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VARIOUS SHADOWING PROPERTIES FOR INVERSE LIMIT SYSTEMS

Manseob Lee*

Abstract. Let $f : X \to X$ be a continuous surjection of a compact metric space and let $(X_f, \widetilde{f})$ be the inverse limit of a continuous surjection f on X. We show that for a continuous surjective map f, if f has the asymptotic average, the average shadowing, the ergodic shadowing property then $\widetilde{f}$ is topologically transitive.

1. Introduction

Let (X, d) be a compact metric space with metric d and let $f : X \to X$ be a continuous surjective map. For $\delta > 0$, a sequence of points $\{x_i\}_{i=0}^{\infty}$ in X is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \geq 0$. We say that f has the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$, there is a point $y \in X$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \geq 0$. The asymptotic average shadowing property introduced by Gu [4]. A sequence $\{x_i\}_{i=0}^{\infty}$ is called an *asymptotic average pseudo orbit* of f if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0.$$

A sequence $\{x_i\}_{i=0}^{\infty}$ is said to be asymptotic average shadowed in average by the point z in X if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) = 0.$$

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We say that f has the *asymptotically average shadowing property* if for every asymptotic average pseudo orbit of f can be asymptotically shadowed in average by some point in X. The average shadowing property was introduced by Blank [2] For $\delta > 0$, a sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is called a δ*-average pseudo orbit* of f if there is $N(\delta) > 0$ such that for all $n \geq N, k \in \mathbb{N} \cup \{0\}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that f has the *average shadowing property* if for any $\epsilon > 0$ there is a $\delta > 0$ such that for every δ-average pseudo orbit $\{x_i\}_{i=0}^{\infty}$ is ϵ-shadowed in average by some $z \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

The notion of ergodic shadowing property for continuous onto maps over compact metric spaces was defined by Fakhari and Ghane in [3]. For any $\delta > 0$, a sequence $\xi = \{x_i\}_{i=0}^{\infty}$ is δ*-ergodic pseudo orbit* of f if for $Np_n(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\} \cap \{0, 1, \ldots, n-1\}$,

$$\lim_{n\to\infty} \frac{\# Np_n(\xi, f, \delta)}{n} = 0.$$

We say that f has the *ergodic shadowing property* if for any $\epsilon > 0$, there is a $\delta > 0$ such that for every δ-ergodic pseudo orbit $\xi = \{x_i\}_{i=0}^{\infty}$ of f there is a point $z \in X$ such that for $Ns_n(\xi, f, z, \epsilon) = \{i : d(f^i(z), x_i) \geq \epsilon\} \cap \{0, 1, \ldots, n-1\}$,

$$\lim_{n\to\infty} \frac{\# Ns_n(\xi, f, z, \epsilon)}{N} = 0.$$

A point $p \in X$ is *periodic* if there is $n > 0$ such that $f^n(p) = p$, where n is called the prime period of p. For $x \in X$ we define the stable set of x as following: $W^s(x) = \{y \in X : d(f^i(x), f^i(y)) \to 0 \text{ as } i \to \infty\}$ and the unstable set of x as following: $W^u(x) = \{y \in X : d(f^i(x), f^i(y)) \to 0$ as $i \to -\infty\}$. Then a point p is *sink* if $W^s(p)$ is a neighborhood of p in X. A point p is *source* if $W^u(p)$ is a neighborhood of p in X.

For any $\delta > 0$, a sequence $\{x^i\}_{i\in\mathbb{N}\cup\{0\}}$ is called a δ *pseudo orbit* of f if $d(f(x^i), x^{i+1}) < \delta$ for all $i \in \mathbb{N} \cup \{0\}$. We say that f is *chain transitive* if for any $\delta > 0$ and $x, y \in X$, there is a δ-pseudo orbit $\{x^i\}_{i=0}^{n} (n \geq 1)$ such that $x^0 = x$ and $x^n = y$. The shadowing property is not equal to the asymptotic average shadowing property, the average shadowing property, and the ergodic shadowing property. In fact, a map f has the asymptotic average shadowing property, the average shadowing property, the ergodic shadowing property then it is chain transitive. Then the map f has neither sinks or sources (see [5]). A More-Smale map have a sink and a source and it has the shadowing property. We say that f is *topologically transitive* if for any open sets U and V, there is $n > 0$ such that $f^n(U) \cap V \neq \emptyset$.

Let $X_f = \{(x_j)_{j=0}^{\infty} : x_j \in X, f(x_j) = x_{j+1}, j \geq 0\}$. For a continuous surjective map $f : X \to X$, we define a map $\widetilde{f} : X_f \to X_f$ by $\widetilde{f}((x_j)_{j=0}^{\infty}) = (f(x_j)_{j=0}^{\infty})$. Then $\widetilde{f}$ is called the *shift map* which is the homeomorphim and $\widetilde{f}^{-1}((x_j)_{j=0}^{\infty}) = (x_{j+1})_{j=0}^{\infty}$ for $(x_j) \in X_f$. We say that $(X_f, \widetilde{f})$ is the *inverse limit* of (X, f). Define a metric $\widetilde{d}$ for X_f by

$$\widetilde{d}(\widetilde{x}, \widetilde{y}) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}$$

for $(x_i)_{i=0}^{\infty}, (y_i)_{i=0}^{\infty} \in X_f$. We say that $\widetilde{f}$ is *topologically transitive* if for any open sets $\widetilde{U}$ and $\widetilde{V}$ in X_f there is $n > 0$ such that $\widetilde{f}^n(\widetilde{U}) \cap \widetilde{V} \neq \emptyset$. By Aoki and Hiraide [1] proved that a continuous surjective map f has the shadowing property then $\widetilde{f}$ has the shadowing property. In the paper, we study that if a continuous surjective map f has the shadowing property with various shadowing properties then the shift map $\widetilde{f}$ is topologically transitive. The following is the main theorem.

THEOREM 1.1. *A surjective continuous map $f : X \to X$ has the shadowing property. If any of the following statements hold:*

(a) *f has the asymptotic average shadowing property,*
(b) *f has the average shadowing property,*
(c) *f has the ergodic shadowing property,*

then $\widetilde{f} : X_f \to X_f$ is topologically transitive.

2. Proof of Theorem 1.1

For any $\delta > 0$, a sequence $\{\widetilde{x}^i\}_{i \in \mathbb{N} \cup \{0\}}$ is called a *δ-pseudo orbit* of f if $\widetilde{d}(\widetilde{f}(\widetilde{x}^i), \widetilde{x}^{i+1}) < \delta$ for all $i \in \mathbb{N} \cup \{0\}$. We say that $\widetilde{f}$ is *chain transitive* if for any $\delta > 0$ and $\widetilde{x}, \widetilde{y} \in X_f$, there is a δ-pseudo orbit $\{\widetilde{x}^i\}_{i=0}^{n} (n \geq 1)$ such that $\widetilde{x}^0 = \widetilde{x}$ and $\widetilde{x}^n = \widetilde{y}$.

LEMMA 2.1. *If a surjective continuous map $f : X \to X$ is chain transitive then $\widetilde{f} : X_f \to X_f$ is chain transitive.*

Proof. Since f is chain transitive, for any $x_j, y_j \in X (j \geq 0)$, and any $\epsilon > 0$, there is a finite $\epsilon/3$-pseudo orbit $\{x_j^i\}_{i=0}^n (n \geq 1)$ such that $x_j^0 = x_j, x_j^n = y_j$ and $d(f(x_j^i), x_j^{i+1}) < \epsilon/3$ for $i = 0, \ldots, n-1$. In particular, $x_0^0 = x, x_0^n = y$ and $d(f(x^i), x^{i+1}) < \epsilon/3$ for $i = 0, \ldots, n-1$. Then for $i = 0, \ldots, n-1$, we have

$$\begin{aligned}\widetilde{d}(\widetilde{f}(\widetilde{x}^i), \widetilde{x}^{i+1}) &= \sum_{j=0}^{\infty} \frac{d(f(x_j^i), x_j^{i+1})}{2^j} \\ &= \sum_{j=0}^{\infty} \frac{d(f(x_j^i), x_j^{i+1})}{2^j} \leq \frac{\epsilon}{3} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{2}{3}\epsilon < \epsilon.\end{aligned}$$

Thus $\{\widetilde{x}^i\}_{i=0}^n (n \geq 1)$ is a finite ϵ-pseudo orbit of $\widetilde{f}$ which means that it is chain transitive. □

It is clear that if f is topologically transitive then it is chain transitive. But, the converse is not true. Indeed, an irrational rotation map is topologically transitive. But, if an oriented preserving rotation map which contains a fixed point then it is not topologically transitive. However, it is chain transitive.

LEMMA 2.2. *If a surjective continuous map $f : X \to X$ is topologically transitive then $\widetilde{f} : X_f \to X_f$ is topologically transitive.*

Proof. Let $\widetilde{U} = (U_j)$ and $\widetilde{V} = (V_j)$ be open sets of X_f. Then we show that there is $n > 0$ such that $\widetilde{f}^n(\widetilde{U}) \cap \widetilde{V} \neq \emptyset$. Suppose, by contradiction, that, for all $k > 0$ such that $\widetilde{f}^k(\widetilde{U}) \cap \widetilde{V} = \emptyset$. Since $\widetilde{f}(\widetilde{U}) = \widetilde{f}((U_j)) = (f(U_j))$ and $\widetilde{V} = (V_j)$, we know that for all $k > 0$

$$\begin{aligned}\widetilde{f}^k(\widetilde{U}) \cap \widetilde{V} &= (f^k(U_j)) \cap (V_j) \\ &= (f^k(U_0), f^k(U_1), \ldots,) \cap (V_0, V_1, \ldots) \\ &= \emptyset.\end{aligned}$$

Then for all $k > 0$ there is $i > 0$ such that $f^k(U_i) \cap V_i = \emptyset$. Since U_j and V_j are arbitrarily open sets in X for all $j \geq 0$ and f is topologically transitive, there is $n > 0$ such that $f^n(U_j) \cap V_j \neq \emptyset$ for all $j \geq 0$ which is a contradiction. □

REMARK 2.3. *Gu* [4, Theorem 3.1] *proved that if a map f has the asymptotic average shadowing property then it is chain transitive, Park and Zhang* [6, Theorem 3.4] *proved that if a map f has the average shadowing property then it is chain transitive and Fakhari and Ghane*[3, Lemma 3.1] *proved that if a map f has the ergodic shadowing property then it is chain transitive.*

LEMMA 2.4. *Let f be a chain transitive map. If f has the shadowing property then it is topologically transitive.*

Proof. The proof is similar to [7, Lemma 2.2]. □

Proof of Theorem 1.1. Suppose that f has the asymptotic average (average, ergodic) shadowing property. Then by Remark 2.3, the map f is chain transitive. Since f is chain transitive, by Lemma 2.1 $\widetilde{f}$ is chain transitive. Since f has the shadowing property, by Lemma 2.4, f is topologically transitive. Thus by Lemma 2.2, $\widetilde{f}$ is topologically transitive. □

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Department of Mathematics
Mokwon University
Daejeon 302-729, Republic of Korea
E-mail: `lmsds@mokwon.ac.kr`