Journal of the Korean Data & Information Science Society
2016, **27**(1), 265–274

http://dx.doi.org/10.7465/jkdi.2016.27.1.265
한국데이터정보과학회지

Moments of the ruin time and the total amount of claims until ruin in a diffusion risk process†

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Received 31 December 2015, revised 12 January 2016, accepted 12 January 2016

Abstract

In this paper, we consider a diffusion risk process, in which, its surplus process behaves like a Brownian motion in-between adjacent epochs of claims. We assume that the claims occur following a Poisson process and their sizes are independent and exponentially distributed with the same intensity. Our main goal is to derive the exact formula of the joint moment generating function of the ruin time and the total amount of aggregated claim sizes until ruin in the diffusion risk process. We also provide a method for computing the related first and second moments using the joint moment generating function and the augmented matrix exponential function.

Keywords: Augmented matrix, diffusion risk process, exponential claim, Poisson process.

1. Introduction

We begin by introducing the diffusion risk process of interest, which is referred to as the jump-diffusion risk process in the literature and is denoted by $U = \{U(t), t \geq 0\}$ in this paper. The diffusion risk process U with $U(0) = u \geq 0$, evolves like a Brownian motion in-between adjacent epochs of claims. Denoting by $B = \{B(t), t \geq 0\}$ a standard Brownian motion, by $N = \{N(t), t \geq 0\}$ the Poisson process with intensity α, and by $\{X_i, i = 1, 2, \cdots\}$ independent and exponentially distributed random variables with mean β^{-1}, then the diffusion risk process is represented as

$$U(t) = u + \mu t + \sigma B(t) - \sum_{i=1}^{N(t)} X_i, \tag{1.1}$$

where $u(> 0)$ is the initial surplus and $\sigma(> 0)$ is the diffusion factor. We note that N, B, and X_i's are assumed to be mutually independent.

In relation to the classical risk process without the diffusion factor, that is, the diffusion risk process with $\sigma = 0$ (Gerber and Shiu, 1997; Lee, 2014), obtained some explicit expressions

† This research was supported by the 2015 Sabbatical year research grant of the University of Seoul.
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for the joint distribution of the ruin time, the surplus immediately prior to ruin, and the deficit at ruin. Later, (Zhang and Wang, 2003) extended the results of (Gerber and Shiu, 1997) to derive the exact expression for the joint density function of the three characteristics. This explicit expression facilitate application of the diffusion risk process to research areas such as insurance and finance (Yang and Zhang, 2005; Chi, 2005; Won *et al.*, 2013).

In this paper, we derive the explicit form of the joint moment generating function of the ruin time and the total amount of aggregated claim sizes until ruin of the diffusion risk process, which cannot be found in the literature. For the derivation, we define a transform matrix in relation to the two characteristics which is the minimal non-negative solution of a Riccati equation. This minimality result is given by the results of (Ahn, 2015) on the shift process modulated by the Markov modulated Brownian motion (MMBM). The MMBM is an extension of the diffusion risk process and the shift process is introduced in Section 2. Then, solving the Riccati equation, we derive the exact form the joint moment generating function. We also provide a method for computing the related first and second moment using the joint moment generating function and the augmented matrix exponential function, which was used in (Ahn *et al.*, 2015) for getting moments in the risk model modulated by an Markov modulated fluid flow.

The remainder of this paper is organized as follows. In Section 2, we introduce the MMBM and the shift process modulated by the MMBM, and also provide their relations to the diffusion risk model. In Section 3, we derive the exact form of the joint moment generating function of the ruin time and the total amount of aggregated claim sizes until ruin of the diffusion risk process, and introduce a method computing related first and second moment using the augmented matrix exponential function. We illustrate the results using selected examples in Section 4, in which we can observe the dynamic effect of the diffusion factor on the two characteristic of our interest. Finally in Section 5, we discuss our further study.

2. Preliminaries

2.1. Connection of the diffusion risk process to MMBM

Denote by $\tilde{U}$ the embedded process of U, which is obtained by using an embedding technique, that is, replacing downward jumps in U with linear stretches with slope -1. In relation to $\tilde{U}$, we define a new process $\tilde{J} = \{\tilde{J}(t), t \geq 0\}$ such as: $\tilde{J}(t) = b$ if t belongs to an interval when $\tilde{U}$ shows a Brownian path originating from U; $\tilde{J}(t) = j^-$ if t belongs to an interval when $\tilde{U}$ shows a linear path originating from downward jumps. Noting that X_i's are mutually independent and identically exponential-distributed random variables that are independent of N and B, then $\tilde{J}$ becomes a Markov process with the state space $S = \{b, j^-\}$ and the infinitesimal generator $\tilde{Q}$ such as

$$\tilde{Q} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}. \tag{2.1}$$

In Figure 2.1, we illustrate a sample path of U, and also how its corresponding sample path of $(\tilde{U}, \tilde{J})$ is generated. We remark that, with $\zeta(t) = \inf\{y : \int_0^y \chi\{\tilde{J}(u) = b\}du > t\}$, $U(t) = \tilde{U}(\zeta(t))$.

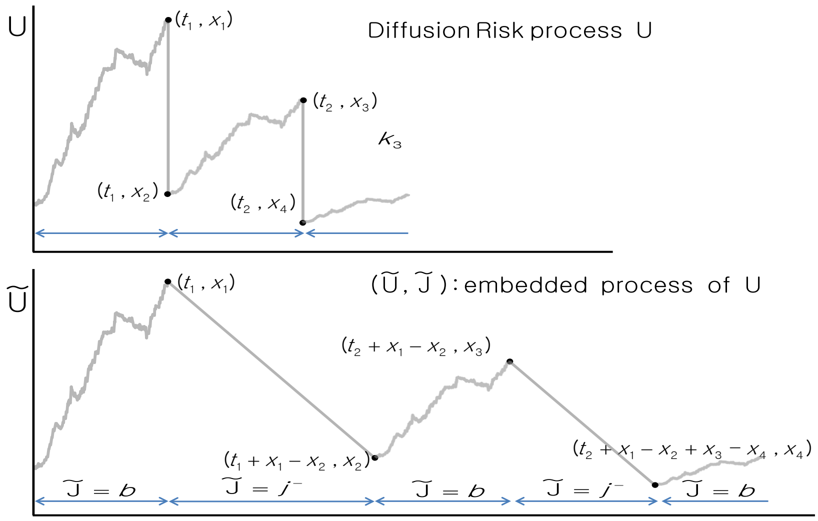

Figure 2.1 Illustration of how a sample path of U is transformed to a sample path of its embedded process $(\tilde{U}, \tilde{J})$.

Then, the process $(\tilde{U}, \tilde{J})$ becomes an MMBM, and its coordinate processes $\tilde{U}$ and $\tilde{J}$ are respectively referred to as the level process and the phase process of the MMBM. Furthermore, the level process $\tilde{U}$ is represented as

$$\tilde{U}(t) = u + \int_0^t \tilde{\mu}_{\tilde{J}(u)} du + \int_0^\infty \tilde{\sigma}_{\tilde{J}(u)} d\tilde{B}(u),$$

where $\tilde{B}$ is a Brownian motion which is independent of $\tilde{J}$, $(\tilde{\mu}_b \;\; \tilde{\mu}_{j^-}) = (\mu \;\; -1)$, and $(\tilde{\sigma}_b \;\; \tilde{\sigma}_{j^-}) = (\sigma \;\; 0)$. For notational convenience, we use $\tilde{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\sigma}}$ to denote the draft vector $(\mu \;\; -1)$ and the diffusion vector $(\sigma \;\; 0)$, respectively.

2.2. Representation of the total amount of aggregated claims before ruin through shift processes

In relation to the MMBM $(\tilde{U}, \tilde{J})$, we define the so-called shift process, denoted by $\tilde{Z}$, which is defined as

$$\tilde{Z}(t) = \int_0^t \tilde{z}_{\tilde{J}(u)} du, \tag{2.2}$$

where $\tilde{z}_b$ and $\tilde{z}_{j^-}$ are given as nonnegative constants, and we let $\tilde{\mathbf{z}} = (\tilde{z}_b \;\; \tilde{z}_{j^-})$.

Define $\tau = \inf\{t > 0 : U(t) \leq 0\}$ and $\tilde{\tau} = \inf\{t > 0 : \tilde{U} \leq 0\}$, which are the first passage times to 0 of U and $\tilde{U}$, respectively. Then, if we let $\tilde{z}_b = 1$ and $\tilde{z}_{j^-} = 0$, then it holds that

$$\tilde{Z}(\tilde{\tau}) = \tau.$$

Furthermore, if we let $\tilde{z}_b = 0$ and $\tilde{z}_{j^-} = 1$, then $\tilde{Z}(\tilde{\tau})$ is the total amount of aggregated claims until ruin of the process U. For notational convenience, we use $\tilde{Z}^{rt}$ and $\tilde{Z}^{cl}$ to denote $\tilde{Z}(\tilde{\tau})$ with $\tilde{\mathbf{z}} = (1 \;\; 0)$ and $\tilde{\mathbf{z}} = (0 \;\; 1)$, respectively.

Letting $\tilde{z}_b = r$ and $\tilde{z}_{j^-} = s$, it holds that

$$\tilde{Z}(\tilde{\tau}) = r\tilde{Z}^{rt} + s\tilde{Z}^{cl},$$

and the expectation $E[e^{-\tilde{Z}(\tilde{\tau})}\ \chi(\tilde{\tau} < \infty)|\tilde{U}(0) = u]$ is equivalent to the joint moment generating function of the ruin time τ and the total amount of aggregated claims until ruin of the process U with respect to transform variables r and s.

The main purpose of this paper is to derive the exact formula of the joint moment generating function which yields the first and second moments of $\tilde{Z}^{rt}$ and $\tilde{Z}^{cl}$.

3. Moments in the diffusion risk process

Define a transform matrix $\tilde{G}(s, a)$ such as: for $i, k \in S = \{b, j^-\}$ and $a \geq 0$,

$$[\tilde{G}(u; r, s)]_{ik} = E[e^{-r\tilde{Z}^{rt} - s\tilde{Z}^{cl}}, \tilde{J}(\tilde{\tau}) = k, \tilde{\tau} < \infty|\tilde{U}(0) = u, \tilde{J}(0) = i].$$

Note that the notation $[A]_{ik}$ is used to denote the (i, k)-th element of a given matrix A. In relation to $\tilde{G}(u; r, s)$, (Ahn (2015)) showed that it has the following exponential form:

$$\tilde{G}(a; r, s) = e^{\tilde{\mathbf{H}}(r,s)u}, \tag{3.1}$$

where $\tilde{\mathbf{H}}(r, s)$ is a 2×2 sub-stochastic matrix, that is, each row-sum is non-positive, and each diagonal element is negative, each off-diagonal element is nonnegative. We provide exact form of the matrix $\tilde{\mathbf{H}}(r, s)$ in the following proposition, for the representation of which, with $a(s) = \mu\sigma^{-1} - (\beta + s)\sigma$ and $b(r) = \sqrt{2r + 2\alpha + \mu^2\sigma^{-2}}$, we define

$$g_1(r, s) = \sqrt{4\left(3[a(s)]^2 + [b(r)]^2\right)^3 - \left(-18[a(s)]^2 b(r) + 2[b(r)]^3 + 27\alpha\beta\sigma\right)^2}, \tag{3.2}$$

$$g_2(r, s) = -18[a(s)]^2 b(r) + 2[b(r)]^3 + 27\alpha\beta\sigma. \tag{3.3}$$

Proposition 3.1 The matrix $\tilde{\mathbf{H}}(r, s)$ is given as

$$\tilde{\mathbf{H}}(r, s) = \begin{pmatrix} [\tilde{\mathbf{H}}(r, s)]_{bb} & [\tilde{\mathbf{H}}(r, s)]_{bj^-} \\ \beta & -\beta - s \end{pmatrix}, \tag{3.4}$$

where

$$\begin{aligned} [\tilde{\mathbf{H}}(r, s)]_{bb} &= \sigma^{-1}[\tilde{\mathbf{\Psi}}^{*(2)}(r, s)]^{*(2)}_{bb} - \sigma^{-2}\mu - \sigma^{-1}\sqrt{2r + 2\alpha + \sigma^{-2}\mu^2}, \\ [\tilde{\mathbf{H}}(r, s)]_{bj^-} &= \sigma^{-1}[\tilde{\mathbf{\Psi}}^{*(2)}(r, s)]^{*(2)}_{bj^-}, \end{aligned}$$

with

$$[\tilde{\mathbf{\Psi}}^{*(2)}(r, s)]^{*(2)}_{bb} = -\frac{1 - \sqrt{3}i}{6\sqrt[3]{2}}\left[g_1(r, s)i + g_2(r, s)\right]^{1/3} - \frac{1 + \sqrt{3}i}{6\sqrt[3]{2}}\left[-g_1(r, s)i + g_2(r, s)\right]^{1/3},$$

$$[\tilde{\mathbf{\Psi}}^{*(2)}(r, s)]^{*(2)}_{bj^-} = 2\beta^{-1}\sigma^{-1}\sqrt{2r + 2\alpha + \mu^2\sigma^{-2}}\ [\tilde{\mathbf{\Psi}}^{*(2)}(r, s)]^{*(2)}_{bb} - \beta^{-1}\sigma^{-1}\left([\tilde{\mathbf{\Psi}}^{*(2)}(r, s)]^{*(2)}_{bb}\right)^2.$$

Proof: In relation to the matrix $\tilde{\mathbf{H}}(r,s)$, we define row vectors $\tilde{\boldsymbol{\Psi}}^*(r,s)$ and $\tilde{\boldsymbol{\Psi}}^{*(1)}(r,s)$ such as:

$$\begin{aligned} \tilde{\boldsymbol{\Psi}}^*(r,s) &= \left(\sigma[\tilde{\mathbf{H}}(r,s)]_{bb} \quad \sigma[\tilde{\mathbf{H}}(r,s)]_{bj^-}\right), \\ \tilde{\boldsymbol{\Psi}}^{*(1)}(r,s) &= \left(-\sigma^{-1}\mu - \sqrt{2r+2\alpha+\sigma^{-2}\mu^2} \quad 0\right). \end{aligned} \tag{3.5}$$

From Theorem 1 of (Ahn, 2015), the matrix $\tilde{\boldsymbol{\Psi}}^{*(2)}(r,s) = \tilde{\boldsymbol{\Psi}}^*(r,s) - \tilde{\boldsymbol{\Psi}}^{*(1)}(r,s)$ is the minimal non-negative solution of the following Riccati equation:

$$A(r,s)Y + YB(r,s) + YC(r,s)Y + D(r,s) = \mathbf{0}, \tag{3.6}$$

where

$$A(r,s) = \mu\sigma^{-2} - \sigma^{-1}\sqrt{2r+2\alpha+\mu^2\sigma^{-2}}, \quad D(r,s) = \begin{pmatrix} 0 & 2\sigma^{-1}\alpha \end{pmatrix},$$
$$B(r,s) = \begin{pmatrix} -\mu\sigma^{-2} - \sigma^{-1}\sqrt{2r+2\alpha+\mu^2\sigma^{-2}} & 0 \\ \beta & -\beta - s \end{pmatrix}, \quad C(r,s) = \begin{pmatrix} \sigma^{-1} \\ 0 \end{pmatrix}.$$

Letting $Y = (y_1, y_2)$, the Riccati equation yields the following simultaneous equation of y_1 and y_2 such as:

$$y_1^3 + \gamma_2(r,s)y_1^2 + \gamma_1(r,s)y_1 - 2\alpha\beta\sigma = 0, \tag{3.7}$$
$$y_2 = 2\beta^{-1}\sigma^{-1}\sqrt{2r+2\alpha+\mu^2\sigma^{-2}}\; y_1 - \beta^{-1}\sigma^{-1}\; y_1^2, \tag{3.8}$$

where

$$\gamma_1(r,s) = 2(2r+2\alpha+\mu^2\sigma^{-2}) - 2(\mu\sigma^{-1} - (\beta+s)\sigma)\sqrt{2r+2\alpha+\mu^2\sigma^{-2}},$$
$$\gamma_2(r,s) = \mu\sigma^{-1} - (\beta+s)\sigma - 3\sqrt{2r+2\alpha+\mu^2\sigma^{-2}}.$$

For simplicity, we temporarily let $a := a(s)$ and $b := b(r)$, and also define the function $f(y)$ such as

$$f(y) = y^3 + \gamma_2(r,s)y^2 + \gamma_1(r,s)y - 2\alpha\beta\sigma = y^3 - (3b-a)y^2 + 2b(b-a)y - 2\alpha\beta\sigma.$$

Then, we can easily verify that $b > a$ irrespective of the sign of a, which yields $f'(0) > 0$, where $f'(y)$ denotes the first derivative of $f(y)$ with respect to y. Furthermore, solving $f'(y) = 0$, we can observe that $f(y)$ has the local maximum at y_1^* and the local minimum at y_2^*, which are given as

$$y_1^* = \frac{3b - a - \sqrt{3b^2+a^2}}{3} \quad \text{and} \quad y_2^* = \frac{3b - a + \sqrt{3b^2+a^2}}{3}. \tag{3.9}$$

In addition, from $b > a$, it holds that

$$0 < y_1^* < 2b.$$

From Theorem 1 of (Ahn, 2015), the simultaneous equations (3.7) and (3.8) should have nonnegative solution. Noting that $f(0) = f(2b) = -2\alpha\beta\sigma < 0$, $f'(0) > 0$, and $y_1^* \neq y_2^*$, the equation $f(y)=0$ should have two different real solutions in $(0,2b)$ and the other in $(2b,\infty)$. Note that $(0,0)$ cannot be a solution of the simultaneous equations. This is satisfied only when $f(y_1^*)>0$, which is equivalent to

$$4\left(3a^2+b^2\right)^3-\left(-18a^2b+2b^3+27c\right)^2>0. \tag{3.10}$$

Under the above condition, the desired minimal solution for y_1 and y_2, denoted by $[\tilde{\boldsymbol{\Psi}}^{*(2)}(r,s)]_{bb}$ and $[\tilde{\boldsymbol{\Psi}}^{*(2)}(r,s)]_{bj-}$, respectively, are given as:

$$\begin{aligned}
&[\tilde{\boldsymbol{\Psi}}^{*(2)}(r,s)]_{bb}\\
&= -\frac{\left(1-\sqrt{3}i\right)\sqrt[3]{\sqrt{4\left(3a^2+b^2\right)^3-\left(-18a^2b+2b^3+27c\right)^2}\; i+\left(-18a^2b+2b^3+27c\right)}}{6\sqrt[3]{2}}\\
&\quad -\frac{\left(1+\sqrt{3}i\right)\sqrt[3]{-\sqrt{4\left(3a^2+b^2\right)^3-\left(-18a^2b+2b^3+27c\right)^2}\; i+\left(-18a^2b+2b^3+27c\right)}}{6\sqrt[3]{2}},
\end{aligned}$$

and

$$[\tilde{\boldsymbol{\Psi}}^{*(2)}(r,s)]_{bj-}=2\beta^{-1}\sigma^{-1}\sqrt{2r+2\alpha+\mu^2\sigma^{-2}}\left([\tilde{\boldsymbol{\Psi}}^{*(2)}(r,s)]_{bb}\right)-\beta^{-1}\sigma^{-1}\left([\tilde{\boldsymbol{\Psi}}^{*(2)}(r,s)]_{bb}\right)^2,$$

where $c=2\alpha\beta\sigma$. For the solution, we refer to Chapter 5 of (Irving (2013)).

Noting $\tilde{\boldsymbol{\Psi}}^{*}(r,s)=\tilde{\boldsymbol{\Psi}}^{*(1)}(r,s)+\tilde{\boldsymbol{\Psi}}^{*(1)}(r,s)$, and also the following relation to be obtained from Theorem 2 of (Ahn (2015)) such as

$$\tilde{\mathbf{H}}(r,s)=\begin{pmatrix}\sigma^{-1}\\ 0\end{pmatrix}\tilde{\boldsymbol{\Psi}}^{*}(r,s)+\begin{pmatrix}0 & 0\\ \beta & -\beta-s\end{pmatrix},$$

we can immediately complete the proof. □

To get the moments of the ruin time and the total amount of aggregated claims until ruin, we need to find the partial derivatives of the transform matrix $\tilde{G}(u;r,s)$ with respect to the transform variables r and s. However, finding the partial derivatives of a matrix exponential function is not as simple as that of a real-valued exponential function, which is due to that commutativity in matrix multiplication does not hold in general. In the following, we introduce a method in (Fung, 2004) to find the derivatives of $\tilde{G}(u;r,s)$ of a matrix exponential form, which uses the matrix exponential of an augmented matrix.

Consider the following system of ordinary differential equations:

$$\frac{d}{dt}\mathbf{y}_t(r,s)=\mathbf{H}(r,s)\mathbf{y}_t(r,s) \tag{3.11}$$

with initial condition $\mathbf{y}_0(r,s)$ at $t=0$. Since $\mathbf{H}(r,s)$ is independent of t, the exact solution at $t=u$ is given by

$$\mathbf{y}_u(r,s)=\tilde{G}(u;r,s)\mathbf{y}_0(r,s) \tag{3.12}$$

with $\tilde{G}(u;r,s)=\exp(\mathbf{H}(r,s)u)$.

For a given function f of (r,s), we define $f^{(r^n)}(r,s) = (\partial^n/\partial r^n)f(r,s)$ and $f^{(r^0)}(r,s) = f(r,s)$. Differentiating Equation (3.11) k times with respect to r, then

$$\begin{aligned} \frac{d}{dt}\mathbf{y}_t^{(r^k)}(r,s) &= \mathbf{H}^{(r^k)}(r,s)\mathbf{y}_t(r,s) + \cdots + \frac{k!}{j!(k-j)!}\mathbf{H}^{(r^j)}(r,s)\mathbf{y}_t^{(r^{k-j})}(r,s) \\ &\quad + \cdots + \mathbf{H}(r,s)\mathbf{y}_t^{(r^k)}(r,s). \end{aligned} \tag{3.13}$$

Letting $\mathbf{z}_t^k(r,s) = \mathbf{y}_t^{(r^k)}(r,s)/k!$ and $\bar{\mathbf{z}}_t(r,s) = [(\mathbf{z}_t^m(r,s))^T, \cdots, (\mathbf{z}_t^0(r,s))^T]^T$ with the superscript T being the transpose operator, the $m+1$ equations in (3.13) with k ranging from 0 to m can be written collectively as

$$\frac{d}{dt}\bar{\mathbf{z}}_t(r,s) = \langle \mathbf{H}(r,s)\rangle_{r^m}\bar{\mathbf{z}}_t(r,s) \tag{3.14}$$

with initial condition $\bar{\mathbf{z}}_0(r,s) = [(\mathbf{y}_0^{(r^m)})^T/m!, \cdots, (\mathbf{y}_0^{(r^0)})^T]^T$ at $t=0$, and $\langle \mathbf{H}(r,s)\rangle_{r^m}$ given as

$$\begin{pmatrix} \mathbf{H}(r,s) & \mathbf{H}^{(r^1)}(r,s) & (1/2!)\mathbf{H}^{(r^2)}(r,s) & \cdots & (1/m!)\mathbf{H}^{(r^m)}(r,s) \\ & \mathbf{H}(r,s) & \mathbf{H}^{(r^1)}(r,s) & \ddots & \vdots \\ & & \mathbf{H}(r,s) & \ddots & (1/2!)\mathbf{H}^{(r^2)}(r,s) \\ & & & \ddots & \mathbf{H}^{(r^1)}(r,s) \\ & & & & \mathbf{H}(r,s) \end{pmatrix}.$$

We note that $\langle \mathbf{H}(r,s)\rangle_{r^m}$ is referred to as the augmented matrix of $\mathbf{H}(r,s)$. This differential equation yields the exact solution for $\bar{\mathbf{z}}_t(r,s)$ at $t=u$ such as

$$\bar{\mathbf{z}}_u(r,s) = \exp(\langle \mathbf{H}(r,s)\rangle_{r^m}u)\bar{\mathbf{z}}_0(r,s). \tag{3.15}$$

On the other hand, differentiating Equation (3.12) k times with respect to r as well, we have

$$\begin{aligned} \mathbf{y}_u^{(r^k)}(r,s) &= \tilde{G}^{(r^k)}(u;r,s)\mathbf{y}_0(r,s) + \cdots + \frac{k!}{j!(k-j)!}\tilde{G}^{(r^j)}(u;r,s)\mathbf{y}_0^{(r^{k-j})}(r,s) \\ &\quad + \cdots + \tilde{G}(u;r,s)\mathbf{y}_0^{(r^k)}(r,s) \end{aligned} \tag{3.16}$$

Hence, by using the definition of $\mathbf{z}_u^k(r,s)$ and $\bar{\mathbf{z}}_u(r,s)$, $\bar{\mathbf{z}}_u(r,s)$ can be written as

$$\bar{\mathbf{z}}_u(r,s) = \langle \tilde{G}(u;r,s)\rangle_{r^m}\bar{\mathbf{z}}_0(r,s), \tag{3.17}$$

where $\langle \tilde{G}(r,s)\rangle_{r^m}$ is given as

$$\begin{pmatrix} \tilde{G}(u;r,s) & \tilde{G}^{(r^1)}(u;r,s) & (1/2!)\tilde{G}^{(r^2)}(u;r,s) & \cdots & (1/m!)\tilde{G}^{(r^m)}(u;r,s) \\ & \tilde{G}(u;r,s) & \tilde{G}^{(r^1)}(u;r,s) & \ddots & \vdots \\ & & \tilde{G}(u;r,s) & \ddots & (1/2!)\tilde{G}^{(r^2)}(u;r,s) \\ & & & \ddots & \tilde{G}^{(r^1)}(u;r,s) \\ & & & & \tilde{G}(u;r,s) \end{pmatrix}.$$

Comparing Equation (3.15) and (3.17), we have

$$\langle \tilde{G}(r,s)\rangle_{r^m} = \exp\left(\langle \mathbf{H}(r,s)\rangle_{r^m} u\right). \tag{3.18}$$

Since we can get the derivatives of $\mathbf{H}(r,s)$ using Proposition 3.1, we can compute the augmented matrix $\langle \mathbf{H}(r,s)\rangle_{r^m}$. Hence, the matrix $\langle \tilde{G}(u;r,s)\rangle_{r^m}$ can be obtained. Similarly, we can get the matrix $\langle \tilde{G}(u;r,s)\rangle_{s^m}$ as well.

4. Examples

We consider the diffusion risk process U with $\alpha = 1$, $\beta = 2$, and $u = 100$, that is, U is given as

$$U(t) = 100 + \mu t + \sigma B(t) - \sum_{i=1}^{N(t)} X_i, \tag{4.1}$$

where B is the standard Brownian motion, N is the Poisson process with intensity 1, and X_i's are independent and exponentially distributed with with mean 1/2. Note that B, N, and $\{X_i\}$ are mutually independent. Then, we determine μ so that the relative security loading θ becomes ± 0.1. The relative security loading θ satisfies

$$1 + \theta = \frac{\mu}{\alpha\beta^{-1}},$$

where μ can be interpreted as the expected total amount of incoming premiums per unit time, and $\alpha\beta^{-1}$ the expected total amount of claims per unit time. From the equation above, we get $\beta = 0.55$ when $\theta = 0.1$, and $\beta = 0.45$ when $\theta = -0.1$. In relation to θ, we note that $P(\tau < \infty) = 1$ or $P(\tau < \infty) < 1$ depending on $\theta \leq 0$ or $\theta > 0$, respectively (See Proposition 2.10 in (Asmussen, 2003)).

With the diffusion parameter σ varying from 0.1 to 100, we compute, using MATLAB, the moments of the ruin time τ of the process U, the moments of the total amount of aggregated claims until ruin, and also the ruin probability, $P(\tau < \infty)$. These quantities can be obtained by using the joint moment generating function which is given as

$$E[e^{-r\tilde{Z}^{rt} - s\tilde{Z}^{cl}} \chi(\tilde{\tau} < \infty)] = (1 \ \ 0)\tilde{G}(u;r,s)\mathbf{1},$$

where $\mathbf{1}$ is the column vector of 1's of appropriate dimension. By definition, the ruin probability $P(\tau < \infty)$ $(= P(\tilde{\tau} < \infty))$ can be obtained by letting $r = s = 0$.

In Table 4.1, we illustrate our computed numbers in the selected cases for possible comparison.

Table 4.1 Ruin probabilities, moments of ruin time and the total amount of aggregated claims until ruin of the diffusion risk process with $\alpha = 1$, $\beta = 0.55$, $\mu = 2$.

σ	Mean(Ruin Time)	Mean (Claim)	Var (Ruin Time)	Var (Claim)	Ruin Prob.
1	5.70	12.24	4339.16	19026.27	0.0109637624
2	19.05	39.14	16189.82	65289.22	0.0355889441
3	54.26	107.31	52692.82	197988.40	0.0996694092
4	110.34	212.66	124456.68	446654.30	0.2012629040
5	174.11	330.03	232626.91	812136.65	0.3167174817

We also provide figures to illustrate the dynamic effect of the diffusion parameter σ on the ruin probability, the moments of the ruin time, and the moments of the total amount of aggregated claims until ruin of the diffusion risk process.

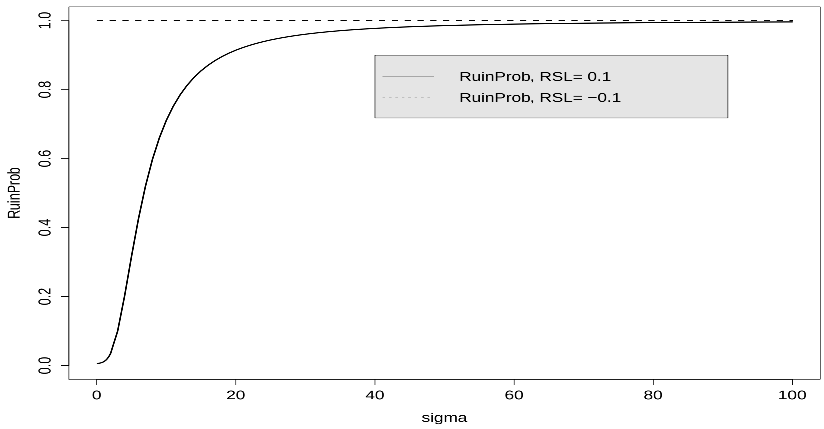

Figure 4.1 Ruin probability of the diffusion risk model.

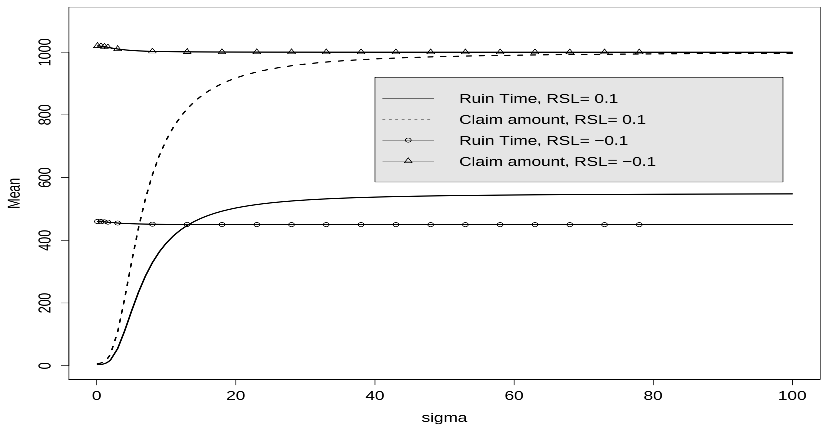

Figure 4.2 Mean of the ruin time and the total claim size until ruin of the diffusion risk model.

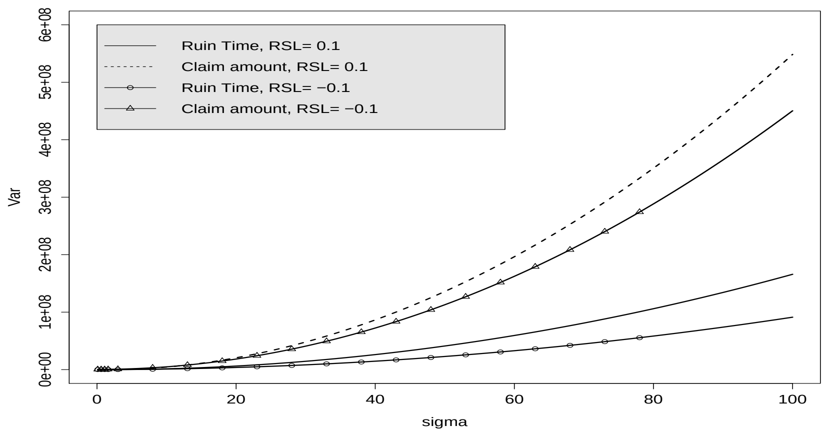

Figure 4.3 Variance of the ruin time and the total claim size until ruin of the diffusion risk model.

5. Concluding remarks

In this paper, we present the exact form of the joint moment generating function of the ruin time and the total amount of aggregated claim sizes until ruin in the diffusion risk process. We also provide a method for the computation of the related first and second moments using the joint moment generating function and the augmented matrix exponential function.

However, if we use the MMBM for modelling a risk process, referred to as the MMBM risk process, it is impossible to get the exact form of the joint moment generating function. Thus, we need a numerical algorithm to get the joint moment generating function of the MMBM risk process, which will be our next study.

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