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ON MULTIPLIERS ON BOOLEAN ALGEBRAS

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Abstract. In this paper, we introduced the notion of multiplier of Boolean algebras and discuss related properties between multipliers and special mappings, like dual closures, homomorphisms on B. We introduce the notions of fixed set $Fix_f(X)$ and normal ideal and obtain interconnection between multipliers and $Fix_f(B)$. Also, we introduce the special multiplier α_p and study some properties. Finally, we show that if B is a Boolean algebra, then the set of all multipliers of B is also a Boolean algebra.

1. Introduction

Boolean algebras play an important role in many fields such as information theory, information retrieval, information access controls and cryptanalysis. In [4] a partial multiplier on a commutative semigroup $(A, \cdot)$ has been introduced as a function F from a nonvoid subset D_F of A into A such that $F(x) \cdot y = x \cdot F(y)$ for all $x, y \in D_F$. In this paper, we introduced the notion of multiplier of Boolean algebras and discuss related properties between multipliers and special mappings, like dual closures, homomorphisms on B. We introduce the notions of fixed set $Fix_f(X)$ and normal ideal and obtain interconnection between multipliers and $Fix_f(B)$. Also, we introduce the special multiplier α_p and study some properties. Finally, we show that if B is a Boolean algebra, then the set of all multipliers of B is also a Boolean algebra.

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2. Preliminaries

DEFINITION 2.1. Let B be a nonempty set endowed with operations $\wedge$ and $\vee$. By a *Boolean algebra* $(B, \wedge, \vee, ', 0, 1)$, we mean a set B satisfying the following conditions, for all $x, y, z \in B$,

DEFINITION 2.2. Let $(B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra. A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

LEMMA 2.3. *Let $(B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra. Define the binary relation $\leq$ as the Definition 2.2. Then $(B, \leq)$ is a poset and for any $x, y \in B$, $x \wedge y$ is the g.l.b. of $\{x, y\}$ and $x \vee y$ is the l.u.b. of $\{x, y\}$.*

LEMMA 2.4. *Let B be a Boolean algebra and $x, y \in B$. If $x \leq y$ and $y \leq x$, then $x = y$.*

LEMMA 2.5. *Let B be a Boolean algebra and $x, y, z \in B$. Then the following properties hold:*

(1) *If $x \leq y$, then $x \wedge z \leq y \wedge z$ and $x \vee z \leq y \vee z'$,*
(2) *$x \leq y$ if and only if $y' \leq x'$.*

THEOREM 2.6. *Let B be a Boolean algebra and $x, y \in B$. Then the following conditions are equivalent:*

(1) $x \leq y$, (2) $x \wedge y' = 0$, (3) $x' \vee y = 1$, (4) $x \wedge y = x$, (5) $x \vee y = y$.

THEOREM 2.7. *Let B be a Boolean algebra and $x, y, z \in B$. Then the following conditions hold:*

(1) *$x \vee y = 0$ if and only if $x = 0$ and $y = 0$,*
(2) *$x \wedge y = 1$ if and only if $x = 1$ and $y = 1$.*

DEFINITION 2.8. Let $f : B_1 \to B_2$ be a function from a Boolean algebra B_1 to a Boolean algebra B_2. Then f is called a *Boolean homomorphism* (or *homomorphism*) if

(1) $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$,
(2) $f(x') = (f(x))'$.

DEFINITION 2.9. Let B be a Boolean algebra and $f : B \to B$ be a function. Then

(1) f is said to be *regular* if $f(0) = 0$.
(2) f is is said to be *isotone* if $x \leq y$ implies $f(x) \leq f(y)$.

THEOREM 2.10. *Let $f : B_1 \to B_2$ be a function from a Boolean algebra B_1 to a Boolean algebra B_2. If f is a Boolean homomorphism, then*

(1) *$f(0) = 0$ and $f(1) = 1$,*
(2) *f is isotone.*

DEFINITION 2.11. An *ideal* is a nonempty subset I of a Boolean algebra B if

(1) If $x \in I$ and $b \in B$, then $x \wedge b \in I$,
(2) If $x, y \in I$, then $x \vee y \in I$.

DEFINITION 2.12. A function f from a Boolean B into itself is a *dual closure* if f is monotone, non-expansive(i.e., $f(x) \leq x$ for all $x \in B$) and idempotent(i.e., $f \circ f = f$).

3. Multipliers on Boolean algebras

In what follows, let B denote a Boolean algebra unless otherwise specified.

DEFINITION 3.1. Let B be a Boolean algebra. A function $f : B \to B$ is called a *multiplier* if it satisfies the following identity

$$f(x \wedge y) = f(x) \wedge y$$

for all $x, y \in B$.

EXAMPLE 3.2. Let $B = \{0, a, b, 1\}$ and $\wedge, \vee$ are two binary operations defined as follows

x	x'
0	1
a	b
b	a
1	0

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	1	b	1
1	1	1	1	1

Then $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra. Define a self-map $f : B \to B$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b, 1 \end{cases}$$

Then it is easy to check that f is a multiplier of a Boolean algebra B.

PROPOSITION 3.3. *Let B be a Boolean algebra and let f be a multiplier on B. Then*

$$f(x) \leq x$$

for all $x \in B$.

Proof. Let f be a multiplier in B. For $x \in B$, we have

$$f(x) = f(x) \wedge f(x) = x \wedge f(f(x)),$$

which implies $f(x) \leq x$. □

PROPOSITION 3.4. *If f is a multiplier on B, then for every $x, y \in B$,*

$$f(x \wedge y) = f(x) \wedge y = x \wedge f(y).$$

Proof. For any $x, y \in B$, $f(x \wedge y) \leq x \wedge y \leq x$ and $f(x) \leq x$, by Proposition 3.3, hence

$$f(x \wedge y) = x \wedge f(x \wedge y) = f(x) \wedge (x \wedge y) = (f(x) \wedge x) \wedge y = f(x) \wedge y,$$

and $f(x \wedge y) = x \wedge f(y)$ by commutativity of $\wedge$. □

PROPOSITION 3.5. *Let B be a Boolean algebra and let f be a multiplier on B. Then $f(0) = 0$.*

Proof. For all $x \in B$, we have

$$f(0) = f(x \wedge 0) = f(x) \wedge 0 = 0,$$

which implies $f(0) = 0$. This completes the proof. □

PROPOSITION 3.6. *Let B be a Boolean algebra and let f be a multiplier on B. Then f is an idempotent on B*, i.e., $f^2(x) = f(x)$.

Proof. For all $x \in B$, we have

$$f^2(x) = f(f(x \wedge x)) = f(f(x) \wedge x)) = f(x \wedge f(x)) = f(x) \wedge f(x) = f(x),$$

which implies that f is an idempotent on B. This completes the proof. □

PROPOSITION 3.7. *Let B be a Boolean algebra and let f be a multiplier on B. Then f is a meet-homomorphism on B.*

Proof. Let f be a multiplier on B. Then by Proposition 3.6, we have $f^2(x) = f(x)$ for all $x \in B$. Now, let $a, b \in B$. Then

$$\begin{aligned} f(a \wedge b) &= f(f(a \wedge b)) = f(f(a) \wedge b)) \\ &= f(b \wedge f(a)) = f(b) \wedge f(a) \\ &= f(a) \wedge f(b), \end{aligned}$$

which implies that f is a meet-homomorphism on B. This completes the proof. □

PROPOSITION 3.8. *Let B be a Boolean algebra and let f be a multiplier on B. If $f(1) = 1$, then f is an identity multiplier in B.*

Proof. Let B be a Boolean algebra and $f(1) = 1$. Then we have from Proposition 3.4,

$$f(x) = f(x \wedge 1) = f(x) \wedge 1 = x \wedge f(1) = x \wedge 1 = x,$$

which implies that f is an identity multiplier in B. □

PROPOSITION 3.9. *Let B be a Boolean algebra and let f be a multiplier on B. If f is a Boolean homomorphism on B and $x \leq y$, then*

(1) $f(x \wedge y') = 0$,
(2) $f(y') \leq x'$,
(3) $f(x) \wedge f(y') = 0$.

Proof. Let $x, y \in B$ be such that $x \leq y$ and let f be a multiplier on B Then f is an isotone by Theorem 2.6 and $f(0) = 0$.

(1) By Theorem 2.6, we have $x \wedge y' = 0$. Thus, we have $f(x \wedge y') = f(0) = 0$.

(2) By Theorem 2.6, we obtain $y \leq x'$ since $x \leq y$, and so $f(y') = (f(y))' \leq (f(x))' = f(x') \leq x'$.

(3) By theorem 2.6, we have

$$\begin{aligned} f(x) \wedge f(y') &\leq f(y) \wedge f(y') \\ &= f(y \wedge f(y')) = f^2(y \wedge y') \\ &= f(y \wedge y') = 0, \end{aligned}$$

which implies $f(x) \wedge f(y') = 0$ by (1). □

Let B be a Boolean algebra and f_1, f_2 two self-maps. We define $f_1 \circ f_2 : B \to B$ by

$$(f_1 \circ f_2)(x) = f_1(f_2(x))$$

for all $x \in B$.

PROPOSITION 3.10. *Let B be a Boolean algebra and let $f_1, f_2, f_3, \cdot\cdot\cdot, f_n$ be multipliers on B. Then $f_1 \circ f_2 \circ f_3 \circ \cdots \circ f_n$ is also a multiplier of B.*

Proof. Let B be a Boolean algebra and f_1, f_2 two multipliers on B. Then we have for all $a, b \in B$

$$\begin{aligned} (f_1 \circ f_2)(a \wedge b) &= f_1(f_2(a \wedge b)) = f_1(f_2(a) \vee b) \\ &= f_1(f_2(a)) \wedge b = (f_1 \circ f_2)(a) \vee b. \end{aligned}$$

This completes the proof. □

Let B be a Boolean algebra and f_1, f_2 two self-maps. We define $f_1 \wedge f_2 : B \to B$ by

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$$

for all $x \in B$.

PROPOSITION 3.11. *Let B be a Boolean algebra and let $f_1, f_2, \cdots, f_n$ be multipliers on B. Then $f_1 \wedge f_2 \wedge \cdots \wedge f_n$ is also a multiplier of B.*

Proof. Let B be a Boolean algebra and f_1, f_2 two multipliers of B. Then we have for all $a, b \in B$

$$\begin{aligned}(f_1 \wedge f_2)(a \wedge b) &= f_1(a \wedge b) \wedge f_2(a \wedge b) \\ &= (f_1(a) \wedge b) \wedge (f_2(a) \wedge b) \\ &= (f_1(a) \wedge f_2(a)) \wedge b \\ &= (f_1 \wedge f_2)(a) \wedge b.\end{aligned}$$

This completes the proof. □

Let B be a Boolean algebra and f_1, f_2 two self-maps. We define $f_1 \vee f_2 : B \to B$ by

$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$$

for all $x \in B$.

PROPOSITION 3.12. *Let B be a Boolean algebra and let $f_1, f_2, \cdots, f_n$ be multipliers of B. Then $f_1 \vee f_2 \vee \cdots \vee f_n$ is also a multiplier of B.*

Proof. Let B be a Boolean algebra and f_1, f_2 two multipliers of B. Then we have for all $a, b \in B$,

$$\begin{aligned}(f_1 \vee f_2)(a \wedge b) &= f_1(a \wedge b) \vee f_2(a \wedge b) \\ &= (f_1(a) \wedge b) \vee (f_2(a) \wedge b) \\ &= (f_1(a) \vee f_2(a)) \wedge b \\ &= (f_1 \vee f_2)(a) \wedge b.\end{aligned}$$

This completes the proof. □

Let $M(B)$ be a set of all multipliers on B and let f be a multiplier on B. Since $f(x) \leq x$, we have $f(x) \leq I(x)$ for all $f \in M(B)$ and $x \in B$, where $I(x) = x$ for all $x \in B$. Also, we obtain $0(x) \leq f(x)$ for all $f \in M(B)$ and $x \in B$, where $0(x) = 0$.

THEOREM 3.13. *Let B be a Boolean algebra and let $M(B)$ be a set of all multipliers on B. Then $(M(B), \wedge, \vee, 0(x), I(x))$ is a bounded distributive lattice.*

Proof. From Proposition 3.11 and 3.12, $\wedge$ and $\vee$ are binary operators on $M(B)$. Define a binary relation " $\leq$ " on $M(B)$ by $f_1 \leq f_2$ if and only if $f_1 \wedge f_2 = f_1$. Then " $\leq$ " is a partial order relation on $M(B)$ and $g.l.b\{f_1, f_2\} = f_1 \wedge f_2, l.u.b\{f_1, f_2\} = f_1 \vee f_2$. Therefore, $(M(B), \wedge, \vee, 0(x), I(x))$ is a bounded lattice. In addition, for any $f_1, f_2, f_3 \in M(B)$ and any $x \in B$,

$$\begin{aligned}(f_1 \wedge (f_2 \vee f_3)(x)) &= f_1(x) \wedge (f_2(x) \vee f_3(x) \\ &= (f_1(x) \wedge f_2(x)) \vee (f_1(x) \vee f_3(x)) \\ &= ((f_1 \wedge f_2)(x)) \vee ((f_1 \wedge f_3)(x)) \\ &= ((f_1 \wedge f_2) \vee (f_1 \wedge f_2))(x).\end{aligned}$$

Therefore, $f_1 \wedge (f_2 \vee f_3) = (f_1 \wedge f_2) \vee (f_1 \wedge f_3)$.
This shows that $(M(B), \wedge, \vee, 0(x), I(x))$ is a bounded distributive lattice. □

THEOREM 3.14. *Let B be a Boolean algebra and $f : B \to B$ be a multiplier of B. Then f is monotone.*

Proof. Let f be a multiplier of B and let $x \leq y$. Then $x \wedge y = x$. Hence $f(x) = f(x \wedge y) = f(x) \wedge y$, i.e., $f(x) \leq y$. Since f is idempotent, we have $f(x) = f(f(x)) = f(f(x) \wedge y) = f(y \wedge f(x)) = f(y) \wedge f(x) = f(x) \wedge f(y)$. This implies $f(x) \leq f(y)$. □

THEOREM 3.15. *Let B be a Boolean algebra and $f : B \to B$ be a multiplier of B. Then the following identities are equivalent,*
(1) *f is an isotone function,*
(2) *$f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in B$.*

Proof. (1) $\Rightarrow$ (2) Let f be an isotone function of B. Then $x \wedge y \leq x$ and $x \wedge y \leq y$ for all $x, y \in L$. Thus, we get $f(x \wedge y) \leq f(x)$ and $f(x \wedge y) \leq f(y)$ for all $x, y \in B$, which implies $f(x \wedge y) \leq f(x) \wedge f(y)$. Also, $f(x) \wedge f(y) \leq f(x) \wedge y = f(x \wedge y)$. Hence we have $f(x \wedge y) = f(x) \wedge f(y)$.

(2) $\Rightarrow$ (1) Let $x, y \in B$ be such that $x \leq y$. Then $f(x) = f(x \wedge y) = x \wedge f(y) \leq f(y)$. Hence f is an isotone function. This completes the proof. □

THEOREM 3.16. *Let B be a Boolean algebra and $f : B \to B$ be a multiplier of B. Then the following identities are equivalent,*
(1) *f is isotone,*
(2) *$f(x \wedge y) = f(x) \wedge f(y)$,*
(3) *$f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in B$.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 3.15, the identities (1) and (2) are equivalent.

(1) $\Rightarrow$ (3) Assume that f is isotone. Then $f(x) \leq f(x \vee y)$ and $f(y) \leq f(x \vee y)$. Also, $f(x) = f((x \vee y) \wedge x) = x \wedge f(x \vee y)$. Similarly, we get $f(y) = y \wedge f(x \vee y)$. Hence we have for $x, y \in B$,

$$\begin{aligned} f(x) \vee f(y) &= (x \wedge f(x \vee y)) \vee (y \wedge (y \wedge f(x \vee y))) \\ &= (x \vee y) \wedge f(x \vee y) \\ &= f(x \vee y) \end{aligned}$$

(3) $\Rightarrow$ (1) Let $x \leq y$ for all $x, y \in B$. Then $y = x \vee y$. Hence we get $f(y) = f(x \vee y) = f(x) \vee f(y) \geq f(x)$, which implies f is isotone. $\square$

Let B_1 and B_2 be two Boolean algebras. Then $B_1 \times B_2$ is also a Boolean algebra with respect to the point-wise operation given by

$$(a, b) \wedge (c, d) = (a \wedge c, b \wedge d)$$

for all $a, c \in B_1$ and $b, d \in B_2$.

Proposition 3.17. *Let B_1 and B_2 be two Boolean algebras. Define a map $f : B_1 \times B_2 \to B_1 \times B_2$ by $f(x, y) = (0, y)$ for all $(x, y) \in B_1 \times B_2$. Then f is a multiplier of $B_1 \times B_2$ with respect to the point-wise operation.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in B_1 \times B_2$. The we have

$$\begin{aligned} f((x_1, y_1) \wedge (x_2, y_2)) &= f(x_1 \wedge x_2, y_1 \wedge y_2) \\ &= (0, y_1 \wedge y_2) \\ &= (0 \wedge x_2, y_1 \wedge y_2) \\ &= (0, y_1) \wedge (x_2, y_2) \\ &= f(x_1, y_1) \wedge (x_2, y_2). \end{aligned}$$

Therefore f is a multiplier of the direct product $B_1 \times B_2$. $\square$

Let B be a Boolean algebra and let f be a multiplier on B. Define a set $Fix_f(B)$ by

$$Fix_f(B) = \{x \in B \mid f(x) = x\}.$$

In the following results, we assume that $Fix_f(B)$ is a nonempty proper subset of B.

Proposition 3.18. *Let B be a Boolean algebra and let f be a multiplier on B. If $f : B \to B$ is a join homomorphism, then $Fix_f(B)$ is a Boolean subalgebra of B.*

Proof. Let $x, y \in Fix_f(B)$. Then $f(x) = x$ and $f(y) = y$. Then $f(x \wedge y) = f(x) \wedge y = x \wedge y$, that is, $x \wedge y \in Fix_f(B)$. Moreover, we have $f(x \vee y) = f(x) \vee f(y) = x \vee y$, which implies $x \vee y \in Fix_f(B)$. Hence $Fix_f(L)$ is a Boolean subalgebra of B. □

PROPOSITION 3.19. *Let B be a Boolean algebra and let f be a multiplier on B. If $x \leq y$ and $y \in Fix_f(B)$, we have $x \in Fix_f(B)$.*

Proof. Let $x \leq y$. Then we have

$$f(x) = f(y \wedge x) = f(y) \wedge x = y \wedge x = x,$$

which implies $x \in Fix_f(B)$. □

PROPOSITION 3.20. *Let B be a Boolean algebra and let f be a multiplier of B. If $x \in Fix_f(B)$ and $y \in B$, we have $x \wedge y \in Fix_f(B)$ for all $x, y \in B$.*

Proof. Let $x \in Fix_f(B)$ and $y \in B$. Then $f(x) = x$. Hence we have

$$f(x \wedge y) = f(x) \wedge y = x \wedge y,$$

which implies $x \wedge y \in Fix_f(B)$. □

PROPOSITION 3.21. *Let B be a lattice and let f_1 and f_2 be isotone multipliers of B. Then $f_1 = f_2$ if and only if $Fix_{f_1}(B) = Fix_{f_2}(B)$.*

Proof. It is obvious that $f_1 = f_2$ implies $Fix_{f_1}(B) = Fix_{f_2}(B)$. Conversely, let $Fix_{f_1}(B) = Fix_{f_2}(B)$ and $x \in B$. By Proposition 3.19, $f_1(x) \in Fix_{f_1}(B) = Fix_{f_2}(B)$ and $f_2(f_1(x)) = f_1(x)$. Similarly, we have $f_1(f_2(x)) = f_2(x)$. Since f_1 and f_2 are isotone, we have $f_2(f_1(x)) \leq f_2(x) = f_1(f_2(x))$, and so $f_2(f_1(x)) \leq f_1(f_2(x))$. Symmetrically, we can also get $f_1(f_2(x)) \leq f_2(f_1(x))$, which implies $f_1(f_2(x)) = f_2(f_1(x))$. Thus, it follows that $f_1(x) = f_2(f_1(x)) = f_1(f_2(x)) = f_2(x)$, that is, $f_1 = f_2$. □

Let us denote the image of B under the multiplier f by $Im(f)$.

PROPOSITION 3.22. *Let f be a multiplier of a lattice B. Then $Im(f) = Fix_f(B)$.*

Proof. Let $x \in Fix_f(B)$. Then $x = f(x) \in Im(f)$. Hence $Fix_f(B) \subseteq Im(f)$. Now let $a \in Im(f)$. Then we get $a = f(b)$ for some $b \in B$. Thus $f(a) = f(f(b)) = f(b) = a$, which implies $Im(f) \subseteq Fix_f(B)$. Therefore, $Im(f) = Fix_f(B)$. This completes the proof. □

Theorem 3.23. *Let f and g be two multipliers of B such that $f \circ g = g \circ f$. Then the following conditions are equivalent.*
(1) $f = g$.
(2) $f(B) = g(B)$.
(3) $Fix_f(B) = Fix_g(B)$.

Proof. (1)$\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (3): Assume that $f(B) = g(B)$ and let $x \in Fix_f(B)$. Then $x = f(x) \in f(B) = g(B)$. Hence $x = g(y)$ for some $y \in B$. Now $g(x) = g(g(y)) = g^2(y) = g(y) = x$. Thus $x \in Fix_g(B)$. Therefore, $Fix_f(B) \subseteq Fix_g(B)$. Similarly, we can obtain $Fix_g(B) \subseteq Fix_f(B)$. Thus $Fix_f(B) = Fix_g(B)$.

(3) $\Rightarrow$ (1): Assume that $Fix_f(B) = Fix_g(B)$. Let $x \in B$. Since $f(x) \in Fix_f(B) = Fix_g(B)$, we have $g(f(x)) = f(x)$. Also, we obtain $g(x) \in Fix_g(B) = Fix_f(B)$. Hence we get $f(g(x)) = g(x)$. Thus we have

$$f(x) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = g(x).$$

Therefore, f and g are equal in the sense of mappings. □

Theorem 3.24. *Let f be a multiplier of a lattice B. Then $Fix_f(B)$ is an ideal of B.*

Proof. By Proposition 3.22, we can see that $x \in Fix_f(B)$ and $y \leq x$ imply $y \in Fix_f(B)$. This means that $Fix_f(B)$ satisfies the condition (1) of Definition 2.11. we need only to show that $x, y \in Fix_f(B)$ implies $x \vee y \in Fix_f(B)$. Let $x, y \in Fix_f(B)$. Then we have $x \vee y = f(x) \vee y = f(x \vee y)$, i.e., $x \vee y \in Fix_f(B)$, which implies that $Fix_f(B)$ satisfies the Definition 2.11. It follows that $Fix_f(B)$ is an ideal of B. □

Theorem 3.25. *Let B be a Boolean algebra. Then the following are equivalent,*
(1) B is a chain,
(2) For every isotone multiplier f, $Fix_f(B)$is a prime ideal of B.

Proof. (1) $\Rightarrow$ (2). Let B be a chain and let f be an isotone multiplier on B. Then $Fix_f(B)$ is an ideal of B by Theorem 3.24. Now, let $x \wedge y \in Fix_f(B)$. Since B is chain, we have $x \leq y$ or $x \leq x$. Assume that $x \leq y$. Then $f(x) \leq f(y)$, and so $f(x) = f(x) \wedge f(y) = f(x \wedge y) = x \wedge y = x$ by Theorem 3.15. It follows that $x \in Fix_f(B)$, which means that $Fix_f(B)$ is a prime ideal of B.

(2) $\Rightarrow$ (1). Let $Fix_f(B)$ be a prime ideal of B for every isotone multiplier of B. For every $x, y \in B$, consider the simple multiplier $f_{x \wedge y}$, which is induced by $x \wedge y$. Then $Fix_{f_{x \wedge y}}(B)$ is a prime ideal by

hypothesis. Note that $x \wedge y \in Fix_{f_{x\wedge y}}(B)$. Hence $x \in Fix_{f_{x\wedge y}}(B)$ or $y \in Fix_{f_{x\wedge y}}(B)$. Assume that $x \in Fix_{f_{x\wedge y}}(B)$. Then $x = f_{x\wedge y}(x) = x \wedge (x \wedge y) = x \wedge y$. So $x \leq y$. This means that B is a chain. □

PROPOSITION 3.26. *For $p \in B$, the mapping $\alpha_p(a) = a \wedge p$ is a multiplier of B.*

Proof. Let $p \in B$. Then we have

$$\alpha_p(a \wedge b) = (a \wedge b) \wedge p = (a \wedge p) \wedge b = \alpha_p(a) \wedge b.$$

This completes the proof. □

PROPOSITION 3.27. *For $p \in B$, the mapping $\beta_p(a) = (a \wedge p) \wedge p$ is a multiplier of B.*

Proof. Let $p \in B$. Then we have

$$\begin{aligned}\beta_p(a \wedge b) &= ((a \wedge b) \wedge p) \wedge p \\ &= ((a \wedge p) \wedge b) \wedge p \\ &= ((a \wedge p) \wedge p) \wedge b \\ &= \beta_p(a) \wedge b.\end{aligned}$$

for all $a, b \in B$. This completes the proof. □

PROPOSITION 3.28. *For $p \in B$, the multiplier $\alpha_p(a) = a \wedge p$ is a meet-homomorphism on B.*

Proof. Let $p \in B$. Then we have

$$\begin{aligned}\alpha_p(a \wedge b) &= (a \wedge b) \wedge p \\ &= (a \wedge p) \wedge (b \wedge p) \\ &= \alpha_p(a) \wedge \alpha_p(b).\end{aligned}$$

for all $a, b \in B$. This completes the proof. □

PROPOSITION 3.29. *Let B be a Boolean algebra. Then α_p is an isotone multiplier on B.*

Proof. Let $a, b \in B$ be such that $a \leq b$. Then $a = a \wedge b$. Thus we have

$$\begin{aligned}\alpha_p(a) &= \alpha_p(a \wedge b) \\ &= \alpha_p(a) \wedge b = (a \wedge p) \wedge b \\ &= (a \wedge p) \wedge (b \wedge p) \\ &= \alpha_p(a) \wedge \alpha_p(b),\end{aligned}$$

which implies $\alpha_p(a) \leq \alpha_p(b)$. This completes the proof. □

We call the multiplier $\alpha_p(a) = a \wedge p$ of Proposition 3.29 as *simple multiplier*. Let us denote $SM(B)$ by the set of all simple multiplier on B. Now we define

$$(\alpha_p \wedge \alpha_q)(x) = \alpha_p(x) \wedge \alpha_q(x), \quad (\alpha_p \vee \alpha_q)(x) = \alpha_p(x) \vee \alpha_q(x)$$

PROPOSITION 3.30. *Let B be a Boolean algebra. If $p \neq q$, then $\alpha_p \neq \alpha_q$.*

Proof. Let $\alpha_p = \alpha_q$. Then $\alpha_p(x) = \alpha_q(x)$ for all $x \in B$. This implies $x \wedge p = x \wedge q$ for all $x \in B$. Now, if $x = p$, then we get $p = p \wedge q$. Hence $p \leq q$. Next, if $x = q$, then $q \wedge p = q$, which means $q \leq p$, and so we get $p = q$, which is a contradiction. Therefore if $p \neq q$, then we have $\alpha_p \neq \alpha_q$. □

LEMMA 3.31. *Let B be a Boolean algebra and let $\alpha_p, \alpha_q \in SM(B)$. Then if $p \leq q$, we have $\alpha_p \leq \alpha_q$.*

Proof. Let $p \leq q$. Then $x \wedge q \leq y \wedge q$, i.e., $\alpha_p \leq \alpha_q$. □

LEMMA 3.32. *Let B be a Boolean algebra and let $\alpha_p, \alpha_q \in SM(B)$. Then we have $\alpha_p \wedge \alpha_q \in SM(B)$ and $\alpha_p \vee \alpha_q \in SM(B)$.*

Proof. Let $\alpha_p, \alpha_q \in SM(B)$. Then

$$\begin{aligned}(\alpha_p \wedge \alpha_q)(x) &= \alpha_p(x) \wedge \alpha_q(x)\\ &= (p \wedge x) \wedge (q \wedge x)\\ &= (p \wedge q) \wedge x\\ &= \alpha_{(p\wedge q)}(x).\end{aligned}$$

Since $p \wedge q \in B$, $\alpha_{(p\wedge q)} \in SM(B)$, which implies $\alpha_p \wedge \alpha_q \in SM(B)$. Also, we have

$$\begin{aligned}(\alpha_p \vee \alpha_q)(x) &= \alpha_p(x) \vee \alpha_q(x)\\ &= (p \wedge x) \vee (q \wedge x)\\ &= (p \vee q) \wedge x\\ &= \alpha_{(p\vee q)}(x).\end{aligned}$$

Since $p \vee q \in B$, $\alpha_{(p\vee q)} \in SM(B)$, which implies $\alpha_p \vee \alpha_q \in SM(B)$. □

THEOREM 3.33. *Let B be a Boolean algebra and let $\alpha_p, \alpha_q \in SM(B)$. Then we have, for every$x, y \in B$,*

(1) $\alpha_p(x \wedge y) = \alpha_p(x) \wedge \alpha_p(y)$,

(2) $\alpha_p(x \vee y) = \alpha_p(x) \vee \alpha_p(y)$,

(3) $\alpha_p(x \sqcup y) = \alpha_p(x) \sqcup \alpha_p(y)$, *where* $x \sqcup y = y \vee (y \vee x)$.

Proof. (1) Let $\alpha_p \in SM(B)$. Then we have

$$\begin{aligned}\alpha_p(x \wedge y) &= \alpha_p(x) \wedge \alpha_p(y)\\ &= (p \wedge x) \wedge (p \wedge y)\\ &= \alpha_p(x) \wedge \alpha_p(y).\end{aligned}$$

(2) Let $\alpha_p \in SM(B)$. Then we have

$$\begin{aligned}\alpha_p(x \vee y) &= \alpha_p(x) \vee \alpha_p(y)\\ &= (p \wedge x) \vee (p \wedge y)\\ &= \alpha_p(x) \vee \alpha_p(y).\end{aligned}$$

(3) Let $\alpha_p \in SM(B)$. Then we have

$$\begin{aligned}\alpha_p(x \sqcup y) &= \alpha_p(y \vee (y \vee x))\\ &= \alpha_p(y) \vee \alpha_p(y \vee x)\\ &= \alpha_p(y) \vee (\alpha_p(y) \vee \alpha_p(x))\\ &= \alpha_p(x) \sqcup \alpha_p(y).\end{aligned}$$

□

THEOREM 3.34. *Let B be a Boolean algebra and let $\alpha_p, \alpha_{p'} \in SM(B)$. Then we have*

(1) $(\alpha_p \vee \alpha_{p'}) = \alpha_0$,

(2) $(\alpha_p \wedge \alpha_{p'}) = \alpha_1$.

Proof. (1) Let B be a Boolean algebra. For every $p \in B$, we have

$$\begin{aligned}(\alpha_p \vee \alpha_{p'})(x) &= (x \wedge p) \vee (x \wedge p')\\ &= x \wedge (p \vee p')\\ &= x \wedge 1 = \alpha_1(x).\end{aligned}$$

(2)

$$\begin{aligned}(\alpha_p \wedge \alpha_{p'})(x) &= (x \wedge p) \wedge (x \wedge p')\\ &= x \wedge (p \wedge p')\\ &= x \wedge 0 = \alpha_0(x).\end{aligned}$$

□

Theorem 3.35. *Let B be a Boolean algebra. Then $SM(B)$ is a Boolean algebra with top element α_1 and bottom element α_0.*

Proposition 3.36. *Let B be a Boolean algebra. Then the simple multiplier α_1 is an identity function of B.*

Proof. For every $a \in B, \alpha_1(a) = a \wedge 1 = a$. This completes the proof. □

Proposition 3.37. *Let B be a Boolean algebra. Then, for each $x \in B$, we have $\alpha_p(x \wedge p) = \alpha_p(x)$.*

Proof. For each $x \in B$, we have

$$\begin{aligned}\alpha_p(x \wedge p) &= \alpha_p(x) \wedge p = (x \wedge p) \wedge p \\ &= x \wedge p = \alpha_p(x)\end{aligned}$$

This completes the proof. □

Theorem 3.38. *Let B be a Boolean algebra and let $B \neq \{0\}$. Then there is no nilpotent multiplier on$B$.*

Proof. For every multiplier f, we have

$$f^n(x) \geq f^{n-1} \geq \cdots \geq f(x) \geq x,$$

for every $x \in B$. If there exists a natural number n such that $f^n = 0$, then we get $f^n(x) = 0$, for all $x \in B$. Thus $x = 0$, for all $x \in B$, which is a contradiction. Hence there is no nilpotent multiplier on B. This completes the proof. □

Lemma 3.39. *If B has n element, then it has at least n multipliers on B.*

Proof. Since α_p is a multiplier, for every $p \in B$, and so B has at least n multipliers. □

Theorem 3.40. *Let B be a Boolean algebra. If $\theta : B \to M(B)$ is a map defined by $\theta(x) = \alpha_x$ for each $x \in B$, then θ is one-to-one and isotone map.*

Proof. Let $\theta(x) = \theta(y)$. Then $\alpha_x = \alpha_y$, and it implies that $x \wedge y = \alpha_y(x) = \alpha_x(x) = x \wedge x = x$ and $y \wedge x = \alpha_x(y) = \alpha_y(y) = y \wedge y = y$. Hence $x \leq y$ and $y \leq x$ imply $x = y$. Let $a \leq b$ in B. Then $a \wedge x \leq b \wedge x$, that is, $\theta(a) = \alpha_a \leq \alpha_b = \theta(b)$. □

Theorem 3.41. *Let $f : B \to B$ is an isotone multiplier of B, then f is a dual closure on B.*

Proof. By Proposition 3.3 and Proposition 3.6, f is non-expensive and idempotent, and so f is a dural closure on B. $\square$

Let B be a Boolean algebra and I be a principal ideal of B generalized by $a \in B$ that is, $I = (a)$.

THEOREM 3.42. *Let B be a Boolean algebra. If f is a simple multiplier of B, then $Fix_f(B)$ is a principal ideal of B.*

Proof. Assume that f is a principal multiplier of B, that is, $f(x) = x \wedge a$, for some $a \in B$. We claim that $Fix_f(B) = \langle a \rangle$. In fact, for any $x \in Fix_f(B)$, we have $x = f(x) = x \wedge a$, and hence $x \leq a$. This means that $x \in \langle a \rangle$. Conversely, let $x \in \langle a \rangle$, that is, $x \leq a$. Then $f(x) = x \wedge a = x$, and hence $x \in Fix_f(B)$. By the above arguments, we have $Fix_f(B) = \langle a \rangle$, and so $Fix_f(B)$ is a principal ideal of B. This completes the proof. $\square$

DEFINITION 3.43. Let B be a Boolean algebra. A non-empty set I of B is called a *normal ideal* if $x \in B$ and $y \in I$ imply $x \wedge y \in I$.

EXAMPLE 3.44. In Example 3.2, let $I = \{0, a\}$. Then it is easy to see that I is a normal ideal on B.

PROPOSITION 3.45. *Let f be a multiplier of a Boolean algebra B. For any normal ideal I of B, both $f(I)$ and $f^{-1}(I)$ are normal ideals of B.*

Proof. Let $x \in B$ and $a \in f(I)$. Then $a = f(s)$ for some $s \in I$. Now $x \wedge a = x \wedge f(s) = f(x \wedge s) \in f(I)$ because $x \wedge s \in I$. Therefore $f(I)$ is a normal ideal of L. Let $x \in B$ and $a \in f^{-1}(I)$. Then $f(a) \in I$. Since I is a normal ideal, we get $f(x \wedge a) = x \wedge f(a) \in I$. Hence $x \wedge a \in f^{-1}(I)$. Therefore $f^{-1}(I)$ is a normal ideal of B. $\square$

PROPOSITION 3.46. *Let f be a multiplier of a Boolean algebra B. Then we have*

(1) *$Fix_f(B)$ is a normal ideal of B.*

(2) *$Im(f)$ is a normal ideal of B.*

Proof. (1) Let $x \in B$ and $a \in Fix_f(B)$. Then $f(a) = a$. Now $f(x \wedge a) = x \wedge f(a) = x \wedge a$. Hence $x \wedge a \in Fix_f(B)$. Therefore, $Fix_f(B)$ is a normal ideal of B.

(2) Let $x \in B$ and $a \in Im(f)$. Then $a = f(b)$ for some $b \in B$. Now $x \wedge a = x \wedge f(b) = f(x \wedge b) \in f(B)$. Therefore, $Im(f)$ is a normal ideal of B. $\square$

Let B be a Boolean algebra and let $f : B \to B$ is a function. Define a set $Kerf$ by

$$Kerf = \{x \in L \mid f(x) = 0\}.$$

PROPOSITION 3.47. *Let f be a multiplier of a Boolean algebra B. If f is a join-homomorphism, $Kerf$ is a Boolean subalgebra on B.*

Proof. Let $x, y \in Kerf$. Then $f(x) = f(y) = 0$, and so $f(x \wedge y) = f(x) \wedge y = 0 \wedge y = 0$, which implies $x \wedge y \in Kerf$. Now, we have $f(x \vee y) = f(x) \vee f(y) = 0 \vee 0 = 0$. This implies $x \vee y \in Kerf$. This completes the proof. □

PROPOSITION 3.48. *Let f be a multiplier of a Boolean algebra B. Then $Kerf$ is a normal ideal of B.*

Proof. Clearly, $0 \in Kerf$. Let $a \in Kerf$ and $x \in L$. Then $f(x \wedge a) = x \wedge f(a) = x \wedge 0 = 0$. Hence $x \wedge a \in Kerf$, which implies that $Kerf$ is a normal ideal of B. □

PROPOSITION 3.49. *Let f be a multiplier of a Boolean algebra B and $x \leq y$. If $y \in Kerf$, then we have $x \in Kerf$.*

Proof. Let $y \in Kerf$ and $x \leq y$. Then $f(x) = f(x \wedge y) = x \wedge f(y) = x \wedge 0 = 0$. Hence $x \in Kerf$. This completes the proof. □

PROPOSITION 3.50. *Let f be a multiplier of a Boolean algebra B. Then we have $Kerf \cap Fix_f(B) = \{0\}$.*

Proof. Let $x \in Kerf \cap Fix_f(B)$. Then $f(x) = 0$ and $f(x) = x$, which implies $x = 0$. Hence $Kerf \cap Fix_f(B) = \{0\}$. This completes the proof. □

PROPOSITION 3.51. *Let f be a multiplier of a Boolean algebra B. Then $Fix_f(L) = \{0\}$ implies $Kerf = B$.*

Proof. Let f be a multiplier of a Boolean algebra B. Then we have $f(x) \in Fix_f(B)$ for all $x \in B$ from Proposition 3.22. Thus, $Fix_f(B) = \{0\}$ implies that $f(x) = 0$ for each $x \in B$. This completes the proof. □

DEFINITION 3.52. Let B be a Boolean algebra and $f : B \to B$ be a function. A nonempty subset I of B is said to be a *f-invariant* if $f(I) \subseteq I$ where $f(I) = \{y \in B \mid y = f(x) \text{ for some } x \in I\}$.

THEOREM 3.53. *Let B be a Boolean algebra and f a multiplier on B. Then every ideal I is a f-invariant.*

Proof. Let I be an ideal of B and let $y \in f(I)$. Then there exists $x \in I$ such that $y = f(x) \leq x$. Since I is an ideal, we get $y \in I$. Thus $f(I) \subseteq I$. □

THEOREM 3.54. *Let $f : B \to B$ is a dual closure. Then f is a multiplier on B.*

Proof. Let $f : B \to B$ be a dual closure and let f be a homomorphism. Then we have, for every $x, y \in B$,

$$\begin{aligned} f(x \wedge y) &= f(x) \wedge f(y) \\ &\leq f(x) \wedge y, \end{aligned}$$

and

$$\begin{aligned} f(x) \wedge y &\leq f(f(x) \wedge y) \\ &= f^2(x) \wedge f(y) \\ &= f(x) \wedge f(y). \end{aligned}$$

□

This implies $f(x \wedge y) = f(x) \wedge y$, that is, f is a multiplier on B.

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