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Modified inverse moment estimation: its principle and applications

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Abstract

In this survey, we present a modified inverse moment estimation of parameters and its applications. We use a specific model to demonstrate its principle and how to apply this method in practice. The estimation of unknown parameters is considered. A necessary and sufficient condition for the existence and uniqueness of maximum-likelihood estimates of the parameters is obtained for the classical maximum likelihood estimation. Inverse moment and modified inverse moment estimators are proposed and their properties are studied. Monte Carlo simulations are conducted to compare the performances of these estimators. As far as the biases and mean squared errors are concerned, modified inverse moment estimator works the best in all cases considered for estimating the unknown parameters. Its performance is followed by inverse moment estimator and maximum likelihood estimator, especially for small sample sizes.

Keywords: inverse moment estimators, maximum likelihood estimates, existence and uniqueness, joint confidence regions, small sample size, Weibull distribution, inverted exponential Pareto distribution, Monte Carlo simulation

1. Introduction

The inverse estimation method was originally proposed by Wang (1992) and was applied to study parameter estimation for Weibull distribution. Different from the regular method of moments, the idea of the inverse moment estimation (IME) is as follows.

For a sample $X_1, \dots, X_n$ from a distribution with unknown parameters, first transform the original sample to a quasi-sample $Y_1, \dots, Y_n$, where Y_i contains the unknown parameters but its distribution does not depend on unknown parameters, that is, Y_i is a pivot variable, $i = 1, \dots, n$. The population moments of the new sample do not dependent on unknown parameters while the sample moments do. Let the population moments of the quasi-sample equal the sample moments and solve the unknown parameters.

Wang (2004) obtained the inverse moment estimators and the interval estimation based on type II progressively censored data under the Weibull distribution. The simulation results showed that the mean square errors of the inverse moment estimators are less than the maximum likelihood estimates (MLE)'s. Gu and Yue (2013) considered the problem of estimating parameters of the generalized exponential distribution based on a complete sample. They proposed the inverse moment estimators of the parameters of the generalized exponential distribution. The precisions of MLEs and IMEs are compared through numerical simulations. Gui (2015) studied the problem of estimating unknown shape and scale parameters of exponentiated half logistic distribution. Inverse moment and modified

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inverse moment estimators were derived. Monte Carlo simulations were conducted to compare the performances.

Wang *et al.* (2010) obtained some exponential inequalities for a negatively orthant dependent sequence and used the exponential inequalities to study the asymptotic approximation of an inverse moment for negatively orthant dependent random variables. Ye and Yang (2013) proposed a new method for dimension reduction in regression using the first two inverse moments. Yang *et al.* (2014) discussed the asymptotic approximation of an inverse moment for nonnegative random variables. Cheng *et al.* (2014) established inverse moment bounds for sample autocovariance matrices based on a detrended time series.

In this paper, we focus on the problem of parameter estimation for the inverted exponential Pareto distribution and demonstrate the inverse estimation and its modified version. We begin with the classical MLE and obtain a necessary and sufficient condition for the existence and uniqueness of MLE of the parameters. We propose inverse moment and modified inverse moment estimators and study their properties. Monte Carlo simulations are used to compare the performances. We also propose the methods for constructing joint confidence regions for the two parameters and study their performances.

Gupta *et al.* (1998) introduced the exponential Pareto distribution Y to model failure time data. The probability density function of Y is given by

$$f(y;\lambda,\alpha) = \alpha\lambda\left[1-(1+y)^{-\lambda}\right]^{\alpha-1}(1+y)^{-(\lambda+1)}, \quad y>0, \tag{1.1}$$

where $\alpha > 0$ and $\lambda > 0$ are two parameters. The corresponding cumulative distribution function is

$$F(y;\lambda,\alpha) = \left[1-(1+y)^{-\lambda}\right]^{\alpha}, \quad y>0. \tag{1.2}$$

A random variable $X = 1/Y$ is said to have the inverted exponential Pareto distribution. Its cumulative distribution function (cdf) and probability density function (pdf) are specified by

$$F(x;\lambda,\alpha) = 1-\left[1-\left(1+\frac{1}{x}\right)^{-\lambda}\right]^{\alpha}, \quad x>0, \tag{1.3}$$

and

$$f(x;\lambda,\alpha) = \frac{\alpha\lambda\left(\frac{1}{x}+1\right)^{-\lambda-1}\left[1-\left(\frac{1}{x}+1\right)^{-\lambda}\right]^{\alpha-1}}{x^2}, \quad x>0, \tag{1.4}$$

respectively, where $\lambda > 0$ and $\alpha > 0$ are the parameters. We denote this distribution as IEPD(λ,α). When $\alpha = 1$, the inverted exponential Pareto distribution reduces to the inverted Pareto distribution. In this paper, we will show that the two-parameter inverted exponential Pareto distribution can be quite effective in modeling lifetime data.

Shawky and Abu-Zinadah (2009) considered the maximum likelihood estimation of the different parameters of an exponential Pareto distribution. Afify (2010) obtained Bayes and classical estimators for two parameters exponentiated Pareto distribution when a sample is available from complete, type I and type II censoring scheme. Ali *et al.* (2010) derived the distribution of the ratio of two independent exponentiated Pareto random variables and studied its properties. Singh *et al.* (2013) proposed maximum likelihood estimators and Bayes estimators of parameters of exponentiated Pareto distribution under general entropy loss function and squared error loss function for progressive type-II censored data with binomial removals.

The rest of this paper is organized as follows. In Section 2, we discuss the classical maximum likelihood estimation of the parameters of the inverted exponential Pareto distribution. In Section 3, we propose the inverse and modified inverse estimation methods to estimate the parameters and study their properties. Joint confidence regions for the two parameters are also proposed in Section 4. Section 5 conducts simulations to compare the estimators and the confidence regions. In Section 6, a numerical example is presented to illustrate the superiorities of the proposed methods. Finally, Section 7 concludes.

2. Maximum likelihood estimation

In this section, we discuss the MLEs of the parameters of inverted exponential Pareto distribution (IEPD) based on a complete sample. Let $X_1, X_2, \ldots, X_n$ be a random sample from IEPD(λ, α) with pdf and cdf as (1.4) and (1.3), respectively. The log-likelihood function is given by

$$L(\lambda,\alpha) = (\alpha-1)\sum_{i=1}^{n}\log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right] - (\lambda+1)\sum_{i=1}^{n}\log\left(\frac{1}{x_i}+1\right) + n\log\alpha + n\log\lambda - 2\sum_{i=1}^{n}\log(x_i)\,. \tag{2.1}$$

The score equations are as:

$$\frac{\partial L(\lambda,\alpha)}{\partial\lambda} = (\alpha-1)\sum_{i=1}^{n}\frac{\left(\frac{1}{x_i}+1\right)^{-\lambda}\log\left(\frac{1}{x_i}+1\right)}{1-\left(\frac{1}{x_i}+1\right)^{-\lambda}} - \sum_{i=1}^{n}\log\left(\frac{1}{x_i}+1\right) + \frac{n}{\lambda}, \tag{2.2}$$

$$\frac{\partial L(\lambda,\alpha)}{\partial\alpha} = \sum_{i=1}^{n}\log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right] + \frac{n}{\alpha}. \tag{2.3}$$

Consider the case when $x_1 = \cdots = x_n = x$, the MLEs $\hat{\lambda}$ and $\hat{\alpha}$ are

$$\hat{\alpha} = -\frac{1}{\log\left[1-\left(\frac{1}{x}+1\right)^{-\lambda}\right]}$$

and $\hat{\lambda}$ is the solution of the equation

$$J(\lambda) = J_1(\lambda) - J_2(\lambda),$$

where

$$J_1(\lambda) = -\frac{\left(\frac{1}{x}+1\right)^{-\lambda}\log\left(\frac{1}{x}+1\right)}{1-\left(\frac{1}{x}+1\right)^{-\lambda}} - \log\left(\frac{1}{x}+1\right) + \frac{1}{\lambda},$$

$$J_2(\lambda) = \frac{1}{\log\left[1-\left(\frac{1}{x}+1\right)^{-\lambda}\right]}\frac{\left(\frac{1}{x}+1\right)^{-\lambda}\log\left(\frac{1}{x}+1\right)}{1-\left(\frac{1}{x}+1\right)^{-\lambda}}.$$

Note that

$$J_1'(\lambda) = \frac{\lambda^2\left(\frac{1}{x}+1\right)^{\lambda}\log^2\left(\frac{1}{x}+1\right) - \left(\left(\frac{1}{x}+1\right)^{\lambda}-1\right)^2}{\lambda^2\left(\left(\frac{1}{x}+1\right)^{\lambda}-1\right)^2} < 0,$$

since $(t-1)^2 > (\log t)^2 t$, for all $t > 1$.

$$J_2'(\lambda) = -\frac{\log^2\left(\frac{1}{x}+1\right)\left[\left(\frac{1}{x}+1\right)^{\lambda}\log\left(1-\left(\frac{1}{x}+1\right)^{-\lambda}\right)+1\right]}{\left[\left(\frac{1}{x}+1\right)^{\lambda}-1\right]^2\log^2\left(1-\left(\frac{1}{x}+1\right)^{-\lambda}\right)} > 0,$$

since $(1/t)\log(1-t)+1 < 0$, for all $t > 0$. $J'(\lambda) < 0$, $J(\lambda)$ is a decreasing function of λ. Moreover, we have $J(0) = \infty$ and $J(\infty) = 0$. Thus, $J(\lambda)$ has no roots in the interval $(0, \infty)$.

In the following, we discuss the existence and uniqueness of MLEs in the case of at least two non-identical observed values of the sample.

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be a random sample from IEPD(λ, α), if the observed values of the sample are not identical, that is $x_i \neq x_{(n)} = \max\{x_1, \ldots, x_n\}$ for at least one $i \in \{1, \ldots, n-1\}$, then MLEs of λ and α exist and unique.*

Proof: From (2.3) we obtain the MLE of α as a function of λ,

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^n \log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right]}. \tag{2.4}$$

The MLE of λ is the root of the following equation

$$G(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^n \frac{\log\left(\frac{1}{x_i}+1\right)}{1-\left(\frac{1}{x_i}+1\right)^{-\lambda}} - \frac{n}{\sum_{i=1}^n \log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right]}\sum_{i=1}^n \frac{\left(\frac{1}{x_i}+1\right)^{-\lambda}\log\left(\frac{1}{x_i}+1\right)}{1-\left(\frac{1}{x_i}+1\right)^{-\lambda}} = 0. \tag{2.5}$$

Firstly, we prove that the equation $G(\lambda) = 0$ has a positive root. We calculate the limits $G(0)$ and $G(+\infty)$ respectively.

$$\begin{aligned}
G(0) &= \frac{n}{\lambda} - \sum_{i=1}^n \frac{\log\left(\frac{1}{x_i}+1\right)}{1-\left(\frac{1}{x_i}+1\right)^{-\lambda}} - \frac{n}{\sum_{i=1}^n \log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right]}\sum_{i=1}^n \frac{\left(\frac{1}{x_i}+1\right)^{-\lambda}\log\left(\frac{1}{x_i}+1\right)}{1-\left(\frac{1}{x_i}+1\right)^{-\lambda}} \\
&= \sum_{i=1}^n \log\left(\frac{1}{x_i}+1\right)\lim_{\lambda\to 0}\left[\frac{1}{\lambda\log\left(\frac{1}{x_i}+1\right)} - \frac{1}{1-\left(\frac{1}{x_i}+1\right)^{-\lambda}}\right] \\
&\quad - \lim_{\lambda\to 0}\frac{n}{\sum_{i=1}^n \lambda\log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right]}\sum_{i=1}^n \frac{\lambda\left(\frac{1}{x_i}+1\right)^{-\lambda}\log\left(\frac{1}{x_i}+1\right)}{1-\left(\frac{1}{x_i}+1\right)^{-\lambda}} \\
&= -\frac{1}{2}\sum_{i=1}^n \log\left(\frac{1}{x_i}+1\right) - \lim_{\lambda\to 0}\frac{n^2}{\sum_{i=1}^n \lambda\log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right]}
\end{aligned}$$

$$= -\frac{1}{2}\sum_{i=1}^{n}\log\left(\frac{1}{x_i}+1\right) - \frac{n^2}{\sum_{i=1}^{n}\left(1/\log\left(\frac{1}{x_i}+1\right)\right)\lim_{\lambda\to 0}\lambda\log\left(\frac{1}{x_i}+1\right)\log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right]}$$
$$= +\infty.$$

$$G(+\infty) = -\sum_{i=1}^{n}\log\left(\frac{1}{x_i}+1\right) - \lim_{\lambda\to\infty}\frac{n\sum_{i=1}^{n}\left(\frac{1}{x_i}+1\right)^{-\lambda}\log\left(\frac{1}{x_i}+1\right)}{\sum_{i=1}^{n}\log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right]}$$
$$= -\sum_{i=1}^{n}\log\left(\frac{1}{x_i}+1\right) + \lim_{\lambda\to\infty}\frac{n\sum_{i=1}^{n}\left(\frac{1}{x_i}+1\right)^{-\lambda}\log\left(\frac{1}{x_i}+1\right)}{\sum_{i=1}^{n}\left(\frac{1}{x_i}+1\right)^{-\lambda}}$$
$$= -\sum_{i=1}^{n}\log\left(\frac{1}{x_i}+1\right) + \lim_{\lambda\to\infty}\frac{n\sum_{i=1}^{n}\left(\frac{1}{x_i}+1\right)^{-\lambda}\left(\frac{1}{x_{(n)}}+1\right)^{\lambda}\log\left(\frac{1}{x_i}+1\right)}{\sum_{i=1}^{n}\left(\frac{1}{x_i}+1\right)^{-\lambda}\left(\frac{1}{x_{(n)}}+1\right)^{\lambda}}$$
$$= -\sum_{i=1}^{n}\log\left(\frac{1}{x_i}+1\right) + n\log\left(\frac{1}{x_{(n)}}+1\right)$$
$$= -\sum_{i=1}^{n}\left[\log\left(\frac{1}{x_i}+1\right) - \log\left(\frac{1}{x_{(n)}}+1\right)\right] < 0.$$

It follows that the equation $G(\lambda) = 0$ has a positive real root.

Secondly, we show that the root is unique. We rewrite $G(\lambda)$ as

$$G(\lambda) = G_1(\lambda) - G_2(\lambda),$$

where

$$G_1(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n}\frac{\log\left(\frac{1}{x_i}+1\right)}{1-\left(\frac{1}{x_i}+1\right)^{-\lambda}} \quad \text{and} \quad G_2(\lambda) = \frac{n}{\sum_{i=1}^{n}\log\left[1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right]}\sum_{i=1}^{n}\frac{\left(\frac{1}{x_i}+1\right)^{-\lambda}\log\left(\frac{1}{x_i}+1\right)}{1-\left(\frac{1}{x_i}+1\right)^{-\lambda}}.$$

$$G_1'(\lambda) = \sum_{i=1}^{n}\left[\frac{\left(\frac{1}{x_i}+1\right)^{-\lambda}\log^2\left(\frac{1}{x_i}+1\right)}{\left(1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right)^2} - \frac{1}{\lambda^2}\right] = \sum_{i=1}^{n}\frac{-\left[\left(\frac{1}{x_i}+1\right)^{\lambda}-1\right]^2 + \lambda^2\left(\frac{1}{x_i}+1\right)^{\lambda}\log^2\left(\frac{1}{x_i}+1\right)}{\lambda^2\left[\left(\frac{1}{x_i}+1\right)^{\lambda}-1\right]^2} < 0,$$

since $(t-1)^2 > t(\log t)^2$ for all $t > 1$.

$$G_2'(\lambda) = \frac{n\left[\left(\sum_{i=1}^{n}-\log\left(1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right)\right)\sum_{i=1}^{n}\frac{\left(\frac{1}{x_i}+1\right)^{\lambda}\log^2\left(\frac{1}{x_i}+1\right)}{\left(\left(\frac{1}{x_i}+1\right)^{\lambda}-1\right)^2} - \left(\sum_{i=1}^{n}\frac{\log\left(\frac{1}{x_i}+1\right)}{\left(\frac{1}{x_i}+1\right)^{\lambda}-1}\right)^2\right]}{\left(\sum_{i=1}^{n}\log\left(1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right)\right)^2}$$
$$\geq \frac{n\left[\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}+1\right)^{-\lambda}\right)\sum_{i=1}^{n}\frac{\left(\frac{1}{x_i}+1\right)^{\lambda}\log^2\left(\frac{1}{x_i}+1\right)}{\left(\left(\frac{1}{x_i}+1\right)^{\lambda}-1\right)^2} - \left(\sum_{i=1}^{n}\frac{\log\left(\frac{1}{x_i}+1\right)}{\left(\frac{1}{x_i}+1\right)^{\lambda}-1}\right)^2\right]}{\left(\sum_{i=1}^{n}\log\left(1-\left(\frac{1}{x_i}+1\right)^{-\lambda}\right)\right)^2}$$
$$\geq 0.$$

Since $-\log(1-t) \geq t$ for all $t > 0$. The second inequality holds by Cauchy-Schwartz inequality. Therefore, $G'(\lambda) < 0$, $G(\lambda)$ is decreasing and $G(\lambda) = 0$ has unique root over $(0, +\infty)$. □

3. Inverse and modified inverse moment estimation

In Statistics, there are many methods available for estimating the parameter(s) of interest. One of the oldest methods is the method of moments. It is based on the assumption that sample moments should provide adequate estimates of the corresponding population moments. Suppose we want to estimate $\theta = (\theta_1, \ldots, \theta_k)$, the procedure is:

(1) Find k population moments, μ_i, $i = 1, 2, \ldots, k$. μ_i will include one or more parameters $\theta_1, \ldots, \theta_k$.

(2) Determine the corresponding k sample moments, m_i, $i = 1, 2, \ldots, k$.

(3) Let $\mu_i = m_i$, $i = 1, 2, \ldots, k$, solve for the parameters. The solution is a moment estimator.

The method of moments is simple and easy to compute. However, the estimator may not be unique or not exist. In this section, we propose an inverse moment estimation. The superiority of the new estimator is its existence and uniqueness.

Definition 1. *Suppose $X \sim F(x, \theta)$, where $\theta = (\theta_1, \ldots, \theta_k)$ is a parameter vector to be estimated. Transform X to a pivotal variable $Y = g(X, \theta)$ whose distribution does not depend on θ. The population moments μ'_i $(i = 1, \ldots, k)$ of Y will not contain θ. The sample $Y_i = g(X_i, \theta)$ $(i = 1, \ldots, k)$ is called quasi-sample since it is a function of sample $(X_1, \ldots, X_n)$ and parameter θ. The moments of the quasi-sample m'_i $(i = 1, \ldots, k)$ is also a function of θ. Let $\mu'_i = m'_i$, $i = 1, 2, \ldots, k$, solve for the parameters. The solution is an inverse moment estimator.*

Let $X_1, \ldots, X_n$ form a sample from IEPD(λ, α) with pdf given in (1.4), it is known that $F(X_i)$, $1 - F(X_i)$, $i = 1, \ldots, n$ follow uniform distribution $U(0, 1)$, and $-\log[1 - F(X_i)]$, $i = 1, \ldots, n$ follow standard exponential distribution Exp(1). By the method of inverse moment estimation, we set

$$\frac{1}{n}\sum_{i=1}^{n}\{-\log[1 - F(X_i)]\} = 1, \tag{3.1}$$

that is,

$$-\frac{\alpha}{n}\sum_{i=1}^{n}\log\left[1 - \left(\frac{1}{X_i} + 1\right)^{-\lambda}\right] = 1. \tag{3.2}$$

Thus, the IME of α is obtained as a function of λ,

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n}\log\left[1 - \left(\frac{1}{X_i} + 1\right)^{-\lambda}\right]}, \tag{3.3}$$

which is identical with the MLE of α.

Lemma 1. *Let $Z_{(1)} \le Z_{(2)} \le \cdots \le Z_{(n)}$ be the order statistics from the standard exponential distribution. Then, the random variables $W_1, W_2, \ldots, W_n$, where*

$$W_i = (n - i + 1)\left(Z_{(i)} - Z_{(i-1)}\right), \quad i = 1, 2, \ldots, n \tag{3.4}$$

with $Z_{(0)} \equiv 0$, are independent and follow standard exponential distributions.

Proof: The proof can be found in Arnold *et al.* (1992). □

Lemma 2. *Let $W_1, W_2, \ldots, W_n$ be iid standard exponential variables, $S_i = W_1 + \cdots + W_i$, $U_i = (S_i/S_{i+1})^i$, $i = 1, 2, \ldots, n-1$, $U_n = W_1 + \cdots + W_n$, then*

(1) $U_1, U_2, \ldots, U_n$ are independent;

(2) $U_1, U_2, \ldots, U_{n-1}$ follow the uniform distribution $U(0,1)$;

(3) $2U_n$ follows $\chi^2(2n)$.

Proof: The proof can be found in Wang (1992). □

Now we consider the IME of λ. For the sample $X_1, \ldots, X_n$ from IEPD(λ, α), for the order statistics $X_{(1)} \le \cdots \le X_{(n)}$, we have

$$-\log[1 - F(X_{(1)})] \le \cdots \le -\log[1 - F(X_{(n)})] \tag{3.5}$$

are n order statistics from standard exponential distribution Exp(1).

Let $Z_{(i)} = -\alpha \log[1 - (1/X_{(i)} + 1)^{-\lambda}]$, $i = 1, \ldots, n$. Thus, $Z_{(1)} \le Z_{(2)} \le \cdots \le Z_{(n)}$ are the first n order statistics from the standard exponential distribution. By Lemma 1, $W_i = (n - i + 1)(Z_{(i)} - Z_{(i-1)})$, $i = 1, 2, \ldots, n$ form a random sample from standard exponential distribution.

Let $S_i = W_1 + \cdots + W_i$, $U_i = (S_i/S_{i+1})^i$, $i = 1, 2, \ldots, n-1$, $U_n = W_1 + \cdots + W_n$, by Lemma 2, we have

$$-2\sum_{i=1}^{n-1} \log U_i = -2\sum_{i=1}^{n-1} i \log\left(\frac{S_i}{S_{i+1}}\right) = 2\sum_{i=1}^{n-1} \log\left(\frac{S_n}{S_i}\right) \sim \chi^2(2n-2), \tag{3.6}$$

where

$$\begin{aligned}\frac{S_n}{S_i} &= \frac{Z_{(1)} + Z_{(2)} + \cdots + Z_{(n)}}{Z_{(1)} + Z_{(2)} + \cdots + Z_{(i-1)} + (n-i+1)Z_{(i)}} \\ &= \frac{\log\left[1 - \left(\frac{1}{X_{(1)}} + 1\right)^{-\lambda}\right] + \log\left[1 - \left(\frac{1}{X_{(2)}} + 1\right)^{-\lambda}\right] + \cdots + \log\left[1 - \left(\frac{1}{X_{(n)}} + 1\right)^{-\lambda}\right]}{\log\left[1 - \left(\frac{1}{X_{(1)}} + 1\right)^{-\lambda}\right] + \cdots + \log\left[1 - \left(\frac{1}{X_{(i-1)}} + 1\right)^{-\lambda}\right] + (n-i+1)\log\left[1 - \left(\frac{1}{X_{(i)}} + 1\right)^{-\lambda}\right]}.\end{aligned}$$

Noting that the mean of $\chi^2(2n-2)$ is $2n-2$. Thus, we obtain an inverse moment equation for λ as follows:

$$\begin{aligned}&\sum_{i=1}^{n-1} \log\left[\frac{\log\left[1 - \left(\frac{1}{X_{(1)}} + 1\right)^{-\lambda}\right] + \log\left[1 - \left(\frac{1}{X_{(2)}} + 1\right)^{-\lambda}\right] + \cdots + \log\left[1 - \left(\frac{1}{X_{(n)}} + 1\right)^{-\lambda}\right]}{\log\left[1 - \left(\frac{1}{X_{(1)}} + 1\right)^{-\lambda}\right] + \cdots + \log\left[1 - \left(\frac{1}{X_{(i-1)}} + 1\right)^{-\lambda}\right] + (n-i+1)\log\left[1 - \left(\frac{1}{X_{(i)}} + 1\right)^{-\lambda}\right]}\right] \\ &= n - 1.\end{aligned} \tag{3.7}$$

Solve the equation and we obtain the inverse estimate $\hat{\lambda}_{IME}$ of λ. Plugging $\hat{\lambda}_{IME}$ into (3.3), we obtain the inverse estimate $\hat{\alpha}_{IME}$. In addition, noting that the mode of $\chi^2(2n-2)$ is $2n-4$, we can obtain a modified equation for λ:

$$\sum_{i=1}^{n-1}\log\left[\frac{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\log\left[1-\left(\frac{1}{X_{(2)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(n)}}+1\right)^{-\lambda}\right]}{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(i-1)}}+1\right)^{-\lambda}\right]+(n-i+1)\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}\right]$$
$$= n-2. \tag{3.8}$$

Solve the equation and we obtain the modified inverse estimate $\hat{\lambda}_{MIME}$ of λ. Plugging $\hat{\lambda}_{MIME}$ into (3.3), we obtain the modified inverse estimate $\hat{\alpha}_{MIME}$.

In the following, we prove the existence and uniqueness of the root in the equation (3.7) and (3.8).

Lemma 3. *The following limits hold: (1)* $\lim_{\lambda\to 0}\log[1-(a+1)^{-\lambda}]/\log[1-(b+1)^{-\lambda}]=1$, *for* $a>0, b>0$. *(2)* $\lim_{\lambda\to\infty}\log[1-(a+1)^{-\lambda}]/\log[1-(b+1)^{-\lambda}]=0$, *for* $a>b>0$. *(3)* $\lim_{\lambda\to\infty}\log[1-(a+1)^{-\lambda}]/\log[1-(b+1)^{-\lambda}]=+\infty$, *for* $b>a>0$.

Lemma 4. *For* $t>0$, $f(t)=(1+te^t-e^t)/(1-e^t)$ *is a decreasing function of* t.

Theorem 2. *Let* $W_i=(n-i+1)(Z_{(i)}-Z_{(i-1)})$, $i=1,2,\dots,n$ *form a sample from standard exponential distribution,* $S_i=W_1+\cdots+W_i$, *then for* $t>0$, *equation* $\sum_{i=1}^{n-1}\log(S_n/S_i)=t$ *has a unique positive solution.*

Proof: By Lemma 3, we obtain

$$\lim_{\lambda\to 0}\frac{S_n}{S_i}=\lim_{\lambda\to 0}\frac{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\log\left[1-\left(\frac{1}{X_{(2)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(n)}}+1\right)^{-\lambda}\right]}{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(i-1)}}+1\right)^{-\lambda}\right]+(n-i+1)\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}$$
$$=\lim_{\lambda\to 0}\frac{\dfrac{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\log\left[1-\left(\frac{1}{X_{(2)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(n)}}+1\right)^{-\lambda}\right]}{\log\left[1-\left(\frac{1}{X_{(n)}}+1\right)^{-\lambda}\right]}}{\dfrac{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(i-1)}}+1\right)^{-\lambda}\right]+(n-i+1)\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}{\log\left[1-\left(\frac{1}{X_{(n)}}+1\right)^{-\lambda}\right]}}$$
$$=\frac{n}{n}=1.$$

Thus, $\lim_{\lambda\to 0}\sum_{i=1}^{n-1}\log(S_n/S_i)=0$. However,

$$\lim_{\lambda\to\infty}\frac{S_n}{S_i}=\lim_{\lambda\to\infty}\frac{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\log\left[1-\left(\frac{1}{X_{(2)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(n)}}+1\right)^{-\lambda}\right]}{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(i-1)}}+1\right)^{-\lambda}\right]+(n-i+1)\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}$$
$$=1+\lim_{\lambda\to\infty}\frac{\log\left[1-\left(\frac{1}{X_{(i+1)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(n)}}+1\right)^{-\lambda}\right]-(n-i)\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]+(n-i)\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}$$

$$= 1 + \lim_{\lambda\to\infty} \frac{\dfrac{\log\left[1-\left(\frac{1}{X_{(i+1)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(n)}}+1\right)^{-\lambda}\right]-(n-i)\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}{\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}}{\dfrac{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\cdots+\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]+(n-i)\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}{\log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}}$$

$$= +\infty.$$

Thus, $\lim_{\lambda\to\infty}\sum_{i=1}^{n-1}\log(S_n/S_i) = \infty$. Therefore, for $t > 0$, equation $\sum_{i=1}^{n-1}\log(S_n/S_i) = t$ exists a positive solution. For the uniqueness of the solution, we consider the derivative of S_n/S_i with respect to λ.

Noting that, for $i = 1, \ldots, n$,

$$\frac{dW_i}{d\lambda} = (n-i+1)\alpha\left[\frac{\left(\frac{1}{X_{(i-1)}}+1\right)^{-\lambda}\log\left(\frac{1}{X_{(i-1)}}+1\right)}{1-\left(\frac{1}{X_{(i-1)}}+1\right)^{-\lambda}} - \frac{\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\log\left(\frac{1}{X_{(i)}}+1\right)}{1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}}\right]$$

$$= W_i\frac{\dfrac{\left(\frac{1}{X_{(i-1)}}+1\right)^{-\lambda}\log\left(\frac{1}{X_{(i-1)}}+1\right)}{1-\left(\frac{1}{X_{(i-1)}}+1\right)^{-\lambda}} - \dfrac{\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\log\left(\frac{1}{X_{(i)}}+1\right)}{1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}}}{\log\left(1-\left(\frac{1}{X_{(i-1)}}+1\right)^{-\lambda}\right)-\log\left(1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right)}.$$

$$\left(\frac{S_n}{S_i}\right)' = \left(1+\frac{W_{i+1}+\cdots+W_n}{W_1+\cdots+W_i}\right)'$$

$$= \frac{1}{\left(\sum_{k=1}^{i}W_k\right)^2}\sum_{j=i+1}^{n}\sum_{k=1}^{i}\left[W_j'W_k - W_jW_k'\right]$$

$$= \frac{1}{\lambda\left(\sum_{k=1}^{i}W_k\right)^2}\sum_{j=i+1}^{n}\sum_{k=1}^{i}W_jW_k\left[A(\lambda)-B(\lambda)\right],$$

where

$$A(\lambda) = \frac{\dfrac{\left(\frac{1}{X_{(j-1)}}+1\right)^{-\lambda}\lambda\log\left(\frac{1}{X_{(j-1)}}+1\right)}{1-\left(\frac{1}{X_{(j-1)}}+1\right)^{-\lambda}} - \dfrac{\left(\frac{1}{X_{(j)}}+1\right)^{-\lambda}\lambda\log\left(\frac{1}{X_{(j)}}+1\right)}{1-\left(\frac{1}{X_{(j)}}+1\right)^{-\lambda}}}{\log\left(1-\left(\frac{1}{X_{(j-1)}}+1\right)^{-\lambda}\right)-\log\left(1-\left(\frac{1}{X_{(j)}}+1\right)^{-\lambda}\right)}$$

and

$$B(\lambda) = \frac{\dfrac{\left(\frac{1}{X_{(k-1)}}+1\right)^{-\lambda}\lambda\log\left(\frac{1}{X_{(k-1)}}+1\right)}{1-\left(\frac{1}{X_{(k-1)}}+1\right)^{-\lambda}} - \dfrac{\left(\frac{1}{X_{(k)}}+1\right)^{-\lambda}\lambda\log\left(\frac{1}{X_{(k)}}+1\right)}{1-\left(\frac{1}{X_{(k)}}+1\right)^{-\lambda}}}{\log\left(1-\left(\frac{1}{X_{(k-1)}}+1\right)^{-\lambda}\right)-\log\left(1-\left(\frac{1}{X_{(k)}}+1\right)^{-\lambda}\right)}.$$

By Cauchy's mean-value theorem, for $j = i+1, \ldots, n$, $k = 1, \ldots, i$, there exist $\xi_1 \in (\lambda\log(1/X_{(j)}+1), \lambda\log(1/X_{(j-1)}+1))$ and $\xi_2 \in (\lambda\log(1/X_{(k)}+1), \lambda\log(1/X_{(k-1)}+1))$ such that

$$A(\lambda) = \frac{1+\xi_1e^{\xi_1}-e^{\xi_1}}{1-e^{\xi_1}}, \qquad B(\lambda) = \frac{1+\xi_2e^{\xi_2}-e^{\xi_2}}{1-e^{\xi_2}}.$$

Note that $\xi_1 < \xi_2$, by Lemma 4, $A(\lambda) - B(\lambda) > 0$, $(S_n/S_i)' > 0$, thus $\sum_{i=1}^{n-1} \log(S_n/S_i)$ is a strictly increasing function of λ, equation $\sum_{i=1}^{n-1} \log(S_n/S_i) = t$ has a unique positive solution. □

4. Joint confidence regions for λ and α

Let $X_1, X_2, \ldots, X_n$ form a sample from IEPD(λ, α), $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are the order statistics from this sample. Let $Z_{(i)} = -\alpha \log[1-(1/X_{(i)}+1)^{-\lambda}]$, $i = 1, \ldots, n$. Thus, $Z_{(1)} \le Z_{(2)} \le \cdots \le Z_{(n)}$ are the first n order statistics from the standard exponential distribution. By Lemma 1, $W_i = (n-i+1)(Z_{(i)} - Z_{(i-1)})$, $i = 1, 2, \ldots, n$ form a sample from standard exponential distribution. Let $S_i = W_1 + \cdots + W_i$, $U_i = (S_i/S_{i+1})^i$, $i = 1, 2, \ldots, n-1$, $U_n = W_1 + \cdots + W_n$. Hence,

$$V = 2S_1 = 2W_1 = 2nZ_{(1)} = -2n\alpha \log\left[1 - \left(\frac{1}{X_{(1)}} + 1\right)^{-\lambda}\right] \sim \chi^2(2), \tag{4.1}$$

and

$$U = 2(S_n - S_1) = 2\sum_{i=2}^{n} W_i = 2\left[Z_{(1)} + \cdots + Z_{(n)} - nZ_{(1)}\right] \sim \chi^2(2n-2). \tag{4.2}$$

It is obvious that U and V are independent. Define

$$T_1 = \frac{U/(2n-2)}{V/2} = \frac{S_n - S_1}{(n-1)S_1} \sim F(2n-2, 2), \tag{4.3}$$

and

$$T_2 = U + V = 2S_n \sim \chi^2(2n). \tag{4.4}$$

We obtain that T_1 and T_2 are independent using the known bank-post office story in statistics.

Let $F_\gamma(v_1, v_2)$ denote the percentile of F distribution with left-tail probability γ and v_1 and v_2 degrees of freedom. Let $\chi^2_\gamma(v)$ denote the percentile of χ^2 distribution with left-tail probability γ and v degrees of freedom.

By using the pivotal variables T_1 and T_2, a joint confidence region for the two parameters λ and α can be constructed as follows.

Theorem 3. *(Method 1) Let $X_1, X_2, \ldots, X_n$ form a sample from IEPD(λ, α), then, based on the pivotal variables T_1 and T_2, a $100(1-\gamma)\%$ joint confidence region for the two parameters λ and α is determined by the following inequalities:*

$$\begin{cases} \lambda_L \le \lambda \le \lambda_U, \\ \dfrac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left[1 - \left(\frac{1}{X_{(i)}} + 1\right)^{-\lambda}\right]} \le \alpha \le \dfrac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left[1 - \left(\frac{1}{X_{(i)}} + 1\right)^{-\lambda}\right]}, \end{cases} \tag{4.5}$$

where λ_L is the root of λ for the equation $T_1 = F_{(1-\sqrt{1-\gamma})/2}(2n-2, 2)$ and λ_U is the root of λ for the equation $T_1 = F_{(1+\sqrt{1-\gamma})/2}(2n-2, 2)$.

Proof: The function of λ is

$$T_1 = \frac{1}{n-1}\frac{\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]+\cdots\log\left[1-\left(\frac{1}{X_{(n)}}+1\right)^{-\lambda}\right]-n\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]}{n\log\left[1-\left(\frac{1}{X_{(1)}}+1\right)^{-\lambda}\right]}$$

and does not depend on α. From Theorem 2, we have $\lim_{\lambda\to 0} T_1 = (1/(n-1))\lim_{\lambda\to 0}(S_n/S_1 - 1) = 0$, $\lim_{\lambda\to\infty} T_1 = (1/(n-1))\lim_{\lambda\to\infty}(S_n/S_1 - 1) = \infty$, $T_1' = (1/(n-1))(S_n/S_i)' > 0$. Therefore, for any $t > 0$, equation $T_1 = t$ has a unique positive root of λ.

$$\begin{aligned}
1-\gamma &= \sqrt{1-\gamma}\sqrt{1-\gamma}\\
&= P\left(F_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2,2) \le T_1 \le F_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2,2)\right)\times P\left(\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n) \le T_2 \le \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)\right)\\
&= P\left(F_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2,2) \le T_1 \le F_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2,2)), \chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n) \le T_2 \le \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)\right)\\
&= P\left(\lambda_L \le \lambda \le \lambda_U, \frac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]} \le \alpha \le \frac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}\right).
\end{aligned}$$

□

However, by Lemma 2, we have

$$T_3 = -2\sum_{i=1}^{n-1}\log U_i = -2\sum_{i=1}^{n-1} i\log\left(\frac{S_i}{S_{i+1}}\right) = 2\sum_{i=1}^{n-1}\log\left(\frac{S_n}{S_i}\right) \sim \chi^2(2n-2). \tag{4.6}$$

T_2 and T_3 are also independent. By using the pivotal variables T_2 and T_3, a joint confidence region for the two parameters λ and α can be constructed as follows.

Theorem 4. *(Method 2) Let $X_1, X_2, \ldots, X_n$ form a sample from IEPD(λ,α), then, based on the pivotal variables T_2 and T_3, a $100(1-\gamma)\%$ joint confidence region for the two parameters λ and α is determined by the following inequalities:*

$$\begin{cases} \lambda_L^* \le \lambda \le \lambda_U^*, \\ \dfrac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]} \le \alpha \le \dfrac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}, \end{cases} \tag{4.7}$$

where λ_L^* is the root of λ for the equation $T_3 = \chi^2_{(1-\sqrt{1-\gamma})/2}(2n-2)$ and λ_U^* is the root of λ for the equation $T_3 = \chi^2_{(1+\sqrt{1-\gamma})/2}(2n-2)$.

Proof: $T_3 = 2\sum_{i=1}^{n-1}\log(S_n/S_i)$ is a function of λ and does not depend on α. From Theorem 2, for any $s > 0$, equation $T_3 = s$ has a unique positive root of λ.

$$\begin{aligned}
1-\gamma &= \sqrt{1-\gamma}\sqrt{1-\gamma}\\
&= P\left(\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2) \le T_3 \le \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2)\right)\times P\left(\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n) \le T_2 \le \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)\right)
\end{aligned}$$

Table 1: Average relative estimates and MSEs of α

n	Methods	$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
	MLE	1.1229 (0.1600)	1.1518 (0.1753)	1.1790 (0.2353)	1.1735 (0.2472)	1.1736 (0.2482)
30	IME	1.0899 (0.1376)	1.1149 (0.1508)	1.1376 (0.2019)	1.1287 (0.2061)	1.1286 (0.2127)
	MIME	1.0457 (0.1166)	1.0656 (0.1254)	1.0835 (0.1662)	1.0721 (0.1688)	1.0696 (0.1734)
	MLE	1.0760 (0.0793)	1.0948 (0.0914)	1.1212 (0.1262)	1.1195 (0.1320)	1.1356 (0.1617)
40	IME	1.0531 (0.0717)	1.0690 (0.0820)	1.0922 (0.1116)	1.0897 (0.1179)	1.1014 (0.1416)
	MIME	1.0220 (0.0638)	1.0345 (0.0717)	1.0543 (0.0964)	1.0499 (0.1017)	1.0590 (0.1212)
	MLE	1.0720 (0.0661)	1.0663 (0.0658)	1.0941 (0.0831)	1.0866 (0.0877)	1.1038 (0.1028)
50	IME	1.0537 (0.0609)	1.0464 (0.0607)	1.0711 (0.0745)	1.0635 (0.0799)	1.0798 (0.0943)
	MIME	1.0290 (0.0551)	1.0199 (0.0549)	1.0418 (0.0660)	1.0329 (0.0711)	1.0472 (0.0831)
	MLE	1.0385 (0.0343)	1.0555 (0.0442)	1.0515 (0.0455)	1.0530 (0.0452)	1.0485 (0.0470)
80	IME	1.0276 (0.0325)	1.0438 (0.0419)	1.0387 (0.0431)	1.0390 (0.0422)	1.0331 (0.0437)
	MIME	1.0129 (0.0306)	1.0275 (0.0390)	1.0214 (0.0401)	1.0207 (0.0392)	1.0141 (0.0407)
	MLE	1.0345 (0.0311)	1.0431 (0.0318)	1.0382 (0.0322)	1.0393 (0.0353)	1.0419 (0.0374)
100	IME	1.0260 (0.0301)	1.0339 (0.0304)	1.0281 (0.0307)	1.0288 (0.0338)	1.0298 (0.0353)
	MIME	1.0143 (0.0287)	1.0210 (0.0287)	1.0144 (0.0290)	1.0144 (0.0319)	1.0147 (0.0332)

MLE = maximum likelihood estimate; IME = inverse moment estimation; MIME = modified inverse moment estimation; MSE = mean square error.

$$= P\left(\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2) \le T_3 \le \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2), \chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n) \le T_2 \le \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)\right)$$

$$= P\left(\lambda_L^* \le \lambda \le \lambda_U^*, \frac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]} \le \alpha \le \frac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left[1-\left(\frac{1}{X_{(i)}}+1\right)^{-\lambda}\right]}\right).$$

□

5. Simulation study

5.1. Comparison of the three estimation methods

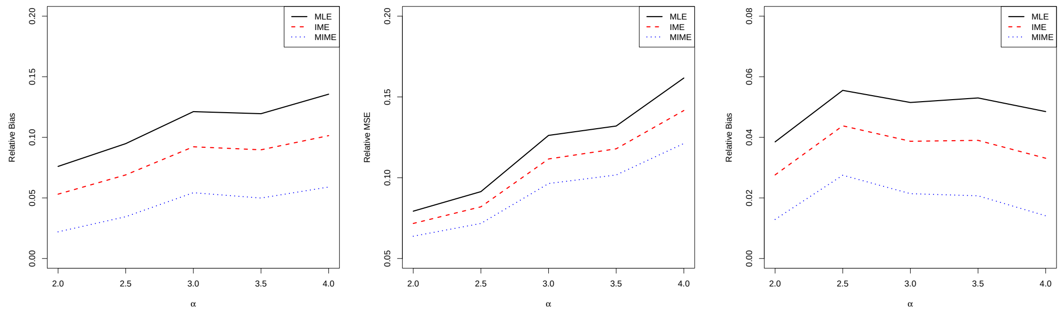
In this section, we conduct simulations to compare the performances of the MLEs, IMEs and MIMEs mainly with respect to their biases and mean squared errors (MSE's), for various sample sizes and for various true parametric values.

Suppose $X \sim \text{IEPD}(\lambda, \alpha)$, the random data can be generated as: $X = 1/[(1-U^{1/\alpha})^{-1/\lambda} - 1]$, where U follows uniform distribution over $[0, 1]$. We obtain $\hat{\lambda}_{MLE}$ by solving Equation (2.5) and $\hat{\alpha}_{MLE}$ by (2.4). The $\hat{\lambda}_{IME}$ and $\hat{\lambda}_{MIME}$ can be obtained by solving (3.7) and (3.8) respectively. The $\hat{\alpha}_{IME}$ and $\hat{\alpha}_{MIME}$ can be obtained from (3.3).

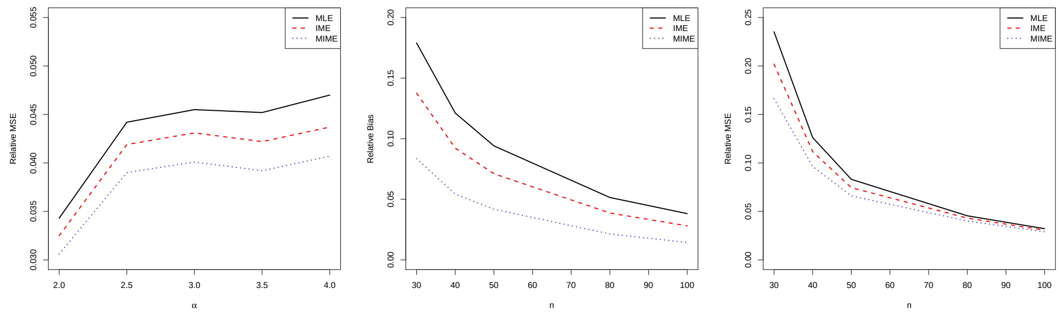
We consider sample sizes $n = 30, 40, 50, 80, 100$ and $\alpha = 2.0, 2.5, 3.0, 3.5, 4.0$. We take $\lambda = 4$ in all our computations. For each combination of sample size n and parameter α, we generate a sample of size n from IEPD($\lambda = 4, \alpha$), and estimate the parameters λ and α by the MLE, IME, MIME methods. The average values of $\hat{\alpha}/\alpha$ and $\hat{\lambda}/4$ as well as the corresponding MSEs over 1,000 replications are computed and reported.

For different cases, Table 1 reports the average values of $\hat{\alpha}/\alpha$ and the corresponding MSE is reported within parenthesis. Figure 1(a), (b), (c), and (d) show the relative biases and the MSEs of the three estimators of α for sample sizes $n = 40$ and $n = 80$. Figure 1(e) and (f) show the relative biases and the MSEs of the three estimators of α for $\alpha = 3.0$. The other cases are also similar.

For different cases, Table 2 reports the average values of $\hat{\lambda}/\lambda = \hat{\lambda}/4$ and the corresponding MSE is reported within parenthesis. Figure 2(a), (b), (c) and (d) show the relative biases and the MSEs of



(a) Relative biases ($n = 40$) (b) Relative MSEs ($n = 40$) (c) Relative biases ($n = 80$)



(d) Relative MSEs ($n = 80$) (e) Relative biases ($\alpha = 3.0$) (f) Relative MSEs ($\alpha = 3.0$)

Figure 1: *Average relative biases and MSEs of α.*

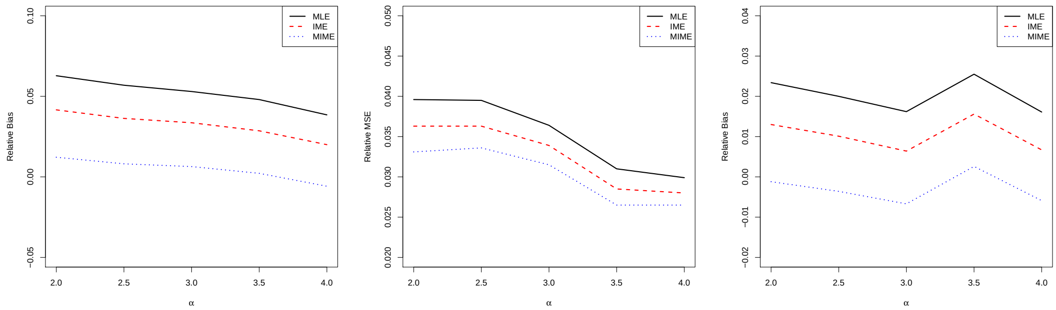
Table 2: Average relative estimates and MSEs of λ

n	Methods	$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
	MLE	1.0742 (0.0576)	1.0590 (0.0491)	1.0678 (0.0472)	1.0524 (0.0414)	1.0557 (0.0459)
30	IME	1.0451 (0.0516)	1.0316 (0.0445)	1.0406 (0.0423)	1.0257 (0.0376)	1.0294 (0.0419)
	MIME	1.0054 (0.0466)	0.9940 (0.0410)	1.0038 (0.0385)	0.9901 (0.0350)	0.9944 (0.0389)
	MLE	1.0628 (0.0396)	1.0569 (0.0395)	1.0530 (0.0364)	1.0480 (0.0310)	1.0385 (0.0299)
40	IME	1.0416 (0.0363)	1.0363 (0.0363)	1.0336 (0.0339)	1.0286 (0.0285)	1.0200 (0.0280)
	MIME	1.0122 (0.0331)	1.0081 (0.0336)	1.0064 (0.0315)	1.0022 (0.0265)	0.9942 (0.0265)
	MLE	1.0407 (0.0298)	1.0370 (0.0251)	1.0360 (0.0261)	1.0346 (0.0237)	1.0366 (0.0245)
50	IME	1.0243 (0.0280)	1.0207 (0.0235)	1.0203 (0.0246)	1.0198 (0.0225)	1.0209 (0.0231)
	MIME	1.0013 (0.0264)	0.9986 (0.0223)	0.9988 (0.0234)	0.9989 (0.0214)	1.0004 (0.0220)
	MLE	1.0234 (0.0179)	1.0200 (0.0147)	1.0162 (0.0145)	1.0255 (0.0136)	1.0161 (0.0139)
80	IME	1.0130 (0.0172)	1.0101 (0.0142)	1.0064 (0.0140)	1.0156 (0.0130)	1.0067 (0.0135)
	MIME	0.9988 (0.0167)	0.9964 (0.0138)	0.9933 (0.0137)	1.0026 (0.0125)	0.9941 (0.0132)
	MLE	1.0193 (0.0123)	1.0178 (0.0130)	1.0185 (0.0115)	1.0189 (0.0105)	1.0177 (0.0104)
100	IME	1.0113 (0.0119)	1.0101 (0.0127)	1.0109 (0.0112)	1.0116 (0.0102)	1.0101 (0.0102)
	MIME	0.9999 (0.0116)	0.9992 (0.0123)	1.0004 (0.0109)	1.0013 (0.0099)	1.0000 (0.0099)

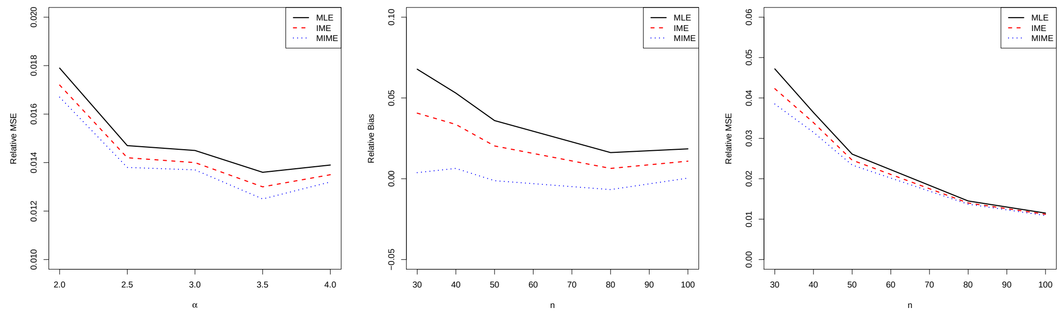
MLE = maximum likelihood estimate; IME = inverse moment estimation; MIME = modified inverse moment estimation; MSE = mean square error.

the three estimators of λ for sample sizes $n = 40$ and $n = 80$. Figure 2(e) and (f) show the relative biases and the MSEs of the three estimators of λ for $\alpha = 3.0$. The other cases are similar.

From Tables 1 and 2, we observe that



(a) Relative biases ($n = 40$) (b) Relative MSEs ($n = 40$) (c) Relative biases ($n = 80$)



(d) Relative MSEs ($n = 80$) (e) Relative biases ($\alpha = 3.0$) (f) Relative MSEs ($\alpha = 3.0$)

Figure 2: *Average relative biases and MSEs of λ.*

- The average relative biases and the average relative MSEs for the three methods decrease as sample size n increases as expected. The asymptotic unbiasedness and consistency of all the estimators are verified.
- For the three methods, the average biases and relative MSEs of $\hat{\lambda}/4$ decrease as α goes up. The average biases and relative MSEs of $\hat{\alpha}/\alpha$ increase as α goes up.
- Considering only MSE's, the estimation of α's are more accurate for smaller values while the estimation of λ's are more accurate for larger values of α.
- MLE and IME overestimate both of the two parameters α and λ. MIME overestimates only α.

As far as the biases and MSEs are concerned, it is clear MIME works the best in all the cases considered to estimate the two parameters. Its performance is followed by IME and MLE, especially for small sample sizes. The three methods are close for larger sample sizes. Considering all the points, MIME is recommended for estimating both the parameters of the IEPD(λ, α) distribution.

5.2. Comparison of the two joint confidence regions

In Section 4, two methods to construct the confidence regions of the two parameters λ and α are proposed. In this section, we conduct simulations to compare the two methods.

First, we assess the precisions of the two methods of interval estimators for the parameter λ. We take sample sizes $n = 30, 40, 50, 80, 100$ and $\alpha = 2.0, 2.5, 3.0, 3.5, 4.0$. We take $\lambda = 4$ in all our

Table 3: Results of the methods for constructing intervals for λ with confidence level 0.95

n	Methods		$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
30	(I)	Mean width	5.6043	5.5489	5.5052	5.4440	5.2401
		Coverage rate	0.955	0.953	0.953	0.942	0.954
	(II)	Mean width	3.3248	3.1778	3.0546	3.0291	2.9178
		Coverage rate	0.961	0.950	0.957	0.947	0.958
40	(I)	Mean width	5.2826	5.1410	5.0569	4.9930	4.8931
		Coverage rate	0.944	0.950	0.943	0.959	0.949
	(II)	Mean width	2.8476	2.7187	2.6228	2.5556	2.5152
		Coverage rate	0.951	0.940	0.957	0.958	0.951
50	(I)	Mean width	4.9759	4.8686	4.7284	4.6879	4.6513
		Coverage rate	0.938	0.959	0.968	0.966	0.950
	(II)	Mean width	2.5179	2.4105	2.3387	2.2844	2.2472
		Coverage rate	0.948	0.943	0.951	0.952	0.960
80	(I)	Mean width	4.4567	4.3395	4.2683	4.1983	4.2200
		Coverage rate	0.954	0.950	0.952	0.949	0.941
	(II)	Mean width	1.9824	1.9010	1.8308	1.7857	1.7608
		Coverage rate	0.938	0.952	0.956	0.949	0.961
100	(I)	Mean width	4.2955	4.1699	4.1204	4.0131	3.9981
		Coverage rate	0.952	0.959	0.953	0.945	0.957
	(II)	Mean width	1.7564	1.6938	1.6432	1.5951	1.5721
		Coverage rate	0.957	0.939	0.951	0.931	0.950

computations. For each combination of sample size n and parameter α, we generate a sample of size n from IEPD($\lambda = 4, \alpha$), and estimate the parameters λ by the two proposed methods (4.5) and (4.7).

The mean widths as well as the coverage rates over 1,000 replications are computed. Here the coverage rate is defined as the rate of the confidence intervals that contain the true value $\lambda = 4$ among these 1,000 confidence intervals. The results are reported in Table 3.

It is observed that:

- The mean widths of the intervals decrease as sample sizes n increase as expected.
- The mean widths of the intervals decrease as the parameter α increases.
- The coverage rates of the two methods are close to the nominal level 0.95.

Considering the mean widths, the interval estimate of λ obtained in method 2 performs better than that obtained in method 1. Method 2 for constructing the interval estimate of λ is recommended.

We consider the two joint confidence regions and the empirical coverage rates and expected areas. The results of the methods for constructing joint confidence regions for (λ, α) with confidence level $\gamma = 0.95$ are reported in Table 4.

It is observed that:

- The mean areas of the joint regions decrease as sample sizes n increase as expected.
- The mean areas of the joint regions increase as the parameter α increases.
- The coverage rates of the two methods are close to the nominal level 0.95.

Considering the mean areas, the joint region of (λ, α) obtained in method 2 performs better than that obtained in method 1. Method 2 is recommended.

Table 4: Results of the methods for constructing joint confidence regions for (λ, α) with confidence level $\gamma = 0.95$

n	Methods		$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
30	(I)	Mean area	18.1447	24.1040	29.3229	37.4723	45.1162
		Coverage rate	0.943	0.953	0.948	0.949	0.942
	(II)	Mean area	7.2611	9.0473	10.4347	12.2616	14.0958
		Coverage rate	0.940	0.951	0.957	0.954	0.949
40	(I)	Mean area	13.9298	17.2864	22.6337	28.3434	32.9806
		Coverage rate	0.950	0.953	0.945	0.938	0.956
	(II)	Mean area	5.2359	6.1751	7.3436	8.6958	9.7041
		Coverage rate	0.952	0.942	0.944	0.946	0.948
50	(I)	Mean area	11.1586	14.1365	17.7005	21.1058	25.5788
		Coverage rate	0.951	0.954	0.937	0.951	0.948
	(II)	Mean area	4.0731	4.8594	5.6845	6.4921	7.3824
		Coverage rate	0.955	0.953	0.942	0.953	0.949
80	(I)	Mean area	7.2142	9.5987	11.8881	13.8864	17.4234
		Coverage rate	0.946	0.944	0.942	0.943	0.935
	(II)	Mean area	4.0370	4.9886	6.1020	7.1601	8.2498
		Coverage rate	0.941	0.951	0.938	0.940	0.936
100	(I)	Mean area	5.8819	7.7936	9.3763	11.3534	13.5202
		Coverage rate	0.954	0.949	0.951	0.950	0.950
	(II)	Mean area	1.8788	2.2596	2.6727	3.0193	3.3800
		Coverage rate	0.950	0.952	0.950	0.942	0.953

6. Real illustrative example

In this section, we consider a real dataset. This data set represents the total seasonal annual rainfall (in inches) recorded at Los Angeles Civic Center during the last 25 years, from 1985 to 2009 (season 1 July–30 June). The observations are

$$12.82, 17.86, 7.66, 12.48, 8.08, 7.35, 11.99, 21.00, 27.36, 8.11, 24.35, 12.44, 12.40, 31.01,$$
$$9.09, 11.57, 17.94, 4.42, 16.42, 9.25, 37.96, 13.19, 3.21, 13.53, 9.08.$$

The dataset has been previously analyzed by Raqab (2006, 2013) and Ahmadi and Balakrishnan (2009). Here we fit the data with IEPD.

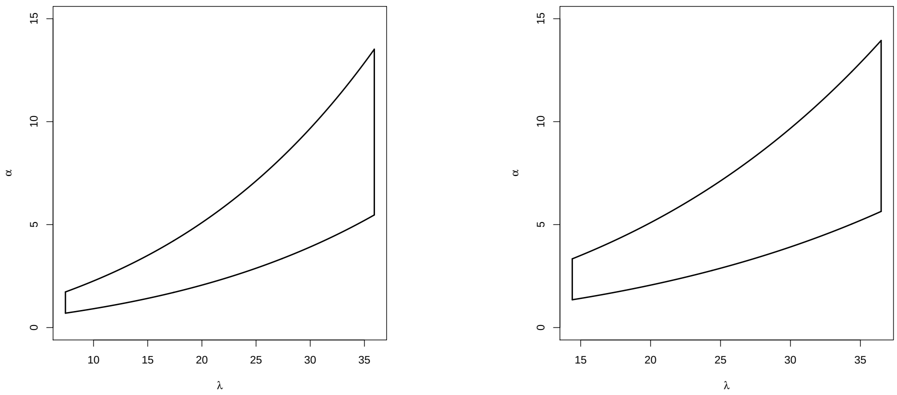
The MLEs of the parameters are $\hat{\lambda}_{MLE} = 25.6869$ and $\hat{\alpha}_{MLE} = 4.9629$ with log-likelihood value -84.1822. The Kolmogorov-Smirnov distance and its corresponding p-value are $D = 0.12$ and $p = 0.9955$, respectively. The inverted exponential Pareto distribution can be effective in modeling the rainfall data.

Using the methods proposed in Section 3, we obtain the following estimates:

$$\hat{\lambda}_{IME} = 24.7618, \quad \hat{\alpha}_{IME} = 4.6793, \quad \hat{\lambda}_{MIME} = 23.7404, \quad \hat{\alpha}_{MIME} = 4.3801.$$

Based on method 1, the 95% joint confidence region for the parameters (λ, α) is given by the following inequalities:

$$\begin{cases} 7.4044 \le \lambda \le 35.9168, \\ \dfrac{-15.17118}{\sum_{i=1}^{25} \log\left(1 - \left(1 + \frac{1}{X_i}\right)^{-\lambda}\right)} \le \alpha \le \dfrac{-37.48729}{\sum_{i=1}^{25} \log\left(1 - \left(1 + \frac{1}{X_i}\right)^{-\lambda}\right)}. \end{cases}$$



(a) Method 1 (b) Method 2

Figure 3: *The 95% joint confidence region of* (λ, α).

Based on method 2, the 95% joint confidence region for the parameters (λ, α) is given by the following inequalities:

$$\begin{cases} 14.3964 \leq \lambda \leq 36.4891, \\ \dfrac{-15.17118}{\sum_{i=1}^{25} \log\left(1 - \left(1 + \frac{1}{X_i}\right)^{-\lambda}\right)} \leq \alpha \leq \dfrac{-37.48729}{\sum_{i=1}^{25} \log\left(1 - \left(1 + \frac{1}{X_i}\right)^{-\lambda}\right)}. \end{cases}$$

Figure 3(a) and (b) show the 95% joint confidence regions of (λ, α).

7. Conclusions and remarks

In this article, we present the modified inverse moment estimation of parameters and its applications. We use the inverted exponential Pareto distribution as a specific model to demonstrate its principle and how to apply this method in practice. The estimation of unknown parameters is investigated. For the classical maximum likelihood estimation, a necessary and sufficient condition for the existence and uniqueness of MLEs of the parameters is obtained. Inverse moment and modified inverse moment estimators are proposed and their properties are studied.

Monte Carlo simulations are conducted to compare the performances of the three estimators. The simulation results show that the modified inverse moment estimator works the best in all the cases considered for estimating the unknown parameters in terms of biases and mean squared errors. Its performance is followed by inverse moment estimator and maximum likelihood estimator, especially for small sample sizes. We also discuss joint confidence regions for unknown parameters. Real rainfall dataset is analyzed and used to illustrate the proposed method.

The method discussed in this paper can be easily extended to other common distributions (such as generalized exponential, and inverse Weibull distribution), which are frequently used in practice. Future research topics should include a comparison of the proposed modified inverse moment estimator with Bayesian estimator, what is the relation between the proposed estimator and sufficient statistics.

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