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ON $PC^\star$-CLOSED SETS

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Abstract. In this paper, the concept of $PC^\star$-closed sets is introduced. $PC^\star$-closed sets contain pre_I^*-open and pre_I^*-closed sets, $\mathcal{RPC}_I$ and pre_I^*-closed sets, $\mathcal{RPC}_I$ and weakly I_{rg}-closed sets.

1. Preliminaries

In many papers, main characterizations of the problems with sets in topology were studied, for example [4, 6, 7, 8, 11, 13]. In 2011, the concept of pre_I^*-open sets was introduced [5]. Pre_I^*-open sets and pre_I^*-closed sets were used for main characterizations of the problems [3, 5]. After then, in 2015, pre_I^*-open and pre_I^*-closed sets were considered to establish some decompositions and also some characterizations [2]. In this paper, $PC^\star$-closed sets are introduced. $PC^\star$-closed sets contain pre_I^*-open and pre_I^*-closed sets, $\mathcal{RPC}_I$ and pre_I^*-closed sets, $\mathcal{RPC}_I$ and weakly I_{rg}-closed sets.

Let (X, ρ) be a topological space and $U \subset X$. The notation $\mathfrak{cl}(U)$ stands for the closure of U and the notation $\mathfrak{int}(U)$ stands for the interior of U.

A family $\mathfrak{I}$ of subsets of a nonempty set X is said to be an ideal [10] if (1) if $V \in \mathfrak{I}$ and $U \subset V$, then $U \in \mathfrak{I}$, (2) if $U, V \in \mathfrak{I}$, then $U \cup V \in \mathfrak{I}$.

$(X, \rho, \mathfrak{I})$ represent an ideal topological space where (X, ρ) is a topological space with an ideal $\mathfrak{I}$ [10]. Let (X, ρ) be a topological space with an ideal $\mathfrak{I}$ and $U \subset X$. $U^\star = \{x \in X : U \cap V \notin \mathfrak{I}$ for each $V \in \rho$ such that $x \in V\}$ is said to be the local function of U with respect to $\mathfrak{I}$ and ρ [10]. It is known that $\mathfrak{cl}^\star(U) = U \cup U^\star$ defines a Kuratowski closure operator for $\rho^\star$ [9].

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Definition 1.1. Let U be a subset of $(X, \rho, \mathfrak{I})$. U is called
(1) a pre_I^*-open set [5] if $U \subset \mathfrak{int}^\star(\mathfrak{cl}(U))$,
(2) a pre_I^*-closed set [3, 5] if $X \backslash U$ is pre_I^*-open,
(3) a pre_I^*-clopen set [2] if U is pre_I^*-open and pre_I^*-closed.

A subset U of (X, ρ) is called semiopen [12] if $U \subset \mathfrak{cl}(\mathfrak{int}(U))$. The complement of a semiopen subset of X is said to be semi-closed [1]. A subset U of (X, ρ) is called regular open [14] if $U = \mathfrak{int}(\mathfrak{cl}(U))$.

Definition 1.2. Let U be a subset of $(X, \rho, \mathfrak{I})$. U is called
(1) an $\mathcal{RPC}_I$-set [2] if there exist a regular open set V_1 and a pre_I^*-clopen set V_2 in X such that $U = V_1 \cap V_2$,
(2) a weakly I_{rg}-closed set [3] if $(\mathfrak{int}(U))^\star \subset V$ whenever $U \subset V$ and V is a regular open set in X.

Theorem 1.3. *([2]) Let U be a subset of $(X, \rho, \mathfrak{I})$. The following are equivalent for U:*
(1) U is a pre_I^*-clopen set in X,
(2) U is an $\mathcal{RPC}_I$-set and a pre_I^*-closed set in X,
(3) U is an $\mathcal{RPC}_I$-set and a weakly I_{rg}-closed set in X.

2. $PC^\star$-closed sets and pre_I^*-clopen sets

In this Section, $PC^\star$-closed sets are introduced. $PC^\star$-closed sets contain pre_I^*-open and pre_I^*-closed sets, $\mathcal{RPC}_I$ and pre_I^*-closed sets, $\mathcal{RPC}_I$ and weakly I_{rg}-closed sets.

Definition 2.1. Let U be a subset of $(X, \rho, \mathfrak{I})$. U is called $PC^\star$-closed if $\mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)) \in \mathfrak{I}$ for every semiopen set V such that $U \subset V$.

Theorem 2.2. *Let U be a subset of $(X, \rho, \mathfrak{I})$. U is $PC^\star$-closed if and only if $(\mathfrak{int}(U))^\star \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)) \in \mathfrak{I}$ for every semiopen set V such that $U \subset V$.*

Proof. $(\Rightarrow)$: Let U be a $PC^\star$-closed subset of X. Suppose that $U \subset V$ and V is a semiopen subset of X. Since U is a $PC^\star$-closed subset of X, then $\mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)) \in \mathfrak{I}$. It is known that $\mathfrak{cl}^\star(\mathfrak{int}(U))$ is equal to $\mathfrak{int}(U) \cup (\mathfrak{int}(U))^\star$. Since

$$\mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)) \in \mathfrak{I},$$

the union of $(\mathfrak{int}(U) \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)))$ and $((\mathfrak{int}(U))^\star \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)))$ is an element of $\mathfrak{I}$.

Thus,

$$((\mathfrak{int}(U))^\star \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)))$$

is a subset of the union of $(\mathfrak{int}(U)\setminus\mathfrak{int}^\star(\mathfrak{cl}(V)))$ and $((\mathfrak{int}(U))^\star\setminus\mathfrak{int}^\star(\mathfrak{cl}(V)))$. Hence, $(\mathfrak{int}(U))^\star \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)) \in \mathfrak{I}$.

$(\Leftarrow)$: Assume that $(\mathfrak{int}(U))^\star \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)) \in \mathfrak{I}$ for every semiopen set V such that $U \subset V$.

Let $U \subset Y$ and Y be a semiopen subset of X. Then $(\mathfrak{int}(U))^\star \setminus \mathfrak{int}^\star(\mathfrak{cl}(Y)) \in \mathfrak{I}$. We have

$$\mathfrak{int}(U) \subset \mathfrak{int}^\star(\mathfrak{cl}(Y)).$$

This implies $\mathfrak{int}(U) \setminus \mathfrak{int}^\star(\mathfrak{cl}(Y)) = \emptyset \in \mathfrak{I}$. Furthermore,

$$\mathfrak{cl}^\star(\mathfrak{int}(U)) \cap (X \setminus \mathfrak{int}^\star(\mathfrak{cl}(Y)))$$

is equal to

$$((\mathfrak{int}(U))^\star \cup \mathfrak{int}(U)) \cap (X \setminus \mathfrak{int}^\star(\mathfrak{cl}(Y)))$$

and so is equal to the union of

$$((\mathfrak{int}(U))^\star \setminus \mathfrak{int}^\star(\mathfrak{cl}(Y))) \text{ and } (\mathfrak{int}(U) \setminus \mathfrak{int}^\star(\mathfrak{cl}(Y))) \in \mathfrak{I}.$$

Hence, U is a $PC^\star$-closed subset of X. □

Theorem 2.3. *Let U be a subset of $(X, \rho, \mathfrak{I})$. If U is pre$_I^*$-open and pre$_I^*$-closed, then U is a $PC^\star$-closed subset of X.*

Proof. Let U be a pre$_I^*$-open and pre$_I^*$-closed subset of $(X, \rho, \mathfrak{I})$. Take a semiopen set V such that $U \subset V$. We have $X\setminus U \subset \mathfrak{int}^\star(\mathfrak{cl}(X\setminus U))$ and then $X \setminus \mathfrak{int}^\star(\mathfrak{cl}(X \setminus U)) \subset U$. This implies $\mathfrak{cl}^\star(X \setminus \mathfrak{cl}(X \setminus U)) \subset U$ and $\mathfrak{cl}^\star(\mathfrak{int}(U)) \subset U$. Consequently, we have $\mathfrak{cl}^\star(\mathfrak{int}(U)) \subset U \subset \mathfrak{int}^\star(\mathfrak{cl}(U))$. Since $U \subset V$, then

$$\mathfrak{cl}^\star(\mathfrak{int}(U)) \subset \mathfrak{int}^\star(\mathfrak{cl}(U)) \subset \mathfrak{int}^\star(\mathfrak{cl}(V)).$$

Thus, $\mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)) = \emptyset \in \mathfrak{I}$ and hence, U is a $PC^\star$-closed subset of X. □

Remark 2.4. In any ideal topological space $(X, \rho, \mathfrak{I})$, a $PC^\star$-closed subset of X need not be pre$_I^*$-closed and pre$_I^*$-open:

Example 2.5. *Let $X = \{a, b, c, d, e\}$, $\rho = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathfrak{I} = \{\emptyset, \{b\}\}$. Then $U = \{b, d\}$ is $PC^\star$-closed in X but U is not pre$_I^*$-closed and pre$_I^*$-open.*

Remark 2.6. Let U be any subset of $(X, \rho, \mathfrak{I})$. We have the following implications in general for U by Theorem 1.3 and 2.3, Remark 2.4 and Example 2.5.

$$\begin{array}{c} PC^{\star}\text{-closed} \\ \uparrow \\ \text{pre}_I^{*}\text{-clopen (pre}_I^{*}\text{-open and pre}_I^{*}\text{-closed)} \\ \uparrow\downarrow \\ \text{an } \mathcal{RPC}_I\text{-set and pre}_I^{*}\text{-closed} \\ \uparrow\downarrow \\ \text{an } \mathcal{RPC}_I\text{-set and weakly } I_{rg}\text{-closed} \end{array}$$

Theorem 2.7. *Let U be a subset of $(X, \rho, \mathfrak{I})$. U is $PC^{\star}$-closed if and only if for every semiopen subset V such that $U \subset V$, there exists a $Y \in \mathfrak{I}$ such that $(\mathfrak{int}(U))^{\star} \subset \mathfrak{int}^{\star}(\mathfrak{cl}(V)) \cup Y$.*

Proof. Follows from Theorem 2.2. □

Theorem 2.8. *Let U be a subset of $(X, \rho, \mathfrak{I})$. U is $PC^{\star}$-closed if and only if for every semiopen subset V such that $U \subset V$, there exists a $Y \in \mathfrak{I}$ such that $\mathfrak{cl}^{\star}(\mathfrak{int}(U)) \subset \mathfrak{int}^{\star}(\mathfrak{cl}(V)) \cup Y$.*

Proof. Follows from Theorem 2.7. □

Theorem 2.9. *Each set in $(X, \rho, \mathfrak{I})$ is $PC^{\star}$-closed if and only if $\mathfrak{cl}^{\star}(\mathfrak{int}(V)) \setminus \mathfrak{int}^{\star}(\mathfrak{cl}(V)) \in \mathfrak{I}$ for every semiopen subset V of X.*

Proof. $(\Rightarrow)$: Suppose that each set in $(X, \rho, \mathfrak{I})$ is $PC^{\star}$-closed. Let V be a semiopen subset of X. This implies that $\mathfrak{cl}^{\star}(\mathfrak{int}(V)) \setminus \mathfrak{int}^{\star}(\mathfrak{cl}(V)) \in \mathfrak{I}$.

$(\Leftarrow)$: Let $\mathfrak{cl}^{\star}(\mathfrak{int}(Y)) \setminus \mathfrak{int}^{\star}(\mathfrak{cl}(Y)) \in \mathfrak{I}$ for every semiopen subset Y of X. Let $U \subset V$ and V be a semiopen subset of X. We have

$$\begin{aligned} & \mathfrak{cl}^{\star}(\mathfrak{int}(U)) \cap (X \setminus \mathfrak{int}^{\star}(\mathfrak{cl}(V))) \\ \subset \quad & \mathfrak{cl}^{\star}(\mathfrak{int}(V)) \cap (X \setminus \mathfrak{int}^{\star}(\mathfrak{cl}(V))) \in \mathfrak{I} \end{aligned}$$

and so $\mathfrak{cl}^{\star}(\mathfrak{int}(U)) \cap (X \setminus \mathfrak{int}^{\star}(\mathfrak{cl}(V))) \in \mathfrak{I}$. Consequently, U is a $PC^{\star}$-closed subset of X. □

Theorem 2.10. *Each set in $(X, \rho, \mathfrak{I})$ is $PC^{\star}$-closed if and only if $(\mathfrak{int}(V))^{\star} \setminus \mathfrak{int}^{\star}(\mathfrak{cl}(V)) \in \mathfrak{I}$ for every semiopen subset V of X.*

Proof. $(\Rightarrow)$: Suppose that each set in $(X, \rho, \mathfrak{I})$ is $PC^{\star}$-closed. Let V be a semiopen subset of X. By Theorem 2.2, we have $(\mathfrak{int}(V))^{\star} \setminus \mathfrak{int}^{\star}(\mathfrak{cl}(V)) \in \mathfrak{I}$.

$(\Leftarrow)$: Let $(\mathfrak{int}(Y))^\star \cap (X \setminus \mathfrak{int}^\star(\mathfrak{cl}(Y))) \in \mathfrak{I}$ for every semiopen subset Y of X. Let $U \subset V$ and V be a semiopen subset of X. This implies that

$$\begin{aligned} & (\mathfrak{int}(U))^\star \cap (X \setminus \mathfrak{int}^\star(\mathfrak{cl}(V))) \\ \subset \quad & (\mathfrak{int}(V))^\star \cap (X \setminus \mathfrak{int}^\star(\mathfrak{cl}(V))) \in \mathfrak{I}. \end{aligned}$$

Hence, $(\mathfrak{int}(U))^\star \cap (X \setminus \mathfrak{int}^\star(\mathfrak{cl}(V))) \in \mathfrak{I}$. By Theorem 2.2, U is a $PC^\star$-closed subset of X. □

3. $PC^\star$-open sets and properties

In this Section, the concept of $PC^\star$-open sets is introduced and properties are studied.

Definition 3.1. Let U be a subset of $(X, \rho, \mathfrak{I})$. U is said to be $PC^\star$-open if $X \setminus U$ is a $PC^\star$-closed subset of X.

Theorem 3.2. *Let U be a set in $(X, \rho, \mathfrak{I})$. U is $PC^\star$-open if and only if for every semi-closed subset V of X such that $V \subset U$, there exists a set $Y \in \mathfrak{I}$ such that $\mathfrak{cl}^\star(\mathfrak{int}(V)) \setminus Y \subset \mathfrak{int}^\star(\mathfrak{cl}(U))$*

Proof. $(\Rightarrow)$: Let U be $PC^\star$-open in X and V be semi-closed in X such that $V \subset U$. This implies that $X \setminus U \subset X \setminus V$, $X \setminus V$ is semiopen and $X \setminus U$ is a $PC^\star$-closed subset of X. Since $X \setminus U$ is a $PC^\star$-closed subset of X, then

$$\mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)) \setminus \mathfrak{int}^\star(\mathfrak{cl}(X \setminus V)) \in \mathfrak{I}.$$

We have

$$\begin{aligned} & \mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)) \cap \mathfrak{cl}^\star(X \setminus \mathfrak{cl}(X \setminus V)) \\ = \quad & \mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)) \cap \mathfrak{cl}^\star(\mathfrak{int}(V)) \in \mathfrak{I}. \end{aligned}$$

Take $Y = \mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)) \cap \mathfrak{cl}^\star(\mathfrak{int}(V))$. Then $\mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)) \subset \mathfrak{int}^\star(\mathfrak{cl}(X \setminus V)) \cup Y$. We have

$$\begin{aligned} & (X \setminus Y) \cap (X \setminus \mathfrak{int}^\star(\mathfrak{cl}(X \setminus V))) \\ \subset \quad & X \setminus (\mathfrak{cl}^\star(\mathfrak{int}(X \setminus U))). \end{aligned}$$

So

$$\begin{aligned} \mathfrak{cl}^\star(\mathfrak{int}(V)) \setminus Y \quad & \subset \quad X \setminus \mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)) \\ & = \quad \mathfrak{int}^\star(\mathfrak{cl}(U)). \end{aligned}$$

Finally, there exists $Y \in \mathfrak{I}$ such that $\mathfrak{cl}^\star(\mathfrak{int}(V)) \setminus Y \subset \mathfrak{int}^\star(\mathfrak{cl}(U))$.

$(\Leftarrow)$: Suppose that for every semi-closed subset D of X such that $D \subset U$, there exists a set $Y \in \mathfrak{I}$ such that $\mathfrak{cl}^\star(\mathfrak{int}(D)) \setminus Y \subset \mathfrak{int}^\star(\mathfrak{cl}(U))$.

Let $X \setminus U \subset V$ and V be a semiopen subset of $(X, \rho, \mathfrak{I})$. Then $X \setminus V \subset U$ and $X \setminus V$ is semi-closed. This implies that there exists a set $Y \in \mathfrak{I}$ such that $\mathfrak{cl}^\star(\mathfrak{int}(X \setminus V)) \setminus Y \subset \mathfrak{int}^\star(\mathfrak{cl}(U))$. We have

$$\mathfrak{cl}^\star(\mathfrak{int}(X \setminus V)) \cap (X \setminus Y) \subset \mathfrak{int}^\star(\mathfrak{cl}(U)).$$

Furthermore, $X \setminus (\mathfrak{int}^\star(\mathfrak{cl}(U))) \subset X \setminus (\mathfrak{cl}^\star(\mathfrak{int}(X \setminus V)) \setminus Y)$. Then

$$\mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)) \subset \mathfrak{int}^\star(\mathfrak{cl}(V)) \cup Y.$$

This implies

$$\mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)) \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)) \subset Y.$$

Hence, $\mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)) \setminus \mathfrak{int}^\star(\mathfrak{cl}(V)) \in \mathfrak{I}$. Finally $X \setminus U$ is $PC^\star$-closed in X and U is $PC^\star$-open. □

THEOREM 3.3. *Let U be a $PC^\star$-closed subset of $(X, \rho, \mathfrak{I})$, $Y \subset \mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus U$ and Y be a semi-closed subset of X. Then $\mathfrak{cl}^\star(\mathfrak{int}(Y)) \in \mathfrak{I}$.*

Proof. Let U be $PC^\star$-closed in $(X, \rho, \mathfrak{I})$, $Y \subset \mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus U$ and Y be a semi-closed subset of X. This implies that $Y \subset X \setminus U$ and $U \subset X \setminus Y$. Since U is a $PC^\star$-closed subset of X, then $\mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus \mathfrak{int}^\star(\mathfrak{cl}(X \setminus Y)) \in \mathfrak{I}$. We have $\mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus (X \setminus \mathfrak{cl}^\star(\mathfrak{int}(Y))) \in \mathfrak{I}$. Furthermore,

$$\mathfrak{cl}^\star(\mathfrak{int}(Y)) \subset \mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus (X \setminus \mathfrak{cl}^\star(\mathfrak{int}(Y))).$$

Hence, $\mathfrak{cl}^\star(\mathfrak{int}(Y)) \in \mathfrak{I}$. □

COROLLARY 3.4. *Let U be a $PC^\star$-closed subset of $(X, \rho, \mathfrak{I})$, $Y \subset \mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus U$ and Y be a semi-closed subset of X. Then $\mathfrak{int}(Y) \in \mathfrak{I}$.*

Proof. Follows by Theorem 3.3. □

REMARK 3.5. Let U be a $PC^\star$-closed subset of $(X, \rho, \mathfrak{I})$, $Y \subset \mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus U$ and Y be a semi-closed subset of X. These conditions do not always imply $\mathfrak{cl}^\star(Y) \in \mathfrak{I}$ or $Y \in \mathfrak{I}$:

EXAMPLE 3.6. *Let $X = \{a, b, c, d, e\}$, $\rho = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{a, b, d\}, X\}$ and $\mathfrak{I} = \{\emptyset, \{c\}\}$. Then $U = \{a, b\}$ is $PC^\star$-closed in X. Also, $Y = \{c, d, e\}$ is semi-closed and $Y \subset \mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus U$. But $\mathfrak{cl}^\star(Y) = Y \notin \mathfrak{I}$.*

THEOREM 3.7. *Let U be $PC^\star$-closed in $(X, \rho, \mathfrak{I})$. Then $\mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus U$ is $PC^\star$-open in $(X, \rho, \mathfrak{I})$.*

Proof. Let U be $PC^\star$-closed in $(X, \rho, \mathfrak{I})$, $Y \subset \mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus U$ and Y be a semi-closed subset of X. By Theorem 3.3, we have $\mathfrak{cl}^\star(\mathfrak{int}(Y)) \in \mathfrak{I}$. Then there exists $\mathfrak{cl}^\star(\mathfrak{int}(Y)) \in \mathfrak{I}$ such that

$$\begin{aligned} \mathfrak{cl}^\star(\mathfrak{int}(Y)) \setminus \mathfrak{cl}^\star(\mathfrak{int}(Y)) &= \emptyset \\ &\subset \mathfrak{int}^\star(\mathfrak{cl}(\mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus U)). \end{aligned}$$

By Theorem 3.2, $\mathfrak{cl}^\star(\mathfrak{int}(U)) \setminus U$ is $PC^\star$-open in $(X, \rho, \mathfrak{I})$. □

THEOREM 3.8. *Let U be $PC^\star$-open in $(X, \rho, \mathfrak{I})$, $\mathfrak{int}^\star(\mathfrak{cl}(U)) \cup (X \setminus U) \subset V$ and V be a semiopen subset of X. Then $\mathfrak{cl}^\star(\mathfrak{int}(X \setminus V)) \in \mathfrak{I}$.*

Proof. Let U be $PC^\star$-open in $(X, \rho, \mathfrak{I})$, $\mathfrak{int}^\star(\mathfrak{cl}(U)) \cup (X \setminus U) \subset V$ and V be a semiopen subset of X.

Since $\mathfrak{int}^\star(\mathfrak{cl}(U)) \subset V$, then

$$\begin{aligned} X \setminus V &\subset X \setminus \mathfrak{int}^\star(\mathfrak{cl}(U)) \\ &= \mathfrak{cl}^\star(\mathfrak{int}(X \setminus U)). \end{aligned}$$

Furthermore, since $X \setminus U \subset V$, then $X \setminus V \subset U$ and also $X \setminus V$ is semi-closed. Then, $X \setminus V \subset (\mathfrak{cl}^\star(\mathfrak{int}(X \setminus U))) \cap U$. By Theorem 3.3, $\mathfrak{cl}^\star(\mathfrak{int}(X \setminus V)) \in \mathfrak{I}$. □

THEOREM 3.9. *Let U be $PC^\star$-closed in $(X, \rho, \mathfrak{I})$, $Y \subset (\mathfrak{int}(U))^\star \setminus U$ and Y be semi-closed in X. Then $\mathfrak{cl}^\star(\mathfrak{int}(Y)) \in \mathfrak{I}$.*

Proof. Follows by Theorem 3.3. □

COROLLARY 3.10. *Let U be $PC^\star$-closed in $(X, \rho, \mathfrak{I})$, $Y \subset (\mathfrak{int}(U))^\star \setminus U$ and Y be semi-closed in X. Then $\mathfrak{int}(Y) \in \mathfrak{I}$ and $(\mathfrak{int}(Y))^\star \in \mathfrak{I}$.*

Proof. Follows by Theorem 3.9. □

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