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ON REFINEMENTS OF HÖLDER'S INEQUALITY II

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Abstract. Generalized Hölder inequality developed by H. Qiang and Z. Hu is further refined. Also, generalized Hölder inequality developed by X. Yang is further refined.

1. Introduction

There have been lots of generalizations, extensions and refinements of the classical Hölder inequality. See, [1] and [2] for classical results, and see for example [3-13] and [14] for recent developments on this topic. Among them is the following theorem of H. Qiang and Z. Hu established in 2011.

Theorem A ([10] Theorem 2.1). Let $a_{ij} > 0$, $p_k > 0$, $\alpha_{kj} \in \mathbb{R}$ $(i = 1, 2, \cdots, n;$ $j = 1, 2, \cdots, m; k = 1, 2, \cdots, s)$, $\sum_{k=1}^{s} \frac{1}{p_k} = 1$ and $\sum_{k=1}^{s} \alpha_{kj} = 0$. Then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \leq \prod_{k=1}^{s} \left\{ \sum_{i=1}^{n} \left(\prod_{j=1}^{m} a_{ij}^{1+p_k \alpha_{kj}} \right) \right\}^{1/p_k} . \tag{1.1}$$

Moreover, for the integral form of the above inequality, if $f_j(x) > 0$ $(j = 1, 2,$ $\cdots, m)$, $x \in [a, b]$, $-\infty < a < b < +\infty$ and $f_j \in C([a, b])$, then

$$\int_a^b \left(\prod_{j=1}^{m} f_j(x) \right) dx \leq \prod_{k=1}^{s} \left(\int_a^b \prod_{j=1}^{m} f_j^{1+p_k \alpha_{kj}}(x)\, dx \right)^{1/p_k} . \tag{1.2}$$

Remark 1. If we set $s = m$ and $\alpha_{kj} = \begin{cases} -1/p_k & (\text{j} \neq \text{k}) \\ 1 - (1/p_k) & (\text{j=k}) \end{cases}$ for $j = 1, 2, \cdots, m$, $k = 1, 2, \cdots, s$, then (1.1) and (1.2) reduce to classical Hölder's inequality of discrete form and integral form respectively. Theorem A extends Hölder's inequality in this sense.

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The objective of this paper is to generalize Theorem A to the case of $\sum_{k=1}^{s}\frac{1}{p_k} \geq 1$ and to improve Theorem A by inserting a quantity $Q\,(\leq 1)$ in the right side of (1.1) or (1.2). See Theorem 2.1 and Theorem 2.2.

As an application, we can insert a new continuum between two sides of classical Hölder inequality. See Theorem 4.1 and Remark 4.3.

2. Refinments

The following theorems are our first results of this paper which refine (1.1) and (1.2) respectively.

Theorem 2.1. *Let* $a_{ij} > 0$, $p_k > 0$, $\alpha_{kj} \in \mathbb{R}$ $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m;\ k = 1, 2, \cdots, s)$, $\sum_{k=1}^{s}\frac{1}{p_k} = r \geq 1$ *and* $\sum_{k=1}^{s}\alpha_{kj} = 0$. *If* $e_i \in \mathbb{R}$ *satisfy* $1 - e_i + e_l \geq 0$ $(i = 1, 2, \cdots, n; l = 1, 2, \cdots, n)$, *then for any pair* $\{k_1, k_2\} \subset \{1, 2, \cdots, s\}$ *we have*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij} \leq n^{1-1/r} Q_1^{1/r} \prod_{k=1}^{s}\left\{\sum_{i=1}^{n}\left(\prod_{j=1}^{m} a_{ij}^{1+p_k\alpha_{kj}}\right)\right\}^{\frac{1}{rp_k}}, \tag{2.1}$$

where $Q_1 = Q_1(k_1, k_2, \{a_{ij}\}) =$

$$\left[1 - \left\{\frac{\sum_{i=1}^{n} e_i \prod_{j=1}^{m} a_{ij}^{1+p_{k_1}\alpha_{k_1 j}}}{\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}^{1+p_{k_1}\alpha_{k_1 j}}} - \frac{\sum_{i=1}^{n} e_i \prod_{j=1}^{m} a_{ij}^{1+p_{k_2}\alpha_{k_2 j}}}{\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}^{1+p_{k_2}\alpha_{k_2 j}}}\right\}^2\right]^{\frac{1}{2\max\left(p_{k_1}, p_{k_2}\right)}}.$$

Theorem 2.2. *Let* $f_j \in C([a, b])$, $f_j(x) > 0$, $p_k > 0$, $\alpha_{kj} \in \mathbb{R}$ $(x \in [a, b];\ j = 1, 2, \cdots, m;\ k = 1, 2, \cdots, s)$, $\sum_{k=1}^{s}\frac{1}{p_k} = r \geq 1$ *and* $\sum_{k=1}^{s}\alpha_{kj} = 0$. *If* $e \in C([a, b])$ *satisfy* $1 - e(x) + e(y) \geq 0$ $(x, y \in [a, b])$, *then for any pair* $\{k_1, k_2\} \subset \{1, 2, \cdots, s\}$ *we have*

$$\int_a^b \prod_{j=1}^{m} f_j(x)\,dx \leq (b-a)^{1-1/r} Q_2^{1/r} \prod_{k=1}^{s}\left(\int_a^b \prod_{j=1}^{m} f_j^{1+p_k\alpha_{kj}}(x)\,dx\right)^{\frac{1}{rp_k}}, \tag{2.2}$$

where $Q_2 = Q_2\left(k_1, k_2, \{f_j\}_{j=1}^{m}\right)$

$$= \left[1 - \left\{\frac{\int_a^b e(x)\prod_{j=1}^{m} f_j^{q_{k_1,j}}(x)dx}{\int_a^b \prod_{j=1}^{m} f_j^{q_{k_1,j}}(x)dx} - \frac{\int_a^b e(x)\prod_{j=1}^{m} f_j^{q_{k_2,j}}(x)dx}{\int_a^b \prod_{j=1}^{m} f_j^{q_{k_2,j}}(x)dx}\right\}^2\right]^{\frac{1}{2\max\left(p_{k_1}, p_{k_2}\right)}}$$

with $q_{k_1,j} = 1 + p_{k_1}\alpha_{k_1 j}$ *and* $q_{k_2,j} = 1 + p_{k_2}\alpha_{k_2 j}$.

Remark 2. The case $r = 1$ with $e_j = constant$ for all j (resp. $e(x) = constant$ and all x) of (2.1) (resp. (2.2)) reduces to (1.1) (resp. (1.2)).

3. Proofs

We now are going to prove Theorem 2.1 and Theorem 2.2. Since our process of two proofs are same, we give a proof of Theorem 2.2 left that of Theorem 2.1. We make use of the following lemma whose proof we omit is, by replacing summations over $\{1,2,\cdots,n\}$ with integrations over $[a,b]$, exactly similar to that of Theorem 1.1 in [7].

Lemma 3.1. *Let $f_j \in C([a,b])$, $f_j(x) > 0$, $p_j > 0$ $(x \in [a,b];\ j = 1,2,\cdots,m)$, and $\sum_{j=1}^m \frac{1}{p_j} = r \geq 1$. If $e \in C([a,b])$ satisfy $1 - e(x) + e(y) \geq 0$ $(x, y \in [a,b])$, then for any pair $\{j_1, j_2\} \subset \{1,2,\cdots,m\}$ we have*

$$\int_a^b \prod_{j=1}^m f_j(x)dx \leq (b-a)^{1-\min\{r,1\}} M \cdot \prod_{j=1}^m \left(\int_a^b f_j^{p_j}(x)dx\right)^{1/p_j}, \tag{3.1}$$

where $M = M(j_1, j_2 : \{f_j\}_{j=1}^m)$

$$= \left[1 - \left\{\frac{\int_a^b e(x) f_{j_1}^{p_{j_1}}(x)dx}{\int_a^b f_{j_1}^{p_{j_1}}(x)dx} - \frac{\int_a^b e(x) f_{j_2}^{p_{j_2}}(x)dx}{\int_a^b f_{j_2}^{p_{j_2}}(x)dx}\right\}^2\right]^{\frac{1}{2\max\left(p_{j_1}, p_{j_2}\right)}}.$$

Proof of Theorem 2.2. Let $g_k(x) = \left\{\prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x)\right\}^{1/p_k}$ $(k = 1,2,\cdots,s)$, then it is simple to see $\prod_{k=1}^s g_k(x) = \left\{\prod_{j=1}^m f_j(x)\right\}^r$.

First, assume $r = 1$. Then by (3.1),

$$\int_a^b \prod_{i=1}^m f_i(x)dx = \int_a^b \prod_{k=1}^s g_k(x)dx \leq M \cdot \prod_{k=1}^s \left\{\int_a^b g_k^{p_k}(x)dx\right\}^{1/p_k}$$

where $M = M(k_1, k_2 : \{g_k\}_{k=1}^s)$

$$= \left[1 - \left\{\frac{\int_a^b e(x) g_{k_1}^{p_{k_1}}(x)dx}{\int_a^b g_{k_1}^{p_{k_1}}(x)dx} - \frac{\int_a^b e(x) g_{k_2}^{p_{k_2}}(x)dx}{\int_a^b g_{k_2}^{p_{k_2}}(x)dx}\right\}^2\right]^{\frac{1}{2\max\left(p_{j_1}, p_{j_2}\right)}}.$$

With $q_{k_1,j} = 1 + p_{k_1}\alpha_{k_1 j}$ and $q_{k_2,j} = 1 + p_{k_2}\alpha_{k_2 j}$, this M equals

$$\left[1 - \left\{\frac{\int_a^b e(x) \prod_{j=1}^m f_j^{q_{k_1,j}}(x)dx}{\int_a^b \prod_{j=1}^m f_j^{q_{k_1,j}}(x)dx} - \frac{\int_a^b e(x) \prod_{j=1}^m f_j^{q_{k_2,j}}(x)dx}{\int_a^b \prod_{j=1}^m f_j^{q_{k_2,j}}(x)dx}\right\}^2\right]^{\frac{1}{2\max\left(p_{k_1}, p_{k_2}\right)}}$$

$= Q_2 = Q_2(k_1, k_2, \{f_j\}_{j=1}^m)$.

Next, assume $r > 1$. Then by Jensen's inequality and (3.1),

$$\int_a^b \prod_{i=1}^m f_i(x)dx \le (b-a)^{1-1/r}\left[\int_a^b \left\{\prod_{i=1}^m f_i(x)\right\}^r dx\right]^{1/r}$$

$$=(b-a)^{1-1/r}\left\{\int_a^b \prod_{k=1}^s g_k(x)dx\right\}^{1/r}$$

$$\le(b-a)^{1-1/r}\left[M\left(k_1,k_2:\{g_k\}_{k=1}^s\right)\cdot\prod_{k=1}^s\left\{\int_a^b g_k^{p_k}(x)dx\right\}^{1/p_k}\right]^{1/r}$$

$$=(b-a)^{1-1/r}Q_2^{1/r}(k_1,k_2:\{f_j\}_{j=1}^m)\cdot\prod_{k=1}^s\left\{\int_a^b\prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x)\,dx\right\}^{\frac{1}{rp_k}}.$$

The proof is complete. □

4. Another Refinments

We can improve another refinements of Hölder's inequality as consequences of Theorem 2.1 and Theorem 2.2.

Recall the case $r = 1$ and $s = m$ of (2.2) :

$$\int_a^b \prod_{j=1}^m f_j(x)dx \le Q_2\cdot\prod_{k=1}^m\left\{\int_a^b\prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x)dx\right\}^{1/p_k}. \tag{4.1}$$

For a fixed $t \in \mathbb{R}$, if we set $\alpha_{kj} = \begin{cases} -t/p_k & (\text{j} \neq \text{k}) \\ t(1-1/p_k) & (\text{j} = \text{k})\end{cases}$, then we have

$$\prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x) = f_k^{1+p_k(1-1/p_k)t}(x)\prod_{j\neq k}f_j^{1-t}(x) = \{f_k(x)\}^{p_kt}\left\{\prod_{j=1}^m f_j(x)\right\}^{1-t},$$

so that (4.1) becomes

$$\int_a^b \prod_{j=1}^m f_j(x)dx \le Q_2\cdot\prod_{k=1}^m\left[\int_a^b \{f_k(x)\}^{p_kt}\left\{\prod_{j=1}^m f_j(x)\right\}^{1-t}dx\right]^{1/p_k}. \tag{4.2}$$

According to [6], the function $H:\mathbb{R}\longrightarrow\mathbb{R}^+$ defined by

$$H(t) = \prod_{k=1}^m\left[\int_a^b \{f_k(x)\}^{p_kt}\left\{\prod_{j=1}^m f_j(x)\right\}^{1-t}dx\right]^{1/p_k} \tag{4.3}$$

satisfied $tH'(t) \geq 0$, so that in particular

$$H(0) \leq H(t) \leq H(u), \quad 0 \leq t \leq u < \infty. \tag{4.4}$$

Inequality (4.4) with $u = 1$ refines Hölder's inequality because

$$H(0) = \prod_{k=1}^{m} \left\{ \int_a^b \prod_{j=1}^{m} f_j(x) dx \right\}^{1/p_k} = \int_a^b \prod_{j=1}^{m} f_j(x) dx,$$

and

$$H(1) = \prod_{k=1}^{m} \left\{ \int_a^b f_k(x)^{p_k} dx \right\}^{1/p_k}.$$

By (4.2), we obtain a further refinement of (4.4). We state it as the following.

Theorem 4.1. *Let $f_j \in C([a,b])$, $f_j(x) > 0$, $p_k > 0$, $(x \in [a,b];\ j,k = 1,2,\cdots,m)$, and $\sum_{k=1}^{m} \frac{1}{p_k} = 1$. Let $N = \{1,2,\cdots,m\}$ and $e \in C([a,b])$ satisfy $1 - e(x) + e(y) \geq 0$ $(x,y \in [a,b])$. Define a function $H : \mathbb{R} \longrightarrow \mathbb{R}^+$ by*

$$H(t) = \prod_{k=1}^{m} \left[\int_a^b \{f_k(x)\}^{p_k t} \left\{ \prod_{j=1}^{m} f_j(x) \right\}^{1-t} dx \right]^{1/p_k},$$

and define another function $Q_2 : N^2 \times \mathbb{R} \longrightarrow [0,1]$ by

$$Q_2(k_1,k_2;t) = \left[1 - \left\{ \frac{\int_a^b e(x) F(k_1,t;x) dx}{\int_a^b F(k_1,t;x) dx} - \frac{\int_a^b e(x) F(k_2,t;x) dx}{\int_a^b F(k_2,t;x) dx} \right\}^2 \right]^{\frac{1}{2\max\left(p_{k_1},p_{k_2}\right)}}$$

with

$$F(k,t;x) = \{f_k(x)\}^{p_k t} \left\{ \prod_{j=1}^{m} f_j(x) \right\}^{1-t}.$$

Then

$$H(0) \leq Q_2(k_1,k_2;t) \cdot H(t) \leq H(t) \leq H(u), \quad 0 \leq t \leq u < \infty. \tag{4.5}$$

Remark 3. Notinging that $0 \leq Q_2 \leq 1$, the case $u = 1$ of (4.5) inserted two continuums between two sides of classical Hölder inequality.

Remark 4. Of course Theorem 4.1 has its discrete form by replacing integrations with summations, which can be verified by an exactly same manner. Historically, X. Yang in [14] observed a function $h : \mathbb{R} \longrightarrow \mathbb{R}^+$ by

$$h(t) = \prod_{k=1}^{n} \left[\sum_{i=1}^{m} \{a_{ik}^{p_k}\}^t \left\{ \prod_{j=1}^{n} a_{ij} \right\}^{1-t} \right]^{1/p_k} \tag{4.6}$$

for a positive sequence $\{a_{ij}\}$ and another positive sequence $\{p_k\}$ satisfying $\sum_{k=1}^{n} \frac{1}{p_k} = 1$. He verified that

$$h(0) \leq h(t) \leq h(u), \quad 0 \leq t \leq u < \infty, \tag{4.7}$$

the case $u = 1$ of which connects both sides of (discrete) Hölder's inequality by a continuum. We can refine (4.7) as

$$h(0) \leq Q_1(t) \cdot h(t) \leq h(t) \leq h(u), \quad 0 \leq t \leq u < \infty, \tag{4.8}$$

where $Q_1(t) = Q_1(k_1, k_2; t)$ is defined by

$$N_1(t) = \left[1 - \left\{ \frac{\sum_{i=1}^{n} e_i b(k_1, t; i)}{\sum_{i=1}^{n} b(k_1, t; i)} - \frac{\sum_{i=1}^{n} e_i b(k_2, t; i)}{\sum_{i=1}^{n} b(k_2, t; i)} \right\}^2 \right]^{\frac{1}{2\max\left(p_{k_1}, p_{k_2}\right)}}.$$

with $\{k_1, k_2\} \subset \{1, 2, \cdots, n\}$ and

$$b(k, t; i) = \{a_{ik}\}^{p_k t} \left\{ \prod_{j=1}^{m} a_{ij} \right\}^{1-t}.$$

Remark 5. As in [KBe], we can extensively define $H(t)$ unifying (4.3) and (4.6) on a general measure space. By improving Lemma 3.1 to the case of a general measure space, we can obtain an extended result which can be reduced both to (4.5) and (4.8) as special cases. This topic will be considered with various applications in a coming paper of the first author.

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