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SOME INTEGRAL REPRESENTATIONS OF THE CLAUSEN FUNCTION $\mathsf{Cl}_2(x)$ AND THE CATALAN CONSTANT G

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Abstract. The Clausen function $\mathsf{Cl}_2(x)$ arises in several applications. A large number of indefinite integrals of logarithmic or trigonometric functions can be expressed in closed form in terms of $\mathsf{Cl}_2(x)$. Very recently, Choi and Srivatava [3] and Choi [1] investigated certain integral formulas associated with $\mathsf{Cl}_2(x)$. In this sequel, we present an interesting new definite integral formula for the Clausen function $\mathsf{Cl}_2(x)$ by using a known relationship between the Clausen function $\mathsf{Cl}_2(x)$ and the generalized Zeta function $\zeta(s,a)$. Also an interesting integral representation for the Catalan constant G is considered as one of two special cases of our main result.

1. Introduction, Definitions and Preliminaries

Clausen's integral (or, synonymously, Clausen's function) $\mathsf{Cl}_2(x)$ is defined by

$$\mathsf{Cl}_2(x) := \sum_{k=1}^{\infty} \frac{\sin kx}{k^2} = -\int_0^x \log\left[2\sin\left(\frac{1}{2}\eta\right)\right] d\eta \qquad (x \in \mathbb{R}), \tag{1.1}$$

where $\mathbb{R}$ denotes the set of real numbers. This integral was first treated by Clausen in 1832 [4] and has since then been investigated by many authors (see, *e.g.*, [5], [7], [8], [9, Chapter 4], [12, Section 2.4], and many of the references cited therein). Some known properties and special values of the Clausen integral (or the Clausen function) include the periodic properties given by

$$\mathsf{Cl}_2(2n\pi \pm \theta) = \mathrm{Cl}_2(\pm\theta) = \pm\mathrm{Cl}_2(\theta), \tag{1.2}$$

which, for $n = 1$ and with θ replaced by $\pi + \theta$, yields

$$\mathsf{Cl}_2(\pi + \theta) = -\mathrm{Cl}_2(\pi - \theta). \tag{1.3}$$

From the series definition (1.1), it is obvious that

$$\mathrm{Cl}_2(n\pi) = 0 \qquad (n \in \mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}), \tag{1.4}$$

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which, for $n = 1$, gives

$$\int_0^{\pi} \log\left(2\,\sin\frac{1}{2}\theta\right)\,d\theta = 0 \quad \text{and} \quad \int_0^{\pi/2} \log\left(\sin\theta\right)\,d\theta = -\frac{\pi}{2}\,\log 2. \tag{1.5}$$

Setting $\theta = \frac{1}{2}\pi$ in the series definition (1.1), and using the periodic property (1.3), we find that

$$\mathsf{Cl}_2\left(\frac{1}{2}\pi\right) = \mathsf{G} = -\mathrm{Cl}_2\left(\frac{3}{2}\pi\right), \tag{1.6}$$

where G is the Catalan constant defined by

$$\mathsf{G} := \frac{1}{2}\int_0^1 \mathbf{K}(\kappa)d\kappa = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \cong 0.91596\,55941\,\cdots, \tag{1.7}$$

where $\mathbf{K}(\kappa)$ is the complete elliptic integral of the first kind given by

$$\mathbf{K}(\kappa) := \int_0^{\pi/2} \frac{dt}{\sqrt{1-\kappa^2\sin^2 t}} \qquad (|\kappa| < 1). \tag{1.8}$$

From (1.2), (1.3) and (1.4), it suffices to consider $\mathsf{Cl}_2\,(x)$ in the interval $(0,\,\pi)$. Hurwitz (or generalized) Zeta function $\zeta(s,a)$ is defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \qquad (\Re(s) > 1;\; a \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{1.9}$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers and nonpositive integers, respectively.

The Clausen function $\mathsf{Cl}_2(x)$ arises in several applications. A large number of indefinite integrals of logarithmic or trigonometric functions can be expressed in closed form in terms of $\mathsf{Cl}_2(x)$ (see, *e.g.*, [9] and [10, Section 1.6]). Very recently, Choi and Srivatava [3] and Choi [1] investigated certain integral formulas associated with $\mathsf{Cl}_2(x)$. In this sequel, we present an interesting new definite integral formula for the Clausen function $\mathsf{Cl}_2(x)$ by using a known relationship between the Clausen function $\mathsf{Cl}_2(x)$ and the generalized Zeta function $\zeta(s,a)$. Also an interesting integral representation for the Catalan constant G is considered as one of two special cases of our main result.

2. Certain integral representations of the Clausen function $\mathsf{Cl}_2(x)$ and the Catalan constant G

We begin by recalling an interesting relationship between the Clausen function $\mathsf{Cl}_2(x)$ and the generalized Zeta function $\zeta(s,a)$ as in the following lemma (see [1, Lemma 3]).

Lemma 1. *The following relationship holds true*:

$$\mathsf{Cl}_2(x) = -\,x\,\log\left(\frac{x}{2\,\pi}\right) + 2\pi\,\left[\zeta'\left(-1, 1+\frac{x}{2\,\pi}\right) - \zeta'\left(-1, 1-\frac{x}{2\,\pi}\right)\right] \qquad (-2\,\pi < x < 2\,\pi). \tag{2.1}$$

Among various integral representations of $\zeta(s,a)$ (see, *e.g.*, [12, Section 2.2]), we choose to recall a known formula (see, *e.g.*, [12, p. 160, Eq. (23)]) in the following lemma.

Lemma 2. *The following integral formula for* $\zeta(s,a)$ *holds true*:

$$\zeta(s,a) = \frac{\pi\,2^{s-2}}{s-1} \cdot \int_0^\infty \left[t^2 + (2a-1)^2\right]^{\frac{1}{2}(1-s)}\, \frac{\cos\left[(s-1)\,\arctan\left(\frac{t}{2a-1}\right)\right]}{\cosh^2\left(\frac{1}{2}\pi t\right)}\,dt \qquad \left(s \in \mathbb{C}\setminus\{1\};\ \Re(a) > \frac{1}{2}\right). \tag{2.2}$$

Differentiating (2.2) with respect to s and setting $s = -1$ in the resulting identity, and considering the relation in (2.1), we obtain an interesting integral representation for $\mathsf{Cl}_2(x)$ asserted by the following theorem.

Theorem. *The following integral representation for* $\mathsf{Cl}_2(x)$ *holds true*:

$$\mathsf{Cl}_2(x) = -\,x\,\log\frac{x}{2\pi} + \frac{\pi^2}{16}\int_0^\infty \Big[(1+2\,\log 2)\,(p(t;x) - p(t;-x)) + (q(t;x) - q(t;-x)) + 2(r(t;x) - r(t;-x))\Big]\,dt \qquad (-\pi < x < \pi), \tag{2.3}$$

where, for convenience, the functions $p(t;x)$, $q(t;x)$ and $r(t;x)$ *are defined by*

$$p(t;x) := \left(t^2 + \left(1-\frac{x}{\pi}\right)^2\right)\,\frac{\cos\left(2\,\arctan\left(\frac{\pi\,t}{\pi - x}\right)\right)}{\cosh^2\left(\frac{1}{2}\,\pi\,t\right)}, \tag{2.4}$$

$$q(t;x) := \log\left(t^2 + \left(1+\frac{x}{\pi}\right)^2\right)\,p(t;-x) \tag{2.5}$$

and

$$r(t;x) := \arctan\left(\frac{\pi\,t}{\pi - x}\right)\,\left(t^2 + \left(1-\frac{x}{\pi}\right)^2\right)\,\frac{\sin\left(2\,\arctan\left(\frac{\pi\,t}{\pi - x}\right)\right)}{\cosh^2\left(\frac{1}{2}\,\pi\,t\right)}. \tag{2.6}$$

A large number of integral formulas for the Catalan constant G in (1.7) has been presented (see, *e.g.*, [6]). In view of (1.6), setting $x = \pi/2$ in (2.3) yields an interesting new integral formula for the Catalan constant G as in the following corollary.

Corollary. *Each of the following integral formulas holds true*:

$$\begin{aligned}\mathsf{G} = \pi\,\log 2 + \frac{\pi^2}{16} \int_0^\infty \Big[&(1 + 2\,\log 2)\,(p(t;\pi/2) - p(t;-\pi/2)) \\ &+ (q(t;\pi/2) - q(t;-\pi/2)) + 2(r(t;\pi/2) - r(t;-\pi/2))\Big]\,dt.\end{aligned} \tag{2.7}$$

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