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THE RESULTS CONCERNING JORDAN DERIVATIONS

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Abstract. Let R be a 3!-torsion free semiprime ring, and let $D : R \to R$ be a Jordan derivation on a semiprime ring R. In this case, we show that $[D(x), x]D(x) = 0$ if and only if $D(x)[D(x), x] = 0$ for every $x \in R$. In particular, let A be a Banach algebra with $\mathrm{rad}(A)$. If D is a continuous linear Jordan derivation on A, then we see that $[D(x), x]D(x) \in \mathrm{rad}(A)$ if and only if $[D(x), x]D(x) \in \mathrm{rad}(A)$ for all $x \in A$.

1. Introduction

Throughout, R represents an associative ring and A will be a real or complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. Let $\mathrm{rad}(R)$ denote the (*Jacobson*) *radical* of a ring R. And a ring R is said to be (*Jacobson*) *semisimple* if its Jacobson radical $\mathrm{rad}(R)$ is zero. See [1] for the more details.

A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$.

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

Johnson and Sinclair [4] proved that any linear derivation on a semisimple Banach algebra is continuous. A result of Singer and Wermer [5] states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra.

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Vukman [7] proved the following: let R be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he proved that the following holds: let A be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

In this paper, our first aim is to prove the following result in the ring theory in order to apply it to the Banach algebra theory:

let R be a 3!-torsion free semiprime ring, and let $D : R \to R$ be a Jordan derivation on a semiprime ring R. In this case, we show that $[D(x), x]D(x) = 0$ if and only if $D(x)[D(x), x] = 0$ for every $x \in R$. In particular, let A be a Banach algebra with $\mathrm{rad}(A)$ and if D is a continuous linear Jordan derivation on A, then we see that $[D(x), x]D(x) \in \mathrm{rad}(A)$ if and only if $[D(x), x]D(x) \in \mathrm{rad}(A)$ for all $x \in A$.

2. Preliminaries

The following lemma is due to Chung and Luh [3].

LEMMA 2.1. *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \cdots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t = 1, 2, \cdots, n$. Then we have $y_k = 0$ for every positive integer k with $1 \leq k \leq n$.*

The following theorem is due to Brešar [2].

THEOREM 2.2. *Let R be a 2-torsion free semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. In this case, D is a derivation.*

And the following theorem is proved by Vukman in [7] under the condition of the prime ring R.

THEOREM 2.3. *Let R be a 3!-torsionfree prime ring. Let $D : R \longrightarrow R$ be a Jordan derivation on R. In this case, we see that if $[D(x), x]D(x) = 0$ for every $x \in R$, then $D(x) = 0$ for all $x \in R$*

3. Main results

We need the following notations. After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where m is a positive integer. When R is a ring, we shall denote the maps $B : R \times R \longrightarrow R,\ f, g : R \longrightarrow R$ by $B(x,y) \equiv [D(x),y] + [D(y),x], f(x) \equiv [D(x),x],\ g(x) \equiv [f(x),x]$ for all $x, y \in R$ respectively. And we have the basic properties:

$$B(x,y) = B(y,x),$$
$$B(x,yz) = B(x,y)z + yB(x,z) + D(y)[z,x] + [y,x]D(z)$$

for all $x, y \in R$ and $z \in R$.

Theorem 3.1. *Let R be a 3!-torsionfree semiprime ring. Let $D : R \longrightarrow R$ be a Jordan derivation on R. In this case, it follows that*

$$[D(x),x]D(x) = 0 \ \Leftrightarrow\ D(x)[D(x),x] = 0$$

for every $x \in R$.

Proof. (Necessity)

It is sufficient to prove the noncommutative case of R.

Assume that

(3.1) $$[D(x),x]D(x) = f(x)D(x) = 0,\ x \in R.$$

Replacing $x + ty$ for x in (3.1), we have

$$[D(x+ty), x+ty]D(x+ty)$$
$$\equiv f(x)D(x) + t\{B(x,y)D(x) + f(x)D(y)\} + t^2 H(x,y)$$
(3.2) $$+t^3 f(y)D(y) = 0,\ x, y \in R,\ t \in S_2$$

where H denotes the term satisfying the identity (3.2).

From (3.1) and (3.2), we obtain

$$t\{B(x,y)D(x) + f(x)D(y)\} + t^2 H(x,y)$$
(3.3) $$= 0,\ x, y \in R,\ t \in S_2.$$

Since R is 2!-torsionfree, by Lemma 2.1 the relation (3.3) yields

(3.4) $$B(x,y)D(x) + f(x)D(y) = 0,\ x, y \in R.$$

Writing yx for y in (3.4), we get

$$B(x,yx)D(x) + f(x)D(yx)$$
$$= B(x,y)xD(x) + 2yf(x)D(x) + [y,x]D(x)^2 + f(x)D(y)x$$
(3.5) $$+f(x)yD(x) = 0,\ x, y \in R.$$

From (3.1) and (3.5), we obtain

$$B(x,y)xD(x) + [y,x]D(x)^2 + f(x)D(y)x + f(x)yD(x)$$
(3.6) $$= 0,\ x, y \in R.$$

Right multiplication of (3.4) by x leads to

$$B(x,y)D(x)x + f(x)D(y)x = 0,\ x,y \in R. \tag{3.7}$$

From (3.6) and (3.7), we arrive at

$$-B(x,y)f(x) + [y,x]D(x)^2 + f(x)yD(x) = 0,\ x,y \in R. \tag{3.8}$$

Substituting xy for y in (3.8), we have

$$\begin{aligned} &-xB(x,y)f(x) - 2f(x)yf(x) - D(x)[y,x]f(x) \\ &+x[y,x]D(x)^2 + f(x)xyD(x) = 0,\ x,y \in R. \end{aligned} \tag{3.9}$$

Left multiplication of (3.8) by x yields

$$-xB(x,y)f(x) + x[y,x]D(x)^2 + xf(x)yD(x) = 0,\ x,y \in R. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$-2f(x)yf(x) - D(x)[y,x]f(x) + g(x)yD(x) = 0,\ x,y \in R. \tag{3.11}$$

Putting $D(x)y$ instead of y in (3.11), we get

$$\begin{aligned} &-2f(x)D(x)yf(x) - D(x)^2[y,x]f(x) - D(x)f(x)yf(x) \\ &+g(x)D(x)yD(x) = 0,\ x,y \in R. \end{aligned} \tag{3.12}$$

From (3.1) and (3.12), we have

$$\begin{aligned} &-D(x)^2[y,x]f(x) - D(x)f(x)yf(x) + g(x)D(x)yD(x) \\ &= 0,\ x,y \in R. \end{aligned} \tag{3.13}$$

Left multiplication of (3.12) by $D(x)$ gives

$$\begin{aligned} &-2D(x)f(x)yf(x) - D(x)^2[y,x]f(x) + D(x)g(x)yD(x) \\ &= 0,\ x,y \in R. \end{aligned} \tag{3.14}$$

From (3.1) and (3.14), we arrive at

$$\begin{aligned} &D(x)f(x)yf(x) + (g(x)D(x) - D(x)g(x))yD(x) \\ &= 0,\ x,y \in R. \end{aligned} \tag{3.15}$$

Right multiplication of (3.15) by $D(x)$ leads to

$$\begin{aligned} &D(x)f(x)yf(x)D(x) + (g(x)D(x) - D(x)g(x))yD(x)^2 \\ &= 0,\ x,y \in R. \end{aligned} \tag{3.16}$$

From (3.1) and (3.16), we obtain

$$(g(x)D(x) - D(x)g(x))yD(x)^2 = 0,\ x,y \in R. \tag{3.17}$$

Writing $yD(x)$ for y in (3.15), we have

$$D(x)f(x)yD(x)f(x) + (g(x)D(x) - D(x)g(x))yD(x)^2 \tag{3.18}$$
$$= 0,\ x, y \in R.$$

From (3.17) and (3.18), we get

$$D(x)f(x)yD(x)f(x) = 0,\ x, y \in R. \tag{3.19}$$

Since R is semiprime, (3.19) gives

$$D(x)f(x) = 0,\ x \in R.$$

Therefore the necessity is proved.

The inverse statement is symmetrically proved in the expressions proved.

(Sufficiency)

Suppose

$$D(x)[D(x), x] = D(x)f(x) = 0,\ x \in R. \tag{3.20}$$

Replacing $x + ty$ for x in (3.20), we have

$$D(x + ty)[D(x + ty), x + ty]$$
$$\equiv D(x)f(x) + t\{D(y)f(x) + D(x)B(x, y)\} + t^2 I(x, y)$$
$$+t^3 D(y)f(y) = 0,\ x, y \in R,\ t \in S_2 \tag{3.21}$$

where I denotes the term satisfying the identity (3.21).
From (3.20) and (3.21), we obtain

$$t\{D(y)f(x) + D(x)B(x, y)\} + t^2 I(x, y)$$
$$= 0,\ x, y \in R,\ t \in S_2. \tag{3.22}$$

Since R is 2!-torsionfree, by Lemma 2.1 the relation (3.22) yields

$$D(y)f(x) + D(x)B(x, y) = 0,\ x, y \in R. \tag{3.23}$$

Writing xy for y in (3.23), we get

$$xD(y)f(x) + D(x)yf(x) + D(x)xB(x, y) + 2D(x)f(x)y$$
$$+D(x)^2[y, x] = 0,\ x, y \in R. \tag{3.24}$$

From (3.20) and (3.24), it follows from that

$$xD(y)f(x) + D(x)yf(x) + D(x)xB(x, y) + D(x)^2[y, x]$$
$$= 0,\ x, y \in R. \tag{3.25}$$

Left multiplication of (3.23) by x leads to

$$xD(y)f(x) + xD(x)B(x, y) = 0,\ x, y \in R. \tag{3.26}$$

From (3.25) and (3.26), we get

$$D(x)yf(x) + f(x)B(x,y) + D(x)^2[y,x] = 0,\ x,y \in R. \tag{3.27}$$

Right multiplication of (3.27) by x yields

$$D(x)yf(x)x + f(x)B(x,y)x + D(x)^2[y,x]x = 0,\ x,y \in R. \tag{3.28}$$

Replacing yx for y in (3.27), we have

$$\begin{aligned} &D(x)yxf(x) + f(x)B(x,y)x + 2f(x)yf(x) + f(x)[y,x]D(x) \\ &+D(x)^2[y,x]x = 0,\ x,y \in R. \end{aligned} \tag{3.29}$$

From (3.28) and (3.29), we obtain

$$D(x)yg(x) - 2f(x)yf(x) - f(x)[y,x]D(x) = 0,\ x,y \in R. \tag{3.30}$$

Right multiplication of (3.30) by $D(x)$ gives

$$\begin{aligned} &D(x)yg(x)D(x) - 2f(x)yf(x)D(x) - f(x)[y,x]D(x)^2 \\ &= 0,\ x,y \in R. \end{aligned} \tag{3.31}$$

Putting $yD(x)$ instead of y in (3.30), it is obvious that

$$\begin{aligned} &D(x)yD(x)g(x) - 2f(x)yD(x)f(x) - f(x)[y,x]D(x)^2 \\ &-f(x)yf(x)D(x) = 0,\ x,y \in R. \end{aligned} \tag{3.32}$$

From (3.20) and (3.32), we have

$$\begin{aligned} &D(x)yD(x)g(x) - f(x)[y,x]D(x)^2 - f(x)yf(x)D(x) \\ &= 0,\ x,y \in R. \end{aligned} \tag{3.33}$$

From (3.31) and (3.33), we get

$$\begin{aligned} &D(x)y\{g(x)D(x) - D(x)g(x)\} - f(x)yf(x)D(x) \\ &= 0,\ x,y \in R. \end{aligned} \tag{3.34}$$

Left multiplication of (3.34) by $D(x)$ leads to

$$\begin{aligned} &D(x)^2y\{g(x)D(x) - D(x)g(x)\} - D(x)f(x)yf(x)D(x) \\ &= 0,\ x,y \in R. \end{aligned} \tag{3.35}$$

From (3.20) and (3.35), we obtain

$$D(x)^2y\{g(x)D(x) - D(x)g(x)\} = 0,\ x,y \in R. \tag{3.36}$$

Substituting $D(x)y$ for y in (3.34), we have

$$\begin{aligned} &D(x)^2y\{g(x)D(x) - D(x)g(x)\} - f(x)D(x)yf(x)D(x) \\ &= 0,\ x,y \in R. \end{aligned} \tag{3.37}$$

From (3.36) and (3.37), we obtain

$$f(x)D(x)yf(x)D(x) = 0,\ x, y \in R. \tag{3.38}$$

Since R is semiprime, (3.38) gives

$$f(x)D(x) = 0,\ x \in R.$$

Therefore the sufficiency is proved. □

We obtain the equivalent property of continuous Jordan derivations on Banach algebras as the application to the Banach algebra theory.

THEOREM 3.2. *Let A be a Banach algebra with rad(A). Let $D : A \longrightarrow A$ be a continuous linear Jordan derivation. Then we obtain*

$$[D(x), x]D(x) \in rad(A) \iff D(x)[D(x), x] \in rad(A)$$

for every $x \in A$.

Proof. It suffices to prove the case that A is noncommutative. By the result of B.E. Johnson and A.M. Sinclair [4] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [5] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P,\ \hat{x} = x + P$. By the assumption that $[D(x), x]D(x) \in \mathrm{rad}(A),\ x \in A$, we obtain $[D_P(\hat{x}), \hat{x}]D_P(\hat{x}) = 0 \iff D_P(\hat{x})[D_P(\hat{x}), \hat{x}] = 0,\ \hat{x} \in A/P$, since all the assumptions of Theorem 3.1 are fulfilled. Thus we see that

$$[D(x), x]D(x) \in P \iff D(x)[D(x), x] \in P$$

for every $x \in A$ and all primitive ideals of A. Therefore we conclude that

$$[D(x), x]D(x) \in \mathrm{rad}(A) \iff D(x)[D(x), x] \in \mathrm{rad}(A)$$

for every $x \in A$. □

As a special case of Theorem 3.2 we get the following result which characterizes commutative semisimple Banach algebras.

COROLLARY 3.3. *Let A be a semisimple Banach algebra. Then we have*

$$[[y, x], x]][y, x] = 0 \iff [y, x][[y, x], x] = 0$$

for every $x, y \in A$.

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