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Note on a Classical Conservative Method for Scalar Hyperbolic Equations

Yong Hun Lee
Department of Mathematics and Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju 561-756, Korea
e-mail : lyh229@jbnu.ac.kr

Sang Dong Kim*
Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea and Department of Mathematics, University of Wisconsin-Whitewater, USA
e-mail : skim@knu.ac.kr

Abstract. We provide a combination of the forward Euler method and the trapezoidal quadrature rule leads to a two-step conservative numerical method which possesses TV-stable property together with consistency.

1. Introduction

There have been known many conservative methods such as upwind, Lax-Friedrichs, Lax-Wendroff, Richtmyer two-step Lax-Wendroff, MacCormick, Godunov's method and etc. (see [1], [3], [4], [6], [7] for example). It is known that the numerical methods for solving hyperbolic equations depend on how the numerical flux function is chosen or modified.

By introducing a new variable $v = f_x(u(x,t))$, we have an equivalent system of equations for $u_t + f_x(u) = 0$. First of all, we note that the trapezoidal numerical quadrature rule to the system of equations in a time interval with a space cell-average yields a classical conservative numerical method. Next, a two-step conservative numerical algorithm is introduced by the forward Euler method for $u_t = -v$ and the trapezoidal quadrature rule for $v = f_x(u)$. Hence it may be called as a conservative Trapezoidal-Euler method(CTEM) which is of first-order and to-

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tal variational stability. This kind approach may provide a new way to develop a conservative method which may be comparative with many known conservative numerical methods (see [4], for example).

This paper consists of as follows. In section 2, we review the classical conservative form in terms of the trapezoidal quadrature rule, the Taylor expansion applied to the system of ordinary differential equations. In section 3, we show that the new method can be derived by using the trapezoidal quadrature rule and the forward Euler method. We present l_1 contracting and TV-stability for a newly introduced method. In section 4, we provide some model numerical tests by taking the same numerical flux functions as in upwind method. Finally, we present conclusions in this paper.

2. Review on a Conservative Method

In this section, we show how the trapezoidal quadrature rule, the forward Euler method and the Taylor expansion (or a combination of those) lead to a conservative method for a nonlinear problem such that

$$u_t + f_x(u) = 0, \qquad -\infty < x < \infty, \quad t \geq 0, \tag{2.1}$$

$$u(x,0) = u_0(x), \tag{2.2}$$

where the function f is assumed to have a nice required property. Assume that the domain $(-\infty,\infty)\times[0,T]$ of the nonlinear problem (2.1) does have the discrete mesh points (x_j, t_m) by

$$\begin{aligned} x_j &= jh, \quad j = \cdots, -1, 0, 1, 2, \cdots \\ t_m &= m\tau, \quad m = 0, 1, 2, \cdots \end{aligned}$$

where h and τ denote a mesh width and a time step, respectively.

Prior to usage of the forward Euler method, trapezoidal quadrature rule and Taylor expansion, let us review the known strategy for a conservative numerical method. First, integrating (2.1) over the region $[x_{j-1/2}, x_{j+1/2}]\times[t_m, t_{m+1}]$, one has

$$\begin{aligned} &\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{m+1})dt \\ =\ &\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_m)dt + \frac{1}{h}\int_{t_m}^{t_{m+1}} f(u(x_{j+1/2},t)) - f(u(x_{j-1/2},t))dt. \end{aligned} \tag{2.3}$$

For the approximation of the values of the solution $u(x,t)$, we use the notation U_j^m for a conservative method approximating the cell average of $u(x,t_m)$, i.e.,

$$U_j^m \approx \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_m)dx \tag{2.4}$$

and the numerical flux function $F(U_j, U_{j+1})$ playing the role of an average flux through $x_{j+1/2}$ over the time interval $[t_m, t_{m+1}]$, i.e.,

$$F(U_j^m, U_{j+1}^m) \approx \frac{1}{\tau} \int_{t_m}^{t_{m+1}} f(u(x_{j+1/2}, t))dt. \tag{2.5}$$

Then, one has the well-known conservation method from (2.3) such that

$$U_j^{m+1} = U_j^m - \frac{\tau}{h}\Big[F(U_j^m, U_{j+1}^m) - F(U_{j-1}^m, U_j^m)\Big], \tag{2.6}$$

First, with only regularity assumption on the exact solution we will provide the consistency of the algorithm (2.6). We note that the consistency of the upwind method for $F(u, v) = f(u)$ is well known (see p.126 [3], for example) under assumption of Lipschitz continuity of f. For this purpose, it is necessary for us to understand (2.1) in an equivalent algorithm by introducing a new variable as

$$v(x, t) = f_x(u(x, t)). \tag{2.7}$$

Then the nonlinear problem (2.1) is equivalent to

$$v = f_x(u(x, t)), \qquad u(x, 0) = u_0(x), \quad v(x, 0) = f_x(u_0(x)) \tag{2.8}$$

$$u_t = -v(x, t), \tag{2.9}$$

First, now integrating (2.8) on time interval $[t_m, t_{m+1}]$ and applying the trapezoidal numerical quadrature to $v(x, t)$ with respect to time-variable, we have

$$\frac{\tau}{2}\Big[v(x, t_{m+1}) + v(x, t_m)\Big] = \int_{t_m}^{t_{m+1}} f_x(u(x, t))dt + O(\tau^3). \tag{2.10}$$

Then, taking the cell average on $[x_{j-1/2}, x_{j+1/2}]$ of both sides of (2.10), the conservative numerical method for (2.8) can be written as

$$V_j^{m+1} = -V_j^m + \frac{2}{h}\Big[F(U_j^m, U_{j+1}^m) - F(U_{j-1}^m, U_j^m)\Big] \tag{2.11}$$

where the same notation V_j^m is used as in (2.4). Secondly, integrating (2.9) on the time interval $[t_m, t_{m+1}]$ and then using the trapezoidal rule, we have

$$\begin{aligned} u(x, t_{m+1}) &= u(x, t_m) - \int_{t_m}^{t_{m+1}} v(x, t)dt \\ &\approx u(x, t_m) - \frac{\tau}{2}\Big[v(x, t_{m+1}) + v(x, t_m)\Big]. \end{aligned} \tag{2.12}$$

Then, taking the cell average on $[x_{j-1/2}, x_{j+1/2}]$ of both sides in (2.12) which leads to

$$U_j^{m+1} = U_j^m - \frac{\tau}{2}\Big[V_j^{m+1} + V_j^m\Big]. \tag{2.13}$$

Hence, we have the equivalent algorithm (2.11) and (2.13) with the well known conservative numerical method (2.6)

Remark 2.1. One may derive the algorithm (2.6) using the trapezoidal quadrature rule and the Taylor series expansion. From (2.9) it follows that

$$u_t(x,t) = -v(x,t), \quad u_{tt}(x,t) = -v_t(x,t).$$

Hence, with the above relation, the Taylor expansion of u with respect to t leads to

$$\begin{aligned} u(x,t+\tau) &= u(x,t) - \tau v(x,t) - \frac{\tau^2}{2} v_t(x,t) + O(\tau^3) \\ &= u(x,t) - \tau v(x,t) - \frac{\tau^2}{2}\Big[\frac{v(x,t+\tau)-v(x,t)}{\tau} + O(\tau)\Big] + O(\tau^3) \\ &= u(x,t) - \tau v(x,t) - \frac{\tau}{2}\Big[v(x,t+\tau) - v(x,t)\Big] + O(\tau^3). \end{aligned} \tag{2.14}$$

Then, (2.14) evaluated at $t = t_m$ should be taken the cell average over $[x_{j-1/2}, x_{j+1/2}]$ so that it becomes

$$U_j^{m+1} = U_j^m - \tau V_j^m - \frac{\tau}{2}\Big[V_j^{m+1} - V_j^m\Big] = U_j^m - \frac{\tau}{2}\Big[V_j^{m+1} + V_j^m\Big]. \tag{2.15}$$

Now, combining (2.15) with (2.11) leads to (2.6).

3. First-Order Two-Step Algorithm

Now, instead of integrating (2.9), we approximate (2.9) by forward Euler method first of all and then make it in a conservative form on a cell $[x_{j-1/2}, x_{j+1/2}]$. That is, we replace (2.13) with forward Euler method applied to (2.9). Hence, it becomes

$$u(x,t_{m+1}) = u(x,t_m) - \tau v(x,t_m) \tag{3.1}$$

Then, taking the cell average of the both sides in (3.1) on $[x_{j-1/2}, x_{j+1/2}]$ leads to the conservative forward Euler method

$$U_j^{m+1} = U_j^m - \tau V_j^m. \tag{3.2}$$

Therefore, combining (2.11) with (3.2), we obtain the conservative Trapezoidal-Euler method for (2.1) such that

$$V_j^{m+1} = -V_j^m + \frac{2}{h}\Big[F(U_j^m, U_{j+1}^m) - F(U_{j-1}^m, U_j^m)\Big] \tag{3.3}$$

$$U_j^{m+1} = U_j^m - \tau V_j^m, \tag{3.4}$$

with the initial conditions

$$U_j^0 = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx \tag{3.5}$$

$$V_j^0 = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} f_x(u_0(x))dx = \frac{1}{h}\Big[f(u_0(x_{j+1/2})) - f(u_0(x_{j-1/2}))\Big]. \tag{3.6}$$

Note that one may get another expression for (3.3) and (3.4) written as

$$
\begin{aligned}
U_j^{m+2} &= U_j^m - \frac{2\tau}{h}\Big(F(U_j^m, U_{j+1}^m) - F(U_{j-1}^m, U_j^m)\Big) \\
U_j^1 &= U_j^0 - \tau V_j^0.
\end{aligned}
\tag{3.7}
$$

Theorem 3.1. *Let (u, v) is the exact solution to (2.8) and (2.9). Assume that V^m generated by (3.3) and (3.4) converges to the exact solution v. Then the solution U^m generated by (3.3) and (3.4) converges to the exact solution u. Hence, the method is first-order accuracy.*

Proof. Let us denote $(u)_j^m = u(x_j, t_m)$ and $(v)_j^m = v(x_j, t_m)$. Note that $u_t + v = 0$. Then, from (3.4) we have

$$(u)_j^{m+1} - (u)_j^m + \tau(v)_j^m = (u_t)_j^m \tau + (u_{tt})_j^m \frac{\tau^2}{2} + O(\tau^3) + \tau(v)_j^m = O(\tau^2)$$

Therefore, U^m converges to the exact solution u as $\tau \to 0$. □

Assume that we have another sequence S^m satisfying (3.3)-(3.4) or (3.7) and let $W_j^m = U_j^m - S_j^m$ where the sequence U^m satisfying (3.3)-(3.4) or (3.7). For the time being, we assume that $f'(U_j^m) > 0$ for all j, $F(u, v) = f(u)$ and that the CFL condition requires that

$$0 \leq \frac{2\tau}{h} f'(u) \leq 1 \tag{3.8}$$

for all u in the range $\min_j(U_j^m, S_j^m) \leq u \leq \max_j(U_j^m, S_j^m)$.

Consider for $m = 0$. From (3.4), we have

$$U_j^1 = U_j^0 - \tau V_j^0, \quad S_j^1 = S_j^0 - \tau V_j^0.$$

In the sense of the discrete l_1 norm for grid functions $U = \{U_j\}$ defined as $\|U\|_1 = h \sum_{j=-\infty}^{\infty} |U_j|$ it follows that, because of $W_j^1 = W_j^0$,

$$\|W^1\|_1 = \|W^0\|_1. \tag{3.9}$$

Consider for $m = 1$ in (3.3)-(3.4). Then two sequences $\{U^2\}$ and $\{S^2\}$ satisfy

$$
\begin{aligned}
U_j^2 &= U_j^1 + \tau V_j^0 - \frac{2\tau}{h}\Big(f(U_j^0) - f(U_{j-1}^0)\Big) \\
S_j^2 &= S_j^1 + \tau V_j^0 - \frac{2\tau}{h}\Big(f(S_j^0) - f(S_{j-1}^0)\Big)
\end{aligned}
$$

which leads to

$$
\begin{aligned}
W_j^2 &= W_j^1 - \alpha_j^0 W_j^0 + \alpha_{j-1}^0 W_{j-1}^0 \\
&= (1 - \alpha_j^0) W_j^1 + \alpha_{j-1}^0 W_{j-1}^1, \qquad \alpha_j^0 := \frac{2\tau}{h} f'(\theta_j^0)
\end{aligned}
\tag{3.10}
$$

where θ_j is between U_j^0 and S_j^0. Therefore, it follows that

$$\|W^2\|_1 \leq \|W^1\|_1. \tag{3.11}$$

For the case $m \geq 1$, we will use (3.7). The two sequences $\{U^{m+2}\}$ and $\{S^{m+2}\}$ satisfying

$$U_j^{m+2} = U_j^m - \frac{2\tau}{h}\Big(f(U_j^m) - f(U_{j-1}^m)\Big), \quad S_j^{m+2} = S_j^m - \frac{2\tau}{h}\Big(f(S_j^m) - f(S_{j-1}^m)\Big)$$

leads to

$$\begin{aligned} W_j^{m+2} &= W_j^m - \frac{2\tau}{h}\Big(f(U_j^m) - f(S_j^m)\Big) - \frac{2\tau}{h}\Big(f(U_{j-1}^m) - f(S_{j-1}^m)\Big) \\ &= (1-\alpha_j^m)W_j^m + \alpha_{j-1}^m W_{j-1}^m, \qquad \alpha_j^m := \frac{2\tau}{h}f'(\theta_{j-1}^m). \end{aligned} \tag{3.12}$$

where θ_{j-1}^m is between U_{j-1}^m and S_{j-1}^m. Hence , if the CFL condition is $0 \leq \alpha_j^m \leq 1$, we have

$$\|W^{m+2}\|_1 \leq \|W^m\|_1. \tag{3.13}$$

Therefore we have l_1 contracting property which is stated in the following theorem.

Theorem 3.2. *Let $\{U^m\}$ and $\{S^m\}$ be sequences satisfying* (3.7). *Assume that $f'(U_j^m) > 0$ for all j, $F(u,v) = f(u)$ and that the CFL condition $0 \leq \frac{2\tau}{h}f'(u) \leq 1$ for all u in the range $\min_j(U_j^m, S_j^m) \leq u \leq \max_j(U_j^m, S_j^m)$. Then the algorithm* (3.3)-(3.4) *has the following l_1- contracting property*

$$\|U^{m+2} - S^{m+2}\|_1 \leq \|U^m - S^m\|_1, \qquad m \geq 0, \tag{3.14}$$

$$\|U^1 - S^1\|_1 = \|U^0 - S^0\|_1, \quad \|U^2 - S^2\|_1 \leq \|U^1 - S^1\|_1. \tag{3.15}$$

Note. We note that the l_1- contracting property stated in Theorem 3.2 does not say contraction between consecutive approximations, for example, it is not stated that $\|U^3 - S^3\|_1 \leq \|U^2 - S^2\|_1$. Therefore we can say that the algorithm (3.3)-(3.4) does not have a full l_1- contracting property. But we will show that it is TV-stable scheme.

Note that a difference scheme is called TVD(total variation decreasing) if $TV(\mathbf{u}^{n+1}) \leq TV(\mathbf{u}^n)$ where the total variation of grid function $TV(\mathbf{u})$ is defined as

$$TV(\mathbf{u}) = \sum_{j=-\infty}^{\infty} |\delta_+ u_j|, \quad \delta_+ u_k = u_{k+1} - u_k.$$

It is well known that any l_1- contracting numerical method is TVD (see [3] for example). Because the algorithm (3.3)-(3.4) is not a full l_1- contracting numerical method, it is worthwhile to mention that the even or odd sequences generated by (3.7) is TVD.

Corollary 3.3. *Under the assumptions in Theorem 3.2, The even or odd sequence generated by algorithm (3.3)-(3.4) is TVD, that is,*

$$TV(\mathbf{U}^{m+2}) \leq TV(\mathbf{U}^m). \tag{3.16}$$

Proof. It follows from Theorem 3.2 immediately by following Theorem 15.4 in [3]. Hence, one may verify this corollary easily. □

This theorem reveals that the algorithm is TV-stable.

Theorem 3.4. *Under the assumptions in Theorem 3.2, the algorithm* (3.4)-(3.3) *is TV-stable.*

Proof. It comes from Theorem 3.2 immediately that, using (3.14) and (3.15),

$$TV(\mathbf{U}^{2m+1}) \leq TV(\mathbf{U}^{2m-1}) \leq TV(\mathbf{U}^0), \quad TV(\mathbf{U}^{2m}) \leq TV(\mathbf{U}^{2m-2}) \leq TV(\mathbf{U}^0)$$

This completes the proof. □

4. Numerical Example

Because the purpose of this paper is to understand some extension of the conservative upwind method, we will take the nonlinear Burgers' equation and a linear hyperbolic equation. The numerical flux is chosen as the same one in the upwind method so that numerical results are very similar if one choose appropriate CFL relations between those two methods (see Table 1). In Figure 1, we display the numerical results at $t = 0,\ 0.5,\ 5$ and also the enlarged result at $t = 5$. In this simulation, we use the technique known according to the wave direction (see p.113 in [3]). Then we can see through these numerical results that the newly proposed algorithm is an extension of the upwind method, which is slightly better than the upwind method (see numerical results in Example 4.2). In Table 2 and 3, we show the errors of the proposed algorithm at $t = 3$ for the linear equation (4.4) with initial condition (4.6) and (4.7) with the same time step 2τ as mesh width h. In Figure 2, we display the numerical results at $t = 1,\ 2,\ 3$ and $t = 5$ with the CFL number is 0.5.

Example 4.1. Consider the following Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \tag{4.1}$$

with the initial condition

$$u_0(x) = \begin{cases} 1, & \text{for} \quad x < 0 \\ 0, & \text{for} \quad x > 0, \end{cases} \tag{4.2}$$

or

$$u_0(x) = 0.5 + \sin(\pi x). \tag{4.3}$$

Table 1: The ℓ_1 errors of $u_t + uu_x = 0$ with $u_0(x)$ is (4.2) at time $t = 2.0$.

	$1/h$	CFL = 0.4	CFL = 0.5	CFL = 0.6	CFL = 0.8
CTEM	4	4.70928e-2	3.98124e-2	3.81683e-2	1.81170e-2
	16	1.22794e-2	1.02813e-2	9.60702e-3	4.56405e-3
	64	3.06997e-3	2.57035e-3	2.40176e-3	1.14101e-3
	256	7.67492e-4	6.42588e-4	6.00439e-4	2.85253e-4
Upwind	4	6.15263e-2	5.79394e-2	5.47982e-2	4.70928e-2
	16	1.63633e-2	1.53332e-2	1.44231e-2	1.22794e-2
	64	4.09152e-3	3.83377e-3	3.60606e-3	3.06997e-3
	256	1.02288e-3	9.58443e-4	9.01515e-4	7.67492e-4

Example 4.2. Consider the following linear equation:

$$u_t + u_x = 0, \tag{4.4}$$

with the initial condition

$$u_0(x) = \begin{cases} e^{-1000(x+0.7)^2}, & \text{for } -0.8 < x < -0.6 \\ 1, & \text{for } -0.4 < x < -0.2 \\ 1 - 10|x - 0.1|, & \text{for } 0 < x < 0.2 \\ 1 - 100(x - 0.5)^2, & \text{for } 0.4 < x < 0.6 \\ 0, & \text{elsewhere,} \end{cases} \tag{4.5}$$

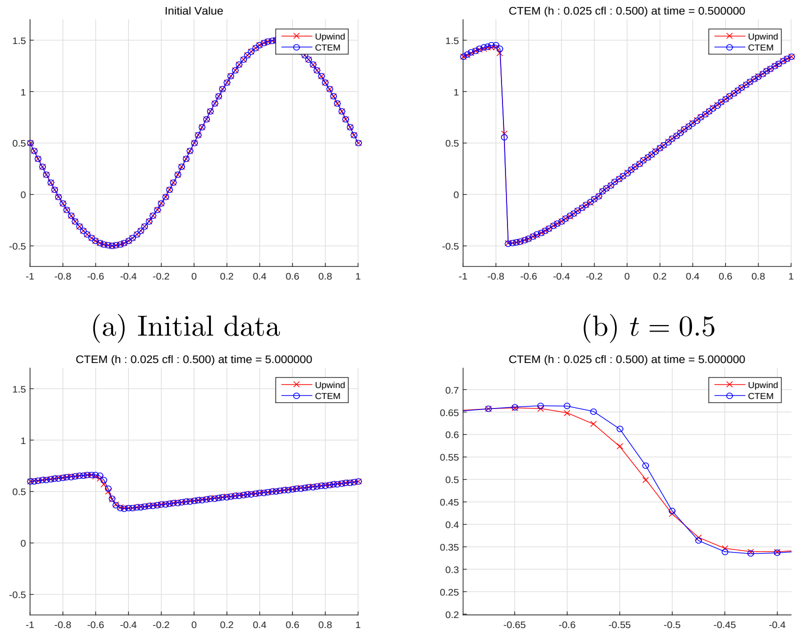
or

$$u_0(x) = \sin(\pi x) \tag{4.6}$$

or

$$u_0(x) = \sin^4(\pi x). \tag{4.7}$$

Figure 1: Numerical results for the Burgers' equation with initial data (4.3).



(a) Initial data (b) $t = 0.5$

(c) $t = 5.0$ (d) $t = 5.0$(enlargement)

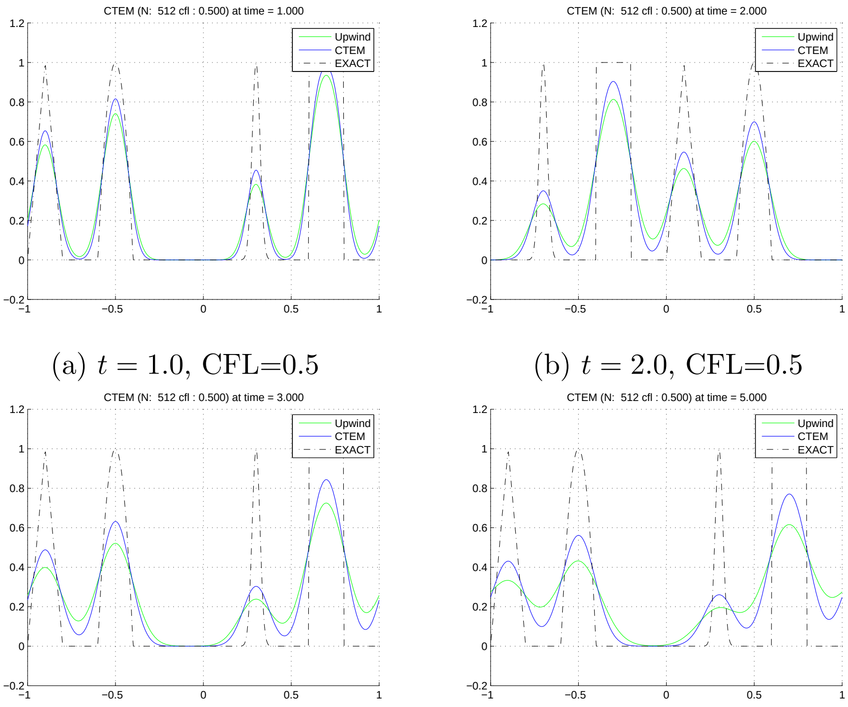
Table 2: The ℓ_1 errors of $u_t + u_x = 0$ with $u_0(x) = \sin(\pi x)$ at time $t = 3.0$.

	$1/h$	CFL = 0.4	CFL = 0.5	CFL = 0.6	CFL = 0.8
CTEM	4	6.85563e-1	5.44001e-1	4.19253e-1	1.63754e-1
	16	4.58387e-1	3.82163e-1	3.06405e-1	1.56640e-1
	64	1.57547e-1	1.31404e-1	1.05443e-1	5.26258e-2
	256	4.29182e-2	3.57721e-2	2.86648e-2	1.43079e-2
	1024	1.09656e-2	9.14056e-3	7.30987e-3	3.65517e-3
Upwind	4	1.07828e-0	1.05965e-0	1.04751e-0	1.02106e-0
	16	6.32898e-1	6.03565e-1	5.75322e-1	5.22228e-1
	64	2.11724e-1	1.99306e-1	1.87037e-1	1.62898e-1
	256	5.73223e-2	5.37929e-2	5.02754e-2	4.32686e-2
	1024	1.46258e-2	1.37153e-2	1.28051e-2	1.09871e-2

Table 3: The ℓ_1 errors of $u_t + u_x = 0$ with $u_0(x) = \sin^4(\pi x)$ at time $t = 3.0$.

	$1/h$	CFL = 0.4	CFL = 0.5	CFL = 0.6	CFL = 0.8
CTEM	4	5.68064e-1	4.38774e-1	3.09594e-1	4.63447e-1
	16	5.02896e-1	4.24164e-1	3.38736e-1	1.61255e-1
	64	2.69349e-1	2.29186e-1	1.86867e-1	9.70107e-2
	256	9.70391e-2	8.16007e-2	6.58861e-2	3.35519e-2
	1024	2.74299e-2	2.29205e-2	1.83857e-2	9.24446e-3
Upwind	4	6.22362e-1	6.22319e-1	6.22284e-1	6.22169e-1
	16	6.25953e-1	6.15306e-1	6.05359e-1	5.88969e-1
	64	3.50774e-1	3.35544e-1	3.20591e-1	2.92303e-1
	256	1.27763e-1	1.20723e-1	1.13706e-1	9.97381e-2
	1024	3.64269e-2	3.42285e-2	3.20306e-2	2.76369e-2

Figure 2: Numerical results for the linear equation with initial data (4.5).



(a) $t = 1.0$, CFL=0.5 (b) $t = 2.0$, CFL=0.5

(c) $t = 3.0$, CFL=0.5 (d) $t = 5.0$, CFL=0.5

5. Conclusion

The classical conservative form can be extended by the forward Euler method and the trapezoidal quadrature rule to a new method which is conservative, consistent and TV-stable. But this new method can not cope with many other physical phenomena occurred in system of hyperbolic equations such as a system of Euler equations, the Cauchy problem and etc. However, the idea introducing a new variable appeared in this paper may give us a good guidance for developing a nice conservative method (see [1], [2], [5] for example) which will be covered in a coming paper.

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