-1}+iz_n)}{2} \\ &= i\left(\tfrac{z_{n-1}-iz_n}{2}\right)\left(1 - \textstyle\sum_{j=1}^{n-2} \epsilon_j (\xi_j)^2\right) \quad , \end{aligned}$$

and

$$\begin{aligned} z_n &= i\tfrac{(z_{n-1}-iz_n)-(z_{n-1}+iz_n)}{2} \\ &= i\left(\tfrac{z_{n-1}-iz_n}{2}\right)\left(1 + \textstyle\sum_{j=1}^{n-2} \epsilon_j (\xi_j)^2\right) \quad , \end{aligned}$$

which yields (5).

Conversely, given any $(\xi_1, \cdots, \xi_{n-2}) \in \mathbb{C}^{n-2}$, setting

$$z_1 = 2\xi_1, \cdots, z_{n-2} = 2\xi_{n-2}, z_{n-1} = 1 - \sum_{j=1}^{n-2} \epsilon_j (\xi_j)^2, z_n = 1 + \sum_{j=1}^{n-2} \epsilon_j (\xi_j)^2 \quad (7)$$

gives a point $z = (z_1, \cdots, z_n) \in \left(\mathbb{Q}^{n-2}\right)^*$.

□

Proposition 2.2. *Let H be the hyperplane in $\mathbb{C}P^{n-1} (n \geq 3)$ defined by*

$$H : z_1 - z_2 = 0 \quad . \quad (8)$$

Then $(\mathbb{Q}^{n-2})^* = \mathbb{Q}^{n-2} \setminus H$ is biholomorphic to $\mathbb{C}^{n-2}$ under the correspondence

$$(z_1, \cdots, z_n) = \frac{z_1 - z_2}{2}\left(\sum_{i=1}^{n-2} (\xi_i)^2 + 1, \sum_{i=1}^{n-2} (\xi_i)^2 - 1, 2\xi_1, \cdots, 2\xi_{n-2}\right), \quad (9)$$

where

$$\xi_1 = \frac{z_3}{z_1 - z_2}, \cdots, \xi_{n-2} = \frac{z_n}{z_1 - z_2} \quad . \quad (10)$$

Proof. Proof is exactly the same as the proof of the Proposition 2.1 except for the substitution (10).

□

Proposition 2.3. *Suppose $A \in \mathbb{C}P^{n-1} (n \geq 3)$ satisfies $\sum_{i=1}^{n} \epsilon_i |a_i|^2 < 0$. To such an A, we may assign a real number t lying in the interval $0 \leq t < 1$, with the following properties:*

a) A is equivalent under the induced action of $SO(1, n-1)$ in $\mathbb{C}P^{n-1}$ to $(i, -t, 0, \cdots, 0) = (1, it, 0, \cdots 0)$.

b) $t = 0$ if and only if A is a real vector.

c) If t, s correspond to A, B, then A and B are equivalent if and only if $t = s$.

Proof. Consider first the case that A is a real vector. Since it is nonzero, we may write $A = \lambda R$ where $\lambda \in \mathbb{C}$ and R is a real unit timelike vector in L^n. Choose an oriented basis of L^n with $e_1 = R$. In the new basis, A takes the form $(1, 0, \cdots, 0)$ up to a constant factor λ, which gives a) with $t = 0$.

Suppose next that A is not a real vector. If we write $A = R + iS$, then R and S must be linearly independent.We consider the effect of choosing different homogeneous coordinates for the point A. That amounts to multiplying through by a complex number $re^{i\theta}$, and the effect is to get a new pair of vectors T, U, where $T + iU = re^{i\theta}A$. A direct computation shows that

$$\begin{aligned}&\left(\langle T, U\rangle, \tfrac{\langle T,T\rangle - \langle U,U\rangle}{2}\right)\\ &= r^2 \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \left(\langle R, S\rangle, \tfrac{\langle R,R\rangle - \langle S,S\rangle}{2}\right) \quad .\end{aligned}$$

There is a unique value of $\theta (\mod \pi)$ such that $\langle T, U\rangle = 0$, and $\langle T, T\rangle - \langle U, U\rangle < 0$. Since $\langle T, T\rangle + \langle U, U\rangle < 0$, at least $\langle T, T\rangle < 0$. We may determine r uniquely so that $< T, T >= -1$. Since L^n cannot have two orthogonal timelike vectors, U must be spacelike or lightlike. But since Theorem 1.1[4] tells us that lightlike vectors cannot be orthogonal to timelike vectors, U must be spacelike. Now we may define orthonormal vectors e_1, e_2 by the conditions $T = e_1$, $U = te_2$, where $0 < t < 1$. Complete them into an oriented orthonoraml basis $e_1, \cdots, e_n$ of L^n. Then the point $A = T + iU$ has the coordinate $(1, it, 0, \cdots, 0) = (i, -t, 0, \cdots, 0)$. By combining the above arguments we prove a) and b). The value of t is given by $\sqrt{< U, U >}$. Therefore the value is uniquely determined by the conditions that $A = T + iU$, where $\langle T, U\rangle = 0$, $\langle T, T\rangle = -1$, $\langle U, U\rangle > 0$, $\langle T, T\rangle < 0$.Thus the value of t is clearly $SO(1, n-1)$-invariant,and conversely if $t = s$, then there is $M \in SO(1, n-1)$ such that $MA = B$. This completes the proof. $\square$

Proposition 2.4. *Suppose $A \in \mathbb{C}P^{n-1} (n \geq 3)$ satisfies $\sum_{i=1}^{n} \epsilon_i |a_i|^2 > 0$. Under the induced action of $SO(1, n-1)$ in $\mathbb{C}P^{n-1}$, A satisfies one of the following statements:*

a) A is equivalent to $(0, 0, \cdots, t, i), 0 \leq t \leq 1$, where $t = 0$ occurs only when A is real, and $t = 1$ only when $A \in \mathbb{Q}^{n-2}$.

b) A is equivalent to $(t, i, 0, \cdots, 0), 0 \leq t < 1$, where $t = 0$ occurs only when A is real.

c) A is equivalent to $(1, 1, \sqrt{2}i, 0, \cdots, 0)$.

Remark 1. The equivalence is unique in any case.

Proof. Consider first the case that A is a real vector. Since it is nonzero, we may write $A = \lambda R$, where $\lambda \in \mathbb{C}$ and R is a real unit spacelike vector in L^n. Choose an oriented basis of L^n with either $e_n = R$ or $e_2 = R$. In the new basis, A takes the form either $(0, 0, \cdots, 0, 1)$ or $(0, 1, 0, \cdots, 0)$ which gives the case $t = 0$. Suppose next that A is not a real vector. If we denote $A = R + iS$, then R and S must be linearly independent. If $\langle R, S\rangle = 0$, and $\langle R, R\rangle - \langle S, S\rangle = 0$, then $A \in \mathbb{Q}^{n-2}$ and A is equivalent to $(0, \cdots, 0, 1, i)$ by taking an orthonormal basis of L^n with $R = e_{n-1}$, $S = e_n$. In the opposite case, let $T + iU = re^{i\theta}(R + iS)$. Then we can find a unique value of $\theta(\mod \pi)$ such that $\langle T, U\rangle = 0$, $\langle T, T\rangle - \langle U, U\rangle < 0$. Since $\langle T, T\rangle + \langle U, U\rangle > 0$, U must be spacelike. Here we are excluding the case $T = O$ since A is assumed to be not a real vector. We may determine r uniquely so that $\langle U, U\rangle = 1$. T may be (nonzero) spacelike, timelike or lightlike. IfT is spacelike, define the orthonormal vectors e_{n-1}, e_n by $T = te_{n-1}$, $U = e_n$, where $0 < t < 1$. If T is timelike, define the orthonormal vectors e_1, e_2 by $T = te_1$, $U = e_2$,where $0 < t < 1$. In either case, completing them to an oriented orthonormal basis $e_1, \cdots, e_n$ of L^n, the point $A = T + iU$ has the coordinates $(0, 0, \cdots, t, i)$ or $(t, i, 0, \cdots, 0)$where $0 < t < 1$. If T is lightlike, we can find out another lightlike vector $\tilde{T}$ such that $\langle T, \tilde{T}\rangle = 1$ and $\langle U, \tilde{T}\rangle = 0$. Define

$$e_1 = \frac{T - \tilde{T}}{\sqrt{2}}, \quad e_2 = \frac{T + \tilde{T}}{\sqrt{2}}, \quad e_3 = U \quad .$$

Complete them into an oriented orthonormal basis $e_1, \cdots, e_n$ of L^n. Then the point A has the coordinate $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1, 0, \cdots, 0\right)$. Since T is $SO(1, n-1)$-invariant in any case, the final statement is also true.

□

Proposition 2.5. *Suppose* $A \in \mathbb{C}P^{n-1}(n \geq 3)$ *satisfies* $\sum_{i=1}^n \epsilon_i |a_i|^2 = 0$. *Then* A *is equivalent to* $(1, i, 0, \cdots, 0)$, *or* $(1, 1, 0, \cdots, 0)$, *under the induced* $SO(1, n-1)$*-action in* $\mathbb{C}P^{n-1}$.

Proof. If A is real, then $A = \lambda R$ for some $\lambda \in \mathbb{C}$ and lightlike vector $R \in L^n$. Since there is another lightlike vector $\tilde{R}$ such that $\langle R, \tilde{R}\rangle = 1$, by putting

$$e_1 = \frac{R - \tilde{R}}{\sqrt{2}}, \quad e_2 = \frac{R + \tilde{R}}{\sqrt{2}},$$

A has the coordinate $(1, 1, 0, \cdots, 0)$ in $\mathbb{C}P^{n-1}$. Note that R and $\tilde{R}$ are linearly independent. If A is not real, then there are two linearly independent vectors R, S in L^n such that $A = R + iS$. Note that $\langle R, S\rangle \neq 0$.By multiplying suitable complex number to A we can make $A = T + iU$, where $\langle T, U\rangle = 0$, $\langle T, T\rangle < \langle U, U\rangle$, $\langle U, U\rangle = 1$. Since $\langle T, T\rangle + \langle U, U\rangle = 0$, $\langle T, T\rangle = -1$. Then with $T = e_1$, $U = e_2$, A has the coordinate $(1, i, 0, \cdots, 0)$.

□

Let H be a hyperplane in $\mathbb{C}P^{n-1}$ defined by an equation $\sum \epsilon_i a_i z_i = 0$, $A = (a_1, \cdots, a_n) \in \mathbb{C}_1^n$. Since $\sum \epsilon_i c a_i z_i = 0$ defines the same hyperplane, we may consider $A \in \mathbb{C}P^{n-1}$. Let $M \in SO(1, n-1)$ and $\tilde{Z} = MZ$. Then the equation $\sum \epsilon_i a_i z_i = 0$ is transformed into $\sum \epsilon_i \tilde{a}_i \tilde{z}_i = 0$ using the new coordinate $\tilde{Z}$, where $\tilde{A} = MA$.

Combining this observation with Proposition 2.3, 2.4, and 2.5, we may get the following proposition.

Proposition 2.6. *Let H be a hyperplane in $\mathbb{C}P^{n-1}(n \geq 3)$ defined by an equation $\sum \epsilon_i a_i z_i = 0$. Then H is uniquely determined as one of the following :*

a) If A is a spacelike vector in $\mathbb{C}_1^n$, then there exists $M \in SO(1, n-1)$ such that if we set $\tilde{Z} = MZ$, then H is transformed to $c\tilde{z_{n-1}} - \tilde{z_n} = 0$ for some $c \in \mathbb{C}$ of the form $c = it$, $0 \leq t \leq 1$, $c\tilde{z_1} - \tilde{z_2} = 0$ for some $c \in \mathbb{C}$ of the form $c = it$, $0 \leq t < 1$, or $-\tilde{z_1} + \tilde{z_2} + \sqrt{2}i\tilde{z_3} = 0$;

b) If A is a timelike vector in $\mathbb{C}_1^n$, then there exists $M \in SO(1, n-1)$ such that if we set $\tilde{Z} = MZ$, then H is transformed to $\tilde{z_1} - c\tilde{z_2} = 0$ for some $c \in \mathbb{C}$ of the form $c = it$, $0 \leq t < 1$;

c) If A is a lightlike vector in $\mathbb{C}_1^n$, then there exists $M \in SO(1, n-1)$ such that if we set $\tilde{Z} = MZ$, then H is transformed to $\tilde{z_1} - i\tilde{z_2} = 0$ or $\tilde{z_1} - \tilde{z_2} = 0$.

Proof. Use the transformation of a hyperplane by the $SO(1, n-1)$-action. □

Remark 2. Combining the above proposition together with proposition 1 and 2, we get the following fact: Let H be a tangent hyperplane to $\mathbb{Q}^{n-2}$. Then $\left(\mathbb{Q}^{n-2}\right)^* = \mathbb{Q}^{n-2} \setminus H$ is biholomorphic to $\mathbb{C}^{n-2}$ since H is transformed into either $z_{n-1} - iz_n = 0$ or $z_1 - z_2 = 0$ under a suitable change of coordinates in L^n.

Proposition 2.7. *Let H be a hyperplane in $\mathbb{C}P^{n-1}(n \geq 3)$ defined by an equation $\sum \epsilon_i a_i z_i = 0$, where $A = (a_1, \cdots, c_n)$ is a timelike vector in $\mathbb{C}_1^n$. Then $H \cap \mathbb{Q}^{n-2}$ is isometric to the quadric defined by $(kz_2)^2 + (z_3)^2 + \cdots + (z_n)^2 = 0$ in $\mathbb{C}P^{n-2}$, where $k = \frac{1-c^2}{1-|c|^2}$, $c = it$, $0 \leq t < 1$.*

Proof. Since $\partial\overline{\partial}\log\left(\sum \epsilon_i z_i \overline{z_i}\right)$ is invariant under the action of $SO(1, n-1)$ on $\mathbb{C}P_+^{n-1}$, and the action of $SO(1, n-1)$ on $\mathbb{Q}_+^{n-2}$ is an isometry, we may, by proposition 6, assume H is given by

$$z_1 - cz_2 = 0, \quad c = it, \quad 0 \leq t < 1 \quad .$$

Note that $H \cap \mathbb{Q}^{n-2} = H \cap \mathbb{Q}_+^{n-2}$. Let $d = \sqrt{1 - |c|^2}$, and put

$$\begin{aligned} \tilde{z_1} &= \tfrac{1}{d}\left(z_1 - cz_2\right) \quad , \\ \tilde{z_2} &= \tfrac{1}{d}\left(\overline{c}z_1 - z_2\right) \quad , \\ \tilde{z_3} &= z_3, \cdots, \tilde{z}_n = z_n \quad . \end{aligned}$$

The transformation from Z to $\tilde{Z}$ is in $U(1, n-1)$, that is, an isometry in the indefinite Fubini-Study metric on $\mathbb{C}P_+^{n-1}$. From its converse,

$$\begin{aligned} z_1 &= \tfrac{1}{d}\left(\tilde{z_1} - c\tilde{z_2}\right) \quad , \\ z_2 &= \tfrac{1}{d}\left(\overline{c}\tilde{z_1} - \tilde{z_2}\right) \quad , \\ z_3 &= \tilde{z_3}, \cdots, z_n = \tilde{z_n} \quad . \end{aligned}$$

Since the hyperplane H has the equation $\tilde{z_1}$ in the new coordinate system, it follows that $\mathbb{Q}_+^{n-2} \cap H$ satisfies

$$\begin{aligned} &\widetilde{z_1} = 0 \quad , \\ &\left(\frac{1-c^2}{d^2}\right) \widetilde{z_2}^2 + \widetilde{z_3}^2 + \cdots + \widetilde{z_n}^2 = 0 \quad , \\ &|\widetilde{z_2}|^2 + \cdots + |\widetilde{z_n}|^2 > 0 \quad . \end{aligned}$$

But the restriction of the indefinite Fubini-Study metric on $\mathbb{C}P_+^{n-1}$ to the hyperplane $\widetilde{z_1} = 0$ is just the usual Fubini-Study metric on $\mathbb{C}P^{n-2}$. Hence $\mathbb{Q}^{n-2} \cap H$ is isometric to the quadric

$$k\left(\widetilde{z_2}\right)^2 + \left(\widetilde{z_3}\right)^2 + \cdots + \left(\widetilde{z_n}\right)^2 = 0$$

in $\mathbb{C}P^{n-2}$, where not all $\widetilde{z_i}$'s are zero. □

Proposition 2.8. $\mathbb{Q}^2 \cap H : \sum_{i=1}^4 \epsilon_i a_i z_i = 0$, *where* $A = (a_1, a_2, a_3, a_4))$ *is timelike in* $\mathbb{C}_1^4$, *is a compact surface* S *of genus* 0 *whose Gauss curvature* K *with respect to the indefinite Fubini-Study metric satisfies*

$$\begin{aligned} maxK(p) &= 2 - \tfrac{1}{|k|^4} \\ minK(p) &= 2 - |k|^2 \quad , \end{aligned}$$

where k *is given in Proposition 2.7.*

Proof. We may assume H has the equation $z_1 - itz_2 = 0$, $0 \leq t < 1$. Then $H \cap \mathbb{Q}^2 = H \cap \mathbb{Q}_+^2$. We already know that $H \cap \mathbb{Q}^2$ is isometric to quadric $k\left(z_2\right)^2 + \left(z_3\right)^2 + \left(z_4\right)^2 = 0$ in $\mathbb{C}P^2$. The Gauss curvature of S at any point $p = (z_2, z_3, z_4)$ is given by the formula

$$K(p) = 2 - \frac{|k|^2 \left(|z_2|^2 + |z_3|^2 + |z_4|^2\right)^3}{\left(|k|^2|z_2|^2 + |z_3|^2 + |z_4|^2\right)^3}$$

from which

$$\begin{aligned} maxK(p) &= 2 - \tfrac{1}{|k|^4} \quad , \\ minK(p) &= 2 - |k|^2 \quad . \end{aligned}$$

□

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