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SOME OPTIMAL METHODS WITH EIGHTH-ORDER CONVERGENCE FOR THE SOLUTION OF NONLINEAR EQUATIONS

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Abstract. In this paper we propose a new family of eighth order optimal methods for solving nonlinear equations by using weight function methods. The methods of the family require three function and one derivative evaluations per step and has order of convergence eight, and so they are optimal in the sense of Kung-Traub hypothesis. Precise analysis of convergence is given. Some members of the family are compared with several existing methods to show their performance and as a result to confirm that our methods are as competitive as compared to them.

1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. To solve nonlinear equations, iterative methods such as Newton's method are usually used. Throughout this paper we consider iterative methods to find a simple root ξ, i.e., $f(\xi) = 0$ and $f'(\xi) \neq 0$, of a nonlinear equation $f(x) = 0$, where $f : I \subset \mathbf{R} \to \mathbf{R}$ for an open interval I.

Newton's method for the calculation of ξ is the most widely used iterative method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.1}$$

It is well known (see e.g. Traub [1]) that this method is quadratically convergent.

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Many modified methods of the Newton's method have been proposed to improve the convergence order and computational efficiency, which is of practical importance, in recent years, see [2] and references therein. This was mainly motivated by the aim to achieve as high as possible order of convergence when using a fixed number of function evaluations per step, which is closely related to the optimal order of convergence in the conjecture introduced by Kung and Traub in 1974 [3]. A method using d evaluations is optimal if the order is 2^{d-1}. Newton's method (1.1) is optimal of order two. Optimal methods of order four were discussed in [2]. Optimal methods of order eight have been suggested and compared in the literature, see e.g. the books by Ostrowski [4], Traub [1] and Neta [2]. See also more results by Kim [5, 6], Cordero et al. [7, 8], and J. Džunić et al. [9, 10] who discussed a wide collection of eighth order methods.

In this paper, we develop a new family of optimal eighth-order methods based on weight functions approach. Each member of the family requires three evaluations of the function and one evaluation of its first derivative per iteration which is thus optimal in the sense of Kung-Traub conjecture. In the next section the family of methods is constructed and it is then proved that the family is of order eight. Numerical comparisons are made with several other existing eighth-order methods by experimenting with as many as 500 initial points and through numerical computations to demonstrate the efficiency and the performance of the presented methods.

2. A new family of optimal eighth-order methods

We consider here the new family of methods of the form

$$\begin{cases} y_n & = & x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n & = & x_n - q(r_n)\,\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} & = & z_n - \psi(r_n, t_n)\,\dfrac{f(z_n)}{f'(x_n)}\,, \end{cases} \tag{2.1}$$

where $r_n = \dfrac{f(y_n)}{f(x_n)}$, $t_n = \dfrac{f(z_n)}{f(y_n)}$, and $q(t)$ and $\psi(t, s)$ are real-valued weight functions to be determined later.

For the methods defined by (2.1), we have the following analysis of convergence.

THEOREM 2.1. *Let $\xi \in I$ be a simple zero in an open interval I of a sufficiently differentiable function $f : I \to \mathbb{R}$. Let $e_n = x_n - \xi$. Then the new family of methods defined by (2.1) is of optimal eighth-order when*

$$q(0)=1, q'(0)=1, q''(0)=4, \psi(0,0)=1, \psi_{r_n}(0,0)=2, \psi_{t_n}(0,0)=1,$$
$$\psi_{r_nt_n}(0,0)=4, \psi_{r_nr_n}(0,0)=2+(1/3)q^{(3)}(0), q^{(4)}(0)=-8q^{(3)}(0)+96.$$

Proof. Let $e_n = x_n - \xi, e_n^y = y_n - \xi$, $e_n^z = z_n - \xi$ and $q^{(3)}(0) = \theta$. Using the Taylor expansion of $f(x)$ around $x = \xi$ and taking $f(\xi) = 0$ into account, we get

$$f(x_n) = f'(\xi)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9)\right], \tag{2.2}$$

and

$$f'(x_n) = f'(\xi)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 + O(e_n^8)\right]. \tag{2.3}$$

Dividing (2.2) by (2.3) gives

$$\begin{aligned} u_n &= \frac{f(x_n)}{f'(x_n)} \\ &= e_n - c_2e_n^2 + (-2c_3 + 2c_2^2)e_n^3 + (-3c_4 + 7c_2c_3 - 4c_2^3)e_n^4 \\ &\quad +(10c_2c_4 - 4c_5 + 6c_3^2 - 20c_3c_2^2 + 8c_2^4)e_n^5 \\ &\quad +(17c_4c_3 - 28c_4c_2^2 + 13c_2c_5 - 5c_6 - 33c_2c_3^2 + 52c_3c_2^3 \\ &\quad -16c_2^5)e_n^6 + (-92c_3c_2c_4 + 22c_3c_5 - 18c_3^3 + 126c_3^2c_2^2 \\ &\quad -128c_3c_2^4 + 12c_4^2 + 72c_4c_2^3 - 36c_5c_2^2 - 6c_7 + 16c_2c_6 + 32c_2^6)e_n^7 \\ &\quad +(-7c_8 - 118c_5c_2c_3 + 348c_4c_3c_2^2 + 19c_2c_7 - 64c_2c_4^2 + 31c_4c_5 \\ &\quad -75c_4c_3^2 - 176c_4c_2^4 + 92c_5c_2^3 + 27c_6c_3 - 44c_6c_2^2 + 135c_2c_3^3 \\ &\quad -408c_3^2c_2^3 + 304c_3c_2^5 - 64c_2^7)e_n^8 + O(e_n^9), \end{aligned} \tag{2.4}$$

from which, we get

$$\begin{aligned} e_n^y &= c_2e_n^2 - (2c_2^2 - 2c_3)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 \\ &\quad +(-10c_2c_4 + 4c_5 - 6c_3^2 + 20c_3c_2^2 - 8c_2^4)e_n^5 + (-17c_4c_3 \\ &\quad +28c_4c_2^2 - 13c_2c_5 + 5c_6 + 33c_2c_3^2 - 52c_3c_2^3 + 16c_2^5)e_n^6 \\ &\quad +(92c_3c_2c_4 - 22c_3c_5 + 18c_3^3 - 126c_3^2c_2^2 + 128c_3c_2^4 \\ &\quad -12c_4^2 - 72c_4c_2^3 + 36c_5c_2^2 + 6c_7 - 16c_2c_6 - 32c_2^6)e_n^7 \\ &\quad +(64c_2^7 + 7c_8 + 118c_5c_2c_3 - 348c_4c_3c_2^2 - 19c_2c_7 \\ &\quad +64c_2c_4^2 - 31c_4c_5 + 75c_4c_3^2 + 176c_4c_2^4 - 92c_5c_2^3 \\ &\quad -27c_6c_3 + 44c_6c_2^2 - 135c_2c_3^3 + 408c_3^2c_2^3 \\ &\quad -304c_3c_2^5)e_n^8 + O(e_n^9). \end{aligned} \tag{2.5}$$

Writing the Taylor's expansion for $f(y_n)$ and using (2.5), we obtain

(2.6)

$$
\begin{aligned}
f(y_n) &= f'(\xi)[e_n^y + c_2(e_n^y)^2 + c_3(e_n^y)^3 + c_4(e_n^y)^4 + O((e_n^y)^5)] \\
&= f'(\xi)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 \\
&\quad +(-10c_2c_4 + 4c_5 - 6c_3^2 + 24c_3c_2^2 - 12c_2^4)e_n^5 + (-17c_4c_3 \\
&\quad +34c_4c_2^2 - 13c_2c_5 + 5c_6 + 37c_2c_3^2 - 73c_3c_2^3 + 28c_2^5)e_n^6 \\
&\quad +(18c_3^3 - 64c_2^6 + 6c_7 + 104c_3c_2c_4 - 16c_2c_6 - 22c_3c_5 \\
&\quad -160c_3^2c_2^2 + 206c_3c_2^4 - 12c_4^2 - 104c_4c_2^3 + 44c_5c_2^2)e_n^7 \\
&\quad +(144c_2^7 + 7c_8 + 134c_5c_2c_3 - 455c_4c_3c_2^2 - 19c_2c_7 + 73c_2c_4^2 \\
&\quad -31c_4c_5 + 75c_4c_3^2 + 297c_4c_2^4 - 134c_5c_2^3 - 27c_6c_3 \\
&\quad +54c_6c_2^2 - 147c_2c_3^3 + 582c_3^2c_2^3 - 552c_3c_2^5)e_n^8 + O(e_n^9)].
\end{aligned}
$$

Dividing (2.6) by (2.2) gives

(2.7)

$$
\begin{aligned}
r_n = \frac{f(y_n)}{f(x_n)} &= c_2e_n + (2c_3 - 3c_2^2)e_n^2 + (3c_4 - 10c_2c_3 + 8c_2^3)e_n^3 \\
&\quad +(-14c_2c_4 + 4c_5 - 8c_3^2 + 37c_3c_2^2 - 20c_2^4)e_n^4 \\
&\quad +(-22c_3c_4 + 51c_4c_2^2 - 18c_2c_5 + 55c_2c_3^2 - 118c_3c_2^3 \\
&\quad +5c_6 + 48c_2^5)e_n^5 + (6c_7 + 150c_4c_2c_3 - 22c_2c_6 \\
&\quad -15c_4^2 - 163c_4c_2^3 - 28c_5c_3 + 65c_5c_2^2 - 252c_3^2c_2^2 \\
&\quad +344c_3c_2^4 + 26c_3^3 - 112c_2^6)e_n^6 + O(e_n^7).
\end{aligned}
$$

Using $q(0) = 1$, $q'(0) = 1$, $q''(0) = 4$, $q^{(3)}(0) = \theta$, $q^{(4)}(0) = 96 - 8\theta$ in Taylor's expansion of $q(r_n)$ about 0 gives

$$
q(r_n) = 1 + r_n + 2r_n^2 + \frac{\theta}{6}r_n^3 + \frac{96 - 8\theta}{24}r_n^4 + O(r_n^5). \tag{2.8}
$$

This then yields

$$
\begin{aligned}
e_n^z &= e_n - q(r_n)u_n \\
&= ze_4e_n^4 + ze_5e_n^5 + ze_6e_n^6 + ze_7e_n^7 + ze_8e_n^8 + O(e_n^9),
\end{aligned} \tag{2.9}
$$

where

(2.10)

$$
\begin{aligned}
ze_4 &= (5 - (1/6)\theta)c_2^3 - c_2c_3, \\
ze_5 &= -2c_3^2 + (-\theta + 32)c_2^2c_3 + (2\theta - 40)c_2^4 - 2c_2c_4, \\
ze_6 &= (-7c_3 + (48 - (3/2)\theta)c_2^2)c_4 + (-2\theta + 66)c_2c_3^2 + (15\theta - 294)c_2^3c_3 \\
&\quad +(-(44/3)\theta + 222)c_2^5 - 3c_2c_5, \\
ze_7 &= (-10c_3 + (-2\theta + 64)c_2^2)c_5 - 6c_4^2 + ((22\theta - 424)c_2^3 + (-6\theta196)c_2c_3)c_4
\end{aligned}
$$

$$
\begin{aligned}
&+(44-(4/3)\theta)c_3^3+(42\theta-800)c_2^2c_3^2+(1928-(394/3)\theta)c_2^4c_3\\
&+(84\theta-1060)c_2^6-4c_2c_6,\\
ze_8 \;=\;& ((80-(5/2)\theta)c_2^2-13c_3)c_6+((260-8\theta)c_2c_3+(-555+29\theta)c_2^3\\
&-17c_4)c_5+(145-(9/2)\theta)c_2c_4^2+((-(1139/6)\theta+2749)c_2^4\\
&+(-2297+123\theta)c_2^2c_3+(-6\theta+194)c_3^2)c_4+((2615/3)\theta-10726)c_2^5c_3\\
&-5c_2c_7+(-955+52\theta)c_2c_3^3+(-(935/2)\theta+6640)c_2^3c_3^2\\
&+(4640-(823/2)\theta)c_2^7,
\end{aligned}
$$

so that, after elementary calculation,

$$
\begin{aligned}
f(z_n) &= f'(\xi)[e_n^z+c_2(e_n^z)^2+O((e_n^z)^3)]\\
&= f'(\xi)[fz_4e_n^4+fz_5e_n^5+fz_6e_n^6+fz_7e_n^7+O(e_n^8)],
\end{aligned}
\tag{2.11}
$$

where

(2.12)

$$
\begin{aligned}
fz_4 &= (5-(1/6)\theta)c_2^3-c_2c_3,\\
fz_5 &= -2c_3^2+(-\theta+32)c_2^2c_3+(2\theta-40)c_2^4-2c_2c_4,\\
fz_6 &= (-2\theta+66)c_2c_3^2+((15\theta-294)c_2^3-7c_4)c_3+(-(44/3)\theta+222)c_2^5\\
&\quad +(48-(3/2)\theta)c_4c_2^2-3c_2c_5,\\
fz_7 &= (44-(4/3)\theta)c_3^3+(42\theta-800)c_2^2c_3^2+((-6\theta+196)c_4c_2-10c_5\\
&\quad +(1928-(394/3)\theta)c_2^4)c_3+(84\theta-1060)c_2^6-4c_2c_6\\
&\quad +(22\theta-424)c_4c_2^3+(-2\theta+64)c_5c_2^2-6c_4^2.
\end{aligned}
$$

An easy calculation then produces

(2.13)

$$
t_n \;=\; \frac{f(z_n)}{f(y_n)} \;=\; T_2e_n^2+T_3e_n^3+T_4e_n^4++T_5e_n^5+T_6e_n^6+O(e_n^7)
$$

where

(2.14)

$$
\begin{aligned}
T_2 &= (5-(1/6)\theta)c_2^2-c_3,\\
T_3 &= (20-(2/3)\theta)c2c_3+((5/3)\theta-30)c_2^3-2c_4,\\
T_4 &= (29-\theta)c2c_4+(-(2/3)\theta+19)c_3^2+(-154+(55/6)\theta)c_2^2c_3\\
&\quad +(-(21/2)\theta+137)c_2^4-3c_5,\\
T_5 &= (38-(4/3)\theta)c_2c_5+((-216+(40/3)\theta)c_2^2+(54-2\theta)c_3)c_4\\
&\quad +((50/3)\theta-260)c_2c_3^2+(-73\theta+904)c_2^3c_3-4c_6+((158/3)\theta-576)c_2^5,\\
T_6 &= ((-279+(35/2)\theta)c_2^2+(70-(8/3)\theta)c_3)c_5+(38-(3/2)\theta)c_4^2
\end{aligned}
$$

$$+((1267-(313/3)\theta)c_2^3+(-725+(145/3)\theta)c_2c_3)c_4+(10\theta-145)c_3^3$$
$$+(2222-(377/2)\theta)c_2^2c_3^2+(-4694+(2647/6)\theta)c_2^4c_3-5c_7$$
$$+(-(692/3)\theta+2328+(1/36)\theta^2)c_2^6+(47-(5/3)\theta)c_6c_2,$$

and

$$\frac{f(z_n)}{f'(x_n)} = A_4e_n^4+A_5e_n^5+A_6e_n^6+A_7e_n^7+O(e_n^8), \tag{2.15}$$

where
(2.16)

$$\begin{aligned}
A_4 &= (5-(1/6)\theta)c_2^3-c_2c_3,\\
A_5 &= -2c_3^2+(-\theta+34)c_2^2c_3+((7/3)\theta-50)c_2^4-2c_2c_4,\\
A_6 &= (-7c_3+(-(3/2)\theta+52)c_2^2)c_4+(73-2\theta)c_2c_3^2+(-377+(35/2)\theta)c_2^3c_3\\
&\quad +(322-(58/3)\theta)c_2^5-3c_2c_5,\\
A_7 &= (-10c_3+(-2\theta+70)c_2^2)c_5-6c_4^2+(((77/3)\theta-548)c_2^3\\
&\quad +(220-6\theta)c_2c_3)c_4+(-(4/3)\theta+50)c_3^3+(-1048+49\theta)c_2^2c_3^2\\
&\quad +(-(520/3)\theta+2832)c_2^4c_3+((368/3)\theta-1704)c_2^6-4c_2c_6.
\end{aligned}$$

Using $\psi(0,0)=1, \psi_r(0,0)=2, \psi_t(0,0)=1, \psi_{r,t}(0,0)=4, \psi_{r,r}=2+\frac{1}{3}\theta$ in Taylor's expansion of $\psi(r_n,t_n)$ about $(0,0)$ gives
(2.17)

$$\psi(r_n,t_n) = \frac{1}{2}(2+\frac{1}{3}\theta)r_n^2+4r_nt_n+\frac{1}{2}\psi_{tt}(0,0)t_n^2+2r_nt_n+1+O(r_n^3,t_n^3).$$

Therefore, from (2.9), (2.15) and (2.17) with r_n and t_n replaced with (2.7) and (2.13), respectively, we obtain

$$\begin{aligned}
e_{n+1} &= e_n^z-\psi(r_n,t_n)\frac{f(z_n)}{f'(x_n)}\\
&= E_8e_n^8+O(e_n^9),
\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
E_8 &= \tfrac{1}{432}c_2[6c_3+(\theta-30)c_2^2][\psi_{tt}(0,0)(6c_3+(\theta-30)c_2^2)^2\\
&\quad +12(-6c_3^2-6c_2c_4-(\theta-114)c_2^2c_3+(13\theta-318)c_2^4)],
\end{aligned} \tag{2.19}$$

which completes the proof. □

By Theorem 2.1, our iteration scheme (2.1) with weight functions q and ψ satisfying the conditions

$$\begin{aligned}
&q(0)=1, q'(0)=1, q''(0)=4, q^{(3)}(0)=\theta, q^{(4)}(0)=-8\theta+96,\\
&\psi(0,0)=1, \psi_r(0,0)=2, \psi_t(0,0)=1, \psi_{rt}(0,0)=4,\\
&\psi_{rr}(0,0)=2+(1/3)\theta,
\end{aligned} \tag{2.20}$$

where θ is arbitrary number, produces a family of optimal eighth-order methods. Such q and ψ can take many forms.

For the weight q we consider the rational function form given by

$$q(t) \;=\; \frac{(-12\theta+144+\theta^2)t^2+(-30\theta+288)t-6\theta+144}{(24\theta-288+\theta^2)t^2+(-24\theta+144)t-6\theta+144}. \tag{2.21}$$

For the weight ψ we consider two forms, one, which is given by

$$\psi(r,t) \;=\; -\frac{6(1+(\lambda+2)r)}{(12\lambda+6+\theta)r^2+(-6\lambda+6\lambda t+12t)r-6+6t}, \tag{2.22}$$

and, the other, which is given by

$$\psi(r,t) \;=\; -\frac{12+18r-r\theta}{(-6-18t+t\theta-\theta)r+12+12\lambda t^2-12t}. \tag{2.23}$$

We can easily check that one combination of q with (2.21) and ψ with (2.22), and the other of q with (2.21) and ψ with (2.23) satisfy (2.20), so the first one is denoted OM1, and the second OM2.

In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many multipoint iterative methods have fixed points that are not zeros of the function of interest called extraneous fixed points. The parameters θ and λ can be chosen to position the extraneous fixed points on the imaginary axis or, at least, close to that axis.

In order to find the extraneous fixed points, we rewrite the methods of interest in the form

$$x_{n+1} \;=\; x_n - \frac{f(x_n)}{f'(x_n)}\, H_f(x_n, y_n, z_n), \tag{2.24}$$

where the function H_f for method (2.1) is given by

$$H_f(x_n, y_n, z_n) \;=\; q(r_n) + \psi(r_n, t_n)\, \frac{f(z_n)}{f(x_n)} \tag{2.25}$$

with $q(t)$ and $\psi(r,t)$ are given in (2.21) and (2.22) or (2.23), respectively

To choose the parameters in the methods, the following criterion can be used, which was developed in [11] and is defined below.

Let $E = \{z_1, z_2, ..., z_{n_{\theta,\lambda}}\}$ be the set of the extraneous fixed points corresponding to the values given to θ and λ. We define

$$d(\theta, \lambda) = \max_{z_i \in E} |Re(z_i)|. \tag{2.26}$$

We have looked for the parameters θ and λ which attain the minimum of d(θ, λ). This minimum occurs at $\theta = 9.1, \lambda = -4$ for method OM1, and at $\theta = 8.6, \lambda = -0.3$ for method OM2.

3. Numerical examples

In the this section we give a numerical comparison of our methods OM1, OM2 with other well known optimal eighth order methods. For this purpose, we shall consider the following nonlinear equations.

$f_1(x) = e^x \sin(x) + \log(x^2+1)$;[12]	$\xi = 0$
$f_2(x) = x^6 - x^4 - x^3 - 1$; [13]	$\xi = 1.403602124874216,\ \xi = -1$
$f_3(x) = e^x - 4x^2$; [14]	$\xi = 0.714805912362777$
$f_4(x) = \tan^{-1}(x) - x + 1$; [15]	$\xi = 2.132267725272885$
$f_5(x) = e^{-x} + \cos(x)$; [16]	$\xi = 1.746139530408012$

The following optimal eighth-order methods, which are considered in [17] for numerical experiments, are considered for the comparison.

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\[2ex] z_n = y_n - \left[\dfrac{f(x_n)}{f(x_n) - 2f(y_n)}\right] \dfrac{f(y_n)}{f'(x_n)}, \\[2ex] x_{n+1} = z_n + \dfrac{f(x_n)f(z_n)\left(f(x_n) + 2f(z_n)\right)\left(f(y_n) + f(z_n)\right)}{f'(x_n)f(y_n)\left(2f(x_n)f(y_n) - f(x_n)^2 + f(y_n)^2\right)}, \end{cases}$$

$$
\begin{cases}
u_n = x_n + \alpha f(x_n), \ \alpha \in \mathbb{R}, \\[2ex]
y_n = x_n - \dfrac{\alpha f(x_n) f(x_n)}{f(u_n) - f(x_n)}, \\[2ex]
z_n = y_n - \dfrac{f(y_n)}{-\frac{f(u_n)(x_n - y_n)}{\alpha f(x_n)(\alpha f(x_n) + x_n - y_n)} + \frac{\alpha f(x_n) + x_n - y_n}{\alpha(x_n - y_n)} - \frac{f(y_n)(\alpha f(x_n) + 2x_n - 2y_n)}{(x_n - y_n)(\alpha f(x_n) + x_n - y_n)}}, \\[2ex]
x_{n+1} = z_n - \dfrac{f(z_n)(u_n - x_n)(u_n - y_n)(u_n - z_n)(x_n - y_n)(x_n - z_n)(y_n - z_n)}{a_1 - a_2 f(z_n)(u_n - x_n)(u_n - y_n)(x_n - y_n)},
\end{cases}
$$

where $a_1 = f(y_n)(u_n - x_n)(u_n - z_n)^2(x_n - z_n)^2 + f(y_n)(u_n - x_n)(u_n - z_n)^2(x_n - z_n)^2 + (y_n - z_n)^2(f(u_n)(x_n - y_n)(x_n - z_n)^2 - f(x_n)(u_n - y_n)(u_n - z_n)^2)$, $a_2 = (u_n(x_n + y_n - 2z_n) + x_n(y_n - 2z_n) + z_n(3z_n - 2y_n))$,

$$
\begin{cases}
y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\[2ex]
z_n = y_n - \left[\dfrac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}\right] \dfrac{f(y_n)}{f'(x_n)}, \\[2ex]
x_{n+1} = z_n - \dfrac{f(z_n)}{2f[z_n,\ x_n] - f'(x_n)} \left[1 + \dfrac{f(z_n)}{f(y_n)} + \left(\dfrac{f(y_n)}{f'(x_n)}\right)^3 - \dfrac{2f(z_n)}{f'(x_n)} \right. \\[2ex]
\qquad \left. - \dfrac{31}{4}\left(\dfrac{f(y_n)}{f(x_n)}\right)^4 - \dfrac{3}{2}\left(\dfrac{f(y_n)}{f(x_n)}\right)^3 + \left(\dfrac{f(z_n)}{f(x_n)}\right)^2 + \left(\dfrac{f(z_n)}{f(y_n)}\right)^2 \right],
\end{cases}
$$

$$
\begin{cases}
w_n = x_n + \beta f(x_n), \ \beta \in \mathbb{R} \\
y_n = x_n - \dfrac{\beta f(x_n) f(x_n)}{f(w_n) - f(x_n)}, \\[2ex]
z_n = y_n - \dfrac{f(w_n) f(y_n)(y_n - x_n)}{(f(w_n) - f(y_n))(f(y_n) - f(x_n))}, \\[2ex]
x_{n+1} = z_n - \dfrac{f(w_n) f(y_n) \left(\frac{f(x_n)(z_n - x_n)}{f(z_n) - f(x_n)} - x_n + y_n\right)}{(f(w_n) - f(z_n))(f(y_n) - f(z_n))} + \dfrac{f(y_n)(z_n - y_n)}{f(z_n) - f(y_n)},
\end{cases}
$$

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \left[\dfrac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\right] \dfrac{f(x_n)}{f'(x_n)}, \\ u_n = z_n - \left(\dfrac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} + \dfrac{f(z_n)}{2(f(y_n) - 2f(z_n))}\right)^2 \dfrac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = u_n - \dfrac{3(b_2 + b_3) f(z_n)(u_n - z_n)}{f'(x_n)(b_1(u_n - z_n) + b_2(y_n - x_n) + b_3(z_n - x_n))} \end{cases}$$

and

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \left[\dfrac{f(y_n)}{f(x_n) - 2f(y_n)}\right] \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \left[\dfrac{6f(y_n)^4\{f(x_n) + 5f(y_n)\}}{f(x_n)^5}\right] \dfrac{f(z_n)}{f'(x_n)} \\ \qquad\qquad - \dfrac{f(x_n) + 31f(z_n)}{f(x_n) + 30f(z_n)} \left[\dfrac{f[y_n,\ x_n] f(z_n)}{f[z_n,\ x_n] f[y_n,\ z_n]}\right], \end{cases}$$

which were proposed by Džunić and Petković [18], Khattri and Steihaug [19] (for $\alpha = 1$), Soleymani et al. in [20], Kung and Traub [3] (for $\beta = 1$), Cordero et al. in [21] (for $b_1 = 1,\ b_2 = 1,\ b_3 = 2$) and Heydari et al. [22], respectively called by DP, KS, SM, KT, CM, and HM.

We experimented with the functions $f_i(x)$, $i = 1, .., 5$. We have taken 500 equally spaced points $\{t_i\}_{i=0}^{500}$ in the interval $[-3, 3]$ for $f_i(x)$, $i = 1, .., 5$ as initial points for the methods. Notice that $f_2(x) = 0$ contains two solutions $\xi = 1.403602124874216, \xi = -1$ in $[-3,\ 3]$, and the others only one. If x_0 attempts a root with tolerance $\epsilon = 10^{-5}$ in 14 iterations we have decided it converged to the root, otherwise, it diverged. We have registered the total number of iterations required to converge to a root and also collected the CPU time in seconds required to run each method on all the points using Samsung Notebook NT900X4C. We then computed the average number of iterations required per point and the number of points requiring more than 14 iterations.

We have averaged performance results for the methods in comparison in Tables 1-3 across the 5 test functions. Based on Table 1 we find that the minimum number of divergent points on average is achieved by KS (7.2 out of 500 points) followed by OM1 (38.8 points), DP (45.2 points) and HM (82.8

Method	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	average	Divergence Percentage
OM1	35	87	17	48	6	38.8	7.72%
OM2	50	375	21	65	19	106	21.2%
DP	7	32	2	179	6	45.2	9.04%
KS	19	9	2	0	6	7.2	1.44%
SM	83	136	37	180	118	110.8	22.16%
KT	77	500	500	0	24	220.2	44.04%
CM	123	500	332	190	500	329	65.8%
HM	72	137	33	134	38	82.8	16.56%

TABLE 1. Number of points requiring more than 14 iterations for each test function (1–5) and each of the methods and divergence percentage

Method	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	average
OM1	11.406	2.359	3.328	5.672	7.000	5.953
OM2	4.172	2.454	4.078	4.188	5.344	4.047
DP	5.406	3.094	2.860	2.578	8.578	4.503
KS	6.328	5.156	3.219	5.375	6.203	5.256
SM	4.594	2.860	1.875	4.891	4.781	3.800
KT	12.297	17.250	27.500	24.625	20.890	20.512
CM	14.047	3.609	9.360	12.188	10.094	9.859
HM	8.250	2.875	4.313	5.906	9.063	6.081

TABLE 2. CPU time (in seconds) required for each test function (1–5) and each of the methods

Method	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	average
OM1	3.24	5.35	2.96	3.13	2.19	3.37
OM2	3.58	11.15	3.00	3.47	2.63	4.76
DP	2.61	3.73	2.48	6.23	2.55	3.52
KS	3.04	3.97	2.74	1.64	1.66	2.61
SM	4.25	5.75	3.22	6.30	4.87	4.87
KT	6.42	14	14	6.37	5.88	9.33
CM	5.52	14	13.41	9.19	14	11.22
HM	4.01	5,82	3.24	5.35	3.02	4.28

TABLE 3. Average number of iterations per point for each test function (1–6) and each of the methods

points). All the others have $106 - 329$ number of points requiring more than 14 iterations on average. Since KT and CM have more than 44 percent of divergence, we will remove these methods from further consideration for looking for best performers.

In terms of CPU time (see Table 2), the fastest method is SM (3.8 seconds) followed by OM2 (4.047 seconds), DP (4.503 seconds) and KS (5.256 seconds). The slowest is KT (20.512 seconds), which was removed from further discussion. We note that although SM is the fastest of all the methods considered, it will no longer be considered since it is one of the methods having more than 22 percent of initial points diverged. Consulting the average number of iterations per point on average (see Table 3), we find that KS is best (2.61) followed by OM1 (3.37) and DP (3.52). The worst is CM (11.22).

Considering the top 3 performers in each category and in view of our analysis of the results in Tables 1-3 given above, the best method overall is DP, followed by OM1. KS and OM2.

4. Conclusions

In this paper we presented a new optimal eighth-order family of methods and they have been compared to several existing methods of the same order. Our proposed methods are found to be as competitive as existing methods based on 3 quantitative criteria.

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