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ALGEBRAIC SPECTRAL SUBSPACES OF OPERATORS WITH FINITE ASCENT

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Abstract. Algebraic spectral subspaces were introduced by Johnson and Sinclair via a transfinite sequence of spaces. Laursen simplified the definition of algebraic spectral subspace. Algebraic spectral subspaces are useful in automatic continuity theory of intertwining linear operators on Banach spaces. In this paper, we characterize algebraic spectral subspaces of operators with finite ascent. From this characterization we show that if T is a generalized scalar operator, then T has finite ascent.

1. Introduction

Let X be a vector space over the complex plane $\mathbb{C}$, and let $T : X \to X$ be a linear operator on X. The *surjectivity spectrum* $\sigma_{su}(T)$ is defined by

$$\sigma_{su}(T) = \{\lambda \in \mathbb{C} \mid (T - \lambda)X \neq X\}.$$

In [3], the surjectivity spectrum is called the approximate defect spectrum. The surjectivity spectrum is clearly a purely algebraic notion. Nevertheless, in this paper we shall concentrate on bounded linear operators. In this setting it is easy to relate the surjectivity spectrum to the spectrum $\sigma(T)$.

Let $\mathrm{lat}(T)$ denote the collection of T-invariant subspaces of X. Let $Y \in \mathrm{lat}(T)$, $T|Y$ denote the restriction of T on Y.

Definition 1.1. Let T be a linear operator on a vector space X. And let F be a subset of the complex plane $\mathbb{C}$. Then the *algebraic spectral subspace* $E_T(F)$ is defined by

$$E_T(F) = \mathrm{span}\{Y \in \mathrm{lat}(T) \mid \sigma_{su}(T|Y) \subseteq F\}.$$

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It is clear that $E_T(F)$ is the largest T-invariant subspace of X for which the surjectivity spectrum of T is a subset of F. Equivalently, $E_T(F)$ is the largest T-invariant subspace of X on which all restrictions $T - \lambda,\ \lambda \in \mathbb{C} \setminus F$, are surjective. That is,

$$(T - \lambda)E_T(F) = E_T(F) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus F$$

as well so that the set is the largest linear subspace with this property. In the next remark, we collect a number of results on algebraic spectral subspaces. These results are found in [6].

Remark 1.2. (1) By the definition of the algebraic spectral subspace, it is clear that

$$E_T(F_1) \subseteq E_T(F_2) \quad \text{for} \quad F_1 \subseteq F_2 \subseteq \mathbb{C}.$$

(2) Let A be a linear operator on a vector space X with $AT = TA$. For a given subset F of $\mathbb{C}$ and $\lambda \notin F$, $AE_T(F) \subseteq E_T(F)$. That is, $E_T(F)$ is a hyper-invariant subspace of T.

(3) It is easy to see that $E_T(F) = E_T(F \cap \sigma_{su}(T))$.

(4) Note that $E_T(\mathbb{C} \setminus \{\lambda\}) = E_{T-\lambda}(\mathbb{C} \setminus \{0\})$. Indeed,

$$\begin{aligned}
E_{T-\lambda}(\mathbb{C} \setminus \{0\}) &= \operatorname{span}\{Y \in \operatorname{lat}(T) \mid \sigma_{su}(T - \lambda|Y) \subseteq \mathbb{C} \setminus \{0\}\} \\
&= \operatorname{span}\{Y \in \operatorname{lat}(T) \mid \sigma_{su}(T|Y) - \lambda \subseteq \mathbb{C} \setminus \{0\}\} \\
&= \operatorname{span}\{Y \in \operatorname{lat}(T) \mid \sigma_{su}(T|Y) \subseteq \mathbb{C} \setminus \{\lambda\}\} \\
&= E_T(\mathbb{C} \setminus \{\lambda\}).
\end{aligned}$$

(5) If $\{F_\alpha\}$ is a family of subsets of $\mathbb{C}$, then

$$E_T\big(\bigcap_\alpha F_\alpha\big) = \bigcap_\alpha E_T(F_\alpha).$$

Lemma 1.3. *Let T be a linear operator on a vector space X. And let F be a subset of the complex plane $\mathbb{C}$. Then the algebraic spectral subspace $E_T(F)$ is the union of all sets $M \subseteq X$ such that $M \subseteq (T-\lambda)M$, for all $\lambda \in \mathbb{C} \setminus F$.*

Proof. Denote by Z the union of all sets M with the given property. Clearly Z is a linear subspace of X with the property that

$$Z \subseteq (T - \lambda)Z \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus F.$$

On the other hand, applying the operator $T - \lambda$ to both sides of the above inclusion we get

$$(T - \lambda)Z \subseteq (T - \lambda)((T - \lambda)Z) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus F.$$

Hence the set $(T-\lambda)Z$ has the given property, and we have

$$(T-\lambda)Z \subseteq Z \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus F.$$

by the definition of Z. Thus we have shown that $(T-\lambda)Z = Z$ for all $\lambda \in \mathbb{C} \setminus F$. Since $E_T(F)$ is the largest linear subspace of X with this property, we have

$$Z \subseteq E_T(F).$$

But the inclusion $E_T(F) \subseteq Z$ is obvious. Therefore, $E_T(F) = Z$. □

REMARK 1.4. It is clear from the definition that

$$E_T(F) \subseteq \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T-\lambda)^n X.$$

Sometimes the above inclusion becomes in fact an equality. Indeed, if T is a normal operator on a Hilbert space H, then it is known that [12]

$$E_T(F) = \bigcap_{\lambda \notin F} (T-\lambda)X.$$

Another example is that: if $F = \mathbb{C} \setminus \{0\}$ and T is one-to-one, then by the injectivity of T,

$$\bigcap_{n=1}^{\infty} T^n X = T\Big(\bigcap_{n=1}^{\infty} T^n X\Big).$$

By the maximality of $E_T(\mathbb{C} \setminus \{0\})$, we have

$$E_T(\mathbb{C} \setminus \{0\}) = \bigcap_{n=1}^{\infty} T^n X.$$

Moreover, for a bounded linear operator T on a Banach space X which has no eigenvalues, we will show that the inclusion of the Remark 1.4. becomes in fact an equality. For example, since shift operators and Volterra operators have no eigenvalues, the algebraic spectral subspaces of these operators can be represented by the right hand side of the above remark.

PROPOSITION 1.5. *If T is a bounded linear operator on a Banach space X which has no eigenvalues, then*

$$E_T(F) = \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T-\lambda)^n X$$

for any subset F of $\mathbb{C}$.

Proof. Suppose that $\lambda \notin F$. Let $x \in \bigcap_{n\in\mathbb{N}} (T-\lambda)^n X$. For each $n \in \mathbb{N}$, there is a sequence $\{x_n\} \in X$ such that $x = (T-\lambda)^n x_n$. Then $T-\lambda$ is one to one, we have

$$x_1 = (T-\lambda)x_2 = (T-\lambda)^2 x_3 = \cdots .$$

Thus

$$x_1 \in \bigcap_{n\in\mathbb{N}} (T-\lambda)^n X.$$

But $x = (T-\lambda)x_1$ and we get

$$\bigcap_{n\in\mathbb{N}} (T-\lambda)^n X \subseteq (T-\lambda)(\bigcap_{n\in\mathbb{N}} (T-\lambda)^n X).$$

By Lemma 1.3,

$$\bigcap_{n\in\mathbb{N}} (T-\lambda)^n X \subseteq E_T(\mathbb{C}\setminus\{\lambda\}) \quad \text{for all} \quad \lambda \notin F.$$

Since $E_T(\cdot)$ preserves an arbitrary intersection, we have

$$\bigcap_{\lambda\notin F, n\in\mathbb{N}} (T-\lambda)^n X \subseteq \bigcap_{\lambda\notin F} E_T(\mathbb{C}\setminus\{\lambda\}) = E_T(\bigcap_{\lambda\notin F} \mathbb{C}\setminus\{\lambda\}) = E_T(F).$$

Hence we complete the proof. □

Let $L(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X over the complex plane $\mathbb{C}$. And let $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of T, respectively. Given an arbitrary operator $T \in L(X)$, the *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies

$$(T-\lambda)f(\lambda) = x \quad \text{for all} \quad \lambda \in U.$$

The *local spectrum* $\sigma_T(x)$ of T at x is then defined as

$$\sigma_T(x) = \mathbb{C}\setminus\rho_T(x).$$

Clearly, the local resolvent set $\rho_T(x)$ is open, and the local spectrum $\sigma_T(x)$ is closed. For each $x \in X$, the function $f(\lambda) : \rho(T) \to X$ defined by $f(\lambda) = (T-\lambda)^{-1}x$ is analytic on $\rho(T)$ and satisfies

$$(T-\lambda)f(\lambda) = x \quad \text{for all} \quad \lambda \in \rho(T).$$

Hence the resolvent set $\rho(T)$ is always subset of $\rho_T(x)$ and hence $\sigma_T(x)$ is always subset of $\sigma(T)$.

Given an arbitrary operator $T \in L(X)$ and for any set $F \subseteq \mathbb{C}$, we define the *analytic spectral subspace* of T by

$$X_T(F) = \{x \in X \mid \sigma_T(x) \subseteq F\}.$$

In the next remark, we collect a number of results on analytic spectral subspaces. These results can be found in [1].

REMARK 1.6. (1) By the definition of the analytic spectral subspace, it is clear that

$$X_T(F_1) \subseteq X_T(F_2) \quad \text{for} \quad F_1 \subseteq F_2.$$

(2) It is well known that $X_T(F)$ is a hyper-invariant subspace of T.

(3) It is easy to see that

$$X_T(F) = X_T(F \cap \sigma(T)).$$

(4) For all $\lambda \in \mathbb{C} \setminus F$, $(T - \lambda)X_T(F) = X_T(F)$. This implies that

$$X_T(F) \subseteq E_T(F) \quad \text{for all} \quad F \subseteq \mathbb{C}.$$

(5) If $\{F_\alpha\}$ is a family of subsets of $\mathbb{C}$, then

$$X_T\Big(\bigcap_\alpha F_\alpha\Big) = \bigcap_\alpha X_T(F_\alpha).$$

LEMMA 1.7. *Let T be a bounded linear operator on a Banach space X and let $\lambda \in \mathbb{C}$. Then*

$$X_{T-\lambda}(\mathbb{C} \setminus \{0\}) = X_T(\mathbb{C} \setminus \{\lambda\}).$$

Proof. For each $\lambda \in \mathbb{C}$. Let $x \in X_{T-\lambda}(\mathbb{C} \setminus \{0\})$. Then $\sigma_{T-\lambda}(x) \subseteq \mathbb{C} \setminus \{0\}$. Hence $0 \in \rho_{T-\lambda}(x)$. Therefore, there is an open neighborhood U of 0 and an analytic function $f : U \to X$ with

$$(T - \lambda - \mu)f(\mu) = x \quad \text{for all} \quad \mu \in U.$$

Let $V = U + \lambda$. Then V is an open neighborhood of λ. And define $g : V \to X$ by $g(z) = f(z - \lambda)$ for all $z \in V$. Then for each $z \in V$ there is a unique $\mu \in U$ such that $z = \mu + \lambda$. Then

$$\begin{aligned}(T - z)g(z) &= (T - \mu - \lambda)g(\mu + \lambda) \\ &= (T - \lambda - \mu)f(\mu) \\ &= x\end{aligned}$$

for all $z \in V$. Hence $V \subseteq \rho_T(x)$. Therefore, we have

$$x \in X_T(\mathbb{C} \setminus \{\lambda\}).$$

Similarly the converse inclusion is also true. □

PROPOSITION 1.8. *Let T be a bounded linear operator on a Banach space X. If F is a subset of $\mathbb{C}$, then*

$$\bigcup_{\lambda\in F, n\in\mathbb{N}} \ker(T-\lambda)^n \subseteq X_T(F) \subseteq E_T(F) \subseteq \bigcap_{\lambda\notin F, n\in\mathbb{N}} (T-\lambda)^n X.$$

Proof. It is only to show that

$$\bigcup_{\lambda\in F, n\in\mathbb{N}} \ker(T-\lambda)^n \subseteq X_T(F).$$

For each $\lambda \in F$ and $n \in \mathbb{N}$, let $x \in \ker(T-\lambda)^n$. Then $(T-\lambda)^n x = 0$. Define the function $f : \mathbb{C}\setminus\{\lambda\} \to X$ by

$$f(\mu) = -\frac{1}{(\mu-\lambda)^n}((T-\lambda)^{n-1} + (\mu-\lambda)(T-\lambda)^{n-2} + \cdots + (\mu-\lambda)^{n-1})x$$

for all $\mu \in \mathbb{C}\setminus\{\lambda\}$. Then clearly f is an analytic function on $\mathbb{C}\setminus\{\lambda\}$. And

$$\begin{aligned}(T-\mu)f(\mu) &= ((T-\lambda)-(\mu-\lambda))f(\mu)\\ &= x\end{aligned}$$

for all $\mu \in \mathbb{C}\setminus\{\lambda\}$. Hence $\mathbb{C}\setminus\{\lambda\} \subseteq \rho_T(x)$. Thus we have

$$\sigma_T(x) \subseteq \{\lambda\} \subseteq F.$$

Hence $x \in X_T(F)$. Therefore we have,

$$\ker(T-\lambda)^n \subseteq X_T(F)$$

for all $\lambda \in F$, $n \in \mathbb{N}$. This completes the proof. □

2. Spectral subspaces of operators with finite ascent

We denote by $C^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions $\varphi(z)$, $z = x_1 + ix_2$, $x_1, x_2 \in \mathbb{R}$, defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. That is, with the topology generated by a family of pseudo-norm

$$|\varphi|_{K,m} = \max_{|p|\le m} \sup_{z\in K} |D^p\varphi(z)|,$$

where K is an arbitrary compact subset of $\mathbb{C}$, m a non-negative integer, $p = (p_1, p_2)$, $p_1, p_2 \in \mathbb{N}$, $|p| = p_1 + p_2$ and

$$D^p\varphi = \frac{\partial^{|p|}\varphi}{\partial {x_1}^{p_1} \partial {x_2}^{p_2}}, \quad (z = x_1 + ix_2).$$

An operator $T \in L(X)$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to L(X)$ satisfying $\Phi(1) = I$, the identity operator on X, and $\Phi(z) = T$ where z denotes the identity function on $\mathbb{C}$. Such a continuous function Φ is in fact an operator valued distribution and it is called a *spectral distribution* for T. The class of generalized scalar operators was introduced by [1]. Every linear operator on a finite dimensional space as well as every spectral operator of finite type are generalized scalar operators.

Definition 2.1. An operator T on a Banach space X is said to have *finite ascent* if for any $\lambda \in \mathbb{C}$ there is an $n \in \mathbb{N}$ such that

$$\ker(T-\lambda)^n = \ker(T-\lambda)^{n+1}.$$

Proposition 2.2. *Let T be a bounded linear operator on a Banach space X. If T has finite ascent, then*

$$E_T(F) = \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T-\lambda)^n X$$

for any subset F of $\mathbb{C}$.

Proof. Since $E_T(\cdot)$ preserves arbitrary intersection,

$$E_T(F) = \bigcap_{\lambda \notin F} E_T(\mathbb{C} \setminus \{\lambda\}) \quad \text{for all} \quad F \subseteq \mathbb{C}.$$

By Remark 1.2, $E_T(\mathbb{C} \setminus \{\lambda\}) = E_{T-\lambda}(\mathbb{C} \setminus \{0\})$. Hence it is enough to show that for each $\lambda \in \mathbb{C}$,

$$E_{T-\lambda}(\mathbb{C} \setminus \{0\}) = \bigcap_{n=1}^{\infty} (T-\lambda)^n X.$$

Let $\lambda \in \mathbb{C}$ be given. Then by the assumption, there is a $p \in \mathbb{N}$ such that $\ker(T-\lambda)^p = \ker(T-\lambda)^{p+1}$. Let

$$Y = \bigcap_{n=1}^{\infty} (T-\lambda)^n X.$$

Then we shall show that

$$E_{T-\lambda}(\mathbb{C} \setminus \{0\}) \subseteq Y \subseteq E_{(T-\lambda)^p}(\mathbb{C} \setminus \{0\}) \subseteq E_{T-\lambda}(\mathbb{C} \setminus \{0\}).$$

The inclusion $E_{T-\lambda}(\mathbb{C} \setminus \{0\}) \subseteq Y$ is obvious from the definition of $E_{T-\lambda}(\mathbb{C} \setminus \{0\})$ and Y. Let

$$Z = E_{(T-\lambda)^p}(\mathbb{C} \setminus \{0\}).$$

Since Z is a hyper-invariant subspace of $(T-\lambda)^p$,

$$Z \supseteq (T-\lambda)Z \supseteq (T-\lambda)^2 Z \supseteq \cdots \supseteq (T-\lambda)^p Z = Z.$$

Hence $(T-\lambda)Z = Z$. By the maximality of $E_{T-\lambda}(\mathbb{C}\setminus\{0\})$, we have

$$Z \subseteq E_{T-\lambda}(\mathbb{C}\setminus\{0\}).$$

To show that $Y \subseteq E_{(T-\lambda)^p}(\mathbb{C}\setminus\{0\})$, Observe first that

$$Y = \bigcap_{n=1}^{\infty} ((T-\lambda)^p)^n X$$

and second that

$$\ker(T-\lambda)^p = \ker((T-\lambda)^p)^2.$$

Thus without loss of generality we may assume that $p = 1$, so that $\ker(T-\lambda) = \ker(T-\lambda)^2$.

Let $x \in Y$. Then there is a sequence $\{y_n\}$ in X such that

$$x = (T-\lambda)^n y_n \quad \text{for} \quad n = 1, 2, \cdots,$$

and

$$(T-\lambda)^2((T-\lambda)^{n-2}y_n - (T-\lambda)^{n-1}y_{n+1}) = 0 \quad \text{for} \quad n = 2, 3, \cdots.$$

Hence

$$(T-\lambda)^{n-2}y_n - (T-\lambda)^{n-1}y_{n+1} \in \ker(T-\lambda)^2 = \ker(T-\lambda).$$

Therefore, we have

$$(T-\lambda)((T-\lambda)^{n-2}y_n - (T-\lambda)^{n-1}y_{n+1}) = 0,$$

and

$$(T-\lambda)^{n-1}y_n = (T-\lambda)^n y_{n+1} \quad \text{for} \quad n = 2, 3, \cdots.$$

Let $y = (T-\lambda)y_2$. Then clearly $y \in Y$, and

$$\begin{aligned} (T-\lambda)y &= (T-\lambda)(T-\lambda)^{n-1}y_n \\ &= (T-\lambda)^n y_n \\ &= x. \end{aligned}$$

So $x \in (T-\lambda)Y$ and hence $Y \subseteq (T-\lambda)Y$. Since Y is invariant under $T-\lambda$, $(T-\lambda)Y \subseteq Y$. Therefore, we have

$$(T-\lambda)Y = Y$$

By the maximality of $E_{T-\lambda}(\mathbb{C}\setminus\{0\})$, we have

$$Y \subseteq E_{T-\lambda}(\mathbb{C}\setminus\{0\}).$$

This completes the proof. $\square$

Theorem 2.3. *If T is a generalized scalar operator on a Banach space X, then T has finite ascent.*

Proof. For each $\lambda \in \mathbb{C}$. By Proposition 1.8.,

$$\ker(T-\lambda)^n \subseteq X_T(\{\lambda\}) \subseteq E_T(\{\lambda\})$$

for all $n \in \mathbb{N}$.

On the other hand, in [11] Vrbová proved that if T is a generalized scalar operator, then there is a $p \in \mathbb{N}$ such that

$$X_T(F) = \bigcap_{\mu \notin F} (T-\mu)^p X \quad \text{for all} \quad F \subseteq \mathbb{C}.$$

Therefore, we have

$$\begin{aligned}
(T-\lambda)^p E_T(\{\lambda\}) &\subseteq (T-\lambda)^p \bigcap_{\mu \neq \lambda} (T-\mu)^p X \\
&\subseteq \bigcap_{\mu \in \mathbb{C}} (T-\mu)^p X \\
&= X_T(\emptyset) \\
&= \{0\}.
\end{aligned}$$

Hence $E_T(\{\lambda\}) \subseteq \ker(T-\lambda)^p$. Therefore, we have

$$\ker(T-\lambda)^p = \ker(T-\lambda)^{p+1}.$$

Hence T has finite ascent. This completes the proof. □

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