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Symbolic Algorithm for a System of Differential-Algebraic Equations

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Abstract. In this paper, a symbolic algorithm for solving a regular initial value problem (IVP) for a system of linear differential-algebraic equations (DAEs) with constant coefficients has been presented. Algebra of integro-differential operators is employed to express the given system of DAEs. We compute a canonical form of the given system which produces another simple equivalent system. Algorithm includes computing the matrix Green's operator and the vector Green's function of a given IVP. Implementation of the proposed algorithm in Maple is also presented with sample computations.

1. Introduction

Symbolic computation is playing the central role to solve the mathematical equations, especially the boundary value problems for differential equations. It is an important tool in scientific field, which is a part of *computer algebra*. In twenty-th century, the science and technology had a very swift progress in various fields, especially in computing, the subfield of scientific and technological computing. Although, the symbolic computation is a part of scientific computing, but generally it is considered as different field because scientific computing is usually based on numerical computation, and most of the numerical calculations are carried out with approximate floating point algorithm, whereas symbolic computing is a mathematical computation in which it emphasizes on the exact solution with symbols representing the mathematical objects. One of the big success in the research of symbolic computation is the development of significant software systems.

The proposed symbolic method is established on algebraic structures and it

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has connection with analytic and numeric techniques. As most of the systems of differential-algebraic equations arising in the applications can only be solved numerically, this connection is absolutely critical. The applications of differential-algebraic equations arise naturally in many fields and various dynamic processes, for example, mechanical systems [15, 21, 23], simulation of electric circuits [7, 8, 20, 22] and chemical reactions subject to invariants etc. [1, 4, 6, 9, 13, 14, 24] are often expressed by differential-algebraic equations (DAEs), which consist of algebraic equations and differential operations. Several methods have been introduced by many researchers and engineers for solving an initial value problems (IVPs) for systems of DAEs. Most of them are trying to find an approximate solution of the given system. However, we present a symbolic method that computes the exact solution of a given IVP.

Some *advantages* of the proposed symbolic algorithm over other numerical methods are: the proposed symbolic method computes the exact solution, it works directly on the level of operators and simple to understand the solution. In this method, we solve not only for a particular system but also a generic expression for different vector forcing functions $f(x)$ to produce the vector *Green's function.* The same idea is also applicable to a regular IVP for system of higher-order linear differential-algebraic equations to compute the vector Green's function. A new algorithm is provided for verifying consistency of the inhomogeneous initial conditions. This algorithm will help to implement the manual calculations in commercial packages such as Matlab, Mathematica, Singular, SCIlab etc. Maple implementation of the algorithm is presented in this paper.

1.1. Algebra of integro-differential operators

First we recall some basic concepts of integro-differential algebras and operators, to represent the system of DAEs with initial conditions in operator-based notations, see, for example [11, 17, 18] for further details. Throughout this section $\mathbb{K}$ denotes the field of characteristic zero.

Suppose $\mathcal{F} = C^\infty[a,b]$, for simplicity, and $[a,b]$ is an interval of $\mathbb{R}$. Consider a system of n linear DAEs of the following type,

$$A\mathtt{D}u(x) + Bu(x) = f(x), \tag{1.1}$$

where $\mathtt{D} = \frac{d}{dx}$; $f(x) = (f_1(x), \ldots, f_n(x))^T \in \mathcal{F}^n$ is a vector forcing function and $u(x) = (u_1(x), \ldots, u_n(x))^T \in \mathcal{F}^n$ is unknown vector to be determine; $A, B \in \mathcal{F}^{n\times n}$ are the coefficient matrices. If $A \equiv 0$, then the system in equation (1.1) is purely an algebraic system and there are several methods to find all possible solutions. If A is regular matrix, then the system (1.1) turns out to be a system of ordinary differential equations as

$$\mathtt{D}u(x) + A^{-1}Bu(x) = A^{-1}f(x).$$

Consider a system of the form (1.1) with non-zero singular matrix A to find a solution of a system of DAEs. The matrix differential operator of the system (1.1) is given by

$$T = A\mathtt{D} + B. \tag{1.2}$$

In order to obtain the unique solution, the system (1.1) must have a set of consistent initial conditions. Suppose

$$u(a) = 0, \tag{1.3}$$

is a consistent initial condition at a fixed initial point $a \in \mathbb{R}$.

Definition 1.1 An IVP for a system of DAEs is said to be *regular* if it has a solution, otherwise it called *singular*.

For a given regular matrix differential operator $T = A\mathtt{D} + B$ and initial conditions, the goal is to find an operator G, so-called *matrix Green's operator*, such that $u(x) = Gf(x)$ satisfies $Tu(x) = f(x)$ with $u(a) = 0$. We want to find an explicit formula for the solution corresponding to general linear systems of DAEs with initial conditions in algebraic settings. The traditional tool for achieving this is the classical concept of Moore-Penrose generalized inverse [10, 16, 19] and the variation of parameters [2, 17].

Definition 1.2([17]). The algebraic structure $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ is called an *integro-differential algebra* over $\mathbb{K}$ if $\mathcal{F}$ is a commutative $\mathbb{K}$-algebra with $\mathbb{K}$-linear operators $\mathtt{D}$ and $\mathtt{A}$ such that the following conditions are satisfied

$$\mathtt{D}(\mathtt{A}f) = f, \tag{1.4}$$

$$\mathtt{D}(fg) = (\mathtt{D}f)g + f(\mathtt{D}g), \tag{1.5}$$

$$(\mathtt{AD}f)(\mathtt{AD}g) + \mathtt{AD}(fg) = (\mathtt{AD}f)g + f(\mathtt{AD}g), \tag{1.6}$$

where $\mathtt{D} : \mathcal{F} \to \mathcal{F}$ and $\mathtt{A} : \mathcal{F} \to \mathcal{F}$ are two maps such that $\mathtt{D}$ is a derivation and $\mathtt{A}$ is a $\mathbb{K}$-linear right inverse of $\mathtt{D}$, i.e. $\mathtt{DA} = 1$ (the identity map). The map $\mathtt{A}$ is called an *integral* for $\mathtt{D}$. An integro-differential algebra over $\mathbb{K}$ is called *ordinary* if $\mathrm{Ker}(\mathtt{D}) = \mathbb{K}$.

The operators $\mathtt{J} = \mathtt{AD}$, $\mathtt{E} = 1 - \mathtt{AD}$, are projectors, known as the *initialization* and the *evaluation* of $\mathcal{F}$ respectively. For an ordinary integro-differential algebra, the evaluation can be translated as a multiplicative linear functional(character) $\mathtt{E} : \mathcal{F} \to \mathbb{K}$.

Example 1.3([17]). For $\mathcal{F} = C^\infty(\mathbb{R})$ with $\mathtt{D} = \frac{d}{dx}$ and $\mathtt{A} = \int_a^x$, the operator $\mathtt{E}f(x) = f(a)$ evaluates $f(x)$ at the initialization point a, and $\mathtt{J}f(x) = f(x) - f(a)$ applies the initial condition.

The following proposition shows that the matrix ring $\mathcal{F}^{n\times n}$ is an integro-differential algebra if $\mathcal{F}$ is an integro-differential algebra.

Proposition 1.4. *Let $\mathcal{F}$ be an integro-differential algebra over a field $\mathbb{K}$. Then the matrix ring $\mathcal{F}^{n\times n}$ is again an integro-differential algebra over $\mathbb{K}$.*

Proof. Let $S = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix}$ and $R = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix}$ be from $\mathcal{F}^{n\times n}$. We apply the action of operators $\mathtt{D}, \mathtt{A}$ and $\mathtt{E}$ component wise. Then, we have

$$\mathtt{D}(\mathtt{A}S) = \begin{pmatrix} \mathtt{D}(\mathtt{A}s_{11}) & \cdots & \mathtt{D}(\mathtt{A}s_{1n}) \\ \vdots & \ddots & \vdots \\ \mathtt{D}(\mathtt{A}s_{n1}) & \cdots & \mathtt{D}(\mathtt{A}s_{nn}) \end{pmatrix} = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix} = S.$$

For $i, j = 1, \dots, n$; we have

$$\begin{aligned}
\mathtt{D}(SR) &= \begin{pmatrix} \mathtt{D}(s_{11}r_{11}) + \cdots + \mathtt{D}(s_{1n}r_{n1}) & \cdots & \mathtt{D}(s_{11}r_{1n}) + \cdots + \mathtt{D}(s_{1n}r_{nn}) \\ \vdots & \ddots & \vdots \\ \mathtt{D}(s_{n1}r_{11}) + \cdots + \mathtt{D}(s_{nn}r_{n1}) & \cdots & \mathtt{D}(s_{n1}r_{1n}) + \cdots + \mathtt{D}(s_{nn}r_{nn}) \end{pmatrix} \\
&= \begin{pmatrix} \mathtt{D}(s_{11})r_{11} + \cdots + \mathtt{D}(s_{1n})r_{n1} & \cdots & \mathtt{D}(s_{11})r_{1n} + \cdots + \mathtt{D}(s_{1n})r_{nn} \\ \vdots & \ddots & \vdots \\ \mathtt{D}(s_{n1})r_{11} + \cdots + \mathtt{D}(s_{nn})r_{n1} & \cdots & \mathtt{D}(s_{n1})r_{1n} + \cdots + \mathtt{D}(s_{nn})r_{nn} \end{pmatrix} \\
&+ \begin{pmatrix} s_{11}\mathtt{D}(r_{11}) + \cdots + s_{1n}\mathtt{D}(r_{n1}) & \cdots & s_{11}\mathtt{D}(r_{1n}) + \cdots + s_{1n}\mathtt{D}(r_{nn}) \\ \vdots & \ddots & \vdots \\ s_{n1}\mathtt{D}(r_{11}) + \cdots + s_{nn}\mathtt{D}(r_{n1}) & \cdots & s_{n1}\mathtt{D}(r_{1n}) + \cdots + s_{nn}\mathtt{D}(r_{nn}) \end{pmatrix} \\
&= \begin{pmatrix} \mathtt{D}s_{11} & \cdots & \mathtt{D}s_{1n} \\ \vdots & \ddots & \vdots \\ \mathtt{D}s_{n1} & \cdots & \mathtt{D}s_{nn} \end{pmatrix} \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} + \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix} \begin{pmatrix} \mathtt{D}r_{11} & \cdots & \mathtt{D}r_{1n} \\ \vdots & \ddots & \vdots \\ \mathtt{D}r_{n1} & \cdots & \mathtt{D}r_{nn} \end{pmatrix} \\
&= (\mathtt{D}S)R + S(\mathtt{D}R).
\end{aligned}$$

Now,

$$\begin{aligned}
&(\mathtt{AD}S)(\mathtt{AD}R) + \mathtt{AD}(SR) \\
&= \begin{pmatrix} \mathtt{AD}s_{11}\mathtt{AD}r_{11} + \cdots + \mathtt{AD}s_{1n}\mathtt{AD}r_{n1} & \cdots & \mathtt{AD}s_{11}\mathtt{AD}r_{1n} + \cdots + \mathtt{AD}s_{1n}\mathtt{AD}r_{nn} \\ \vdots & \ddots & \vdots \\ \mathtt{AD}s_{n1}\mathtt{AD}r_{11} + \cdots + \mathtt{AD}s_{nn}\mathtt{AD}r_{n1} & \cdots & \mathtt{AD}s_{n1}\mathtt{AD}r_{1n} + \cdots + \mathtt{AD}s_{nn}\mathtt{AD}r_{nn} \end{pmatrix} \\
&+ \begin{pmatrix} \mathtt{AD}(s_{11}r_{11}) + \cdots + \mathtt{AD}(s_{1n}r_{n1}) & \cdots & \mathtt{AD}(s_{11}r_{1n}) + \cdots + \mathtt{AD}(s_{1n}r_{nn}) \\ \vdots & \ddots & \vdots \\ \mathtt{AD}(s_{n1}r_{11}) + \cdots + \mathtt{AD}(s_{nn}r_{n1}) & \cdots & \mathtt{AD}(s_{n1}r_{1n}) + \cdots + \mathtt{AD}(s_{nn}r_{nn}) \end{pmatrix} \\
&= \begin{pmatrix} (\mathtt{AD}s_{11})r_{11} + s_{11}(\mathtt{AD}r_{11}) & \cdots & (\mathtt{AD}s_{1n})r_{n1} + s_{1n}(\mathtt{AD}r_{n1}) \\ \vdots & \ddots & \vdots \\ (\mathtt{AD}s_{n1})r_{1n} + s_{n1}(\mathtt{AD}r_{1n}) & \cdots & (\mathtt{AD}s_{nn})r_{nn} + s_{nn}(\mathtt{AD}r_{nn}) \end{pmatrix} \\
&= (\mathtt{AD}S)R + S(\mathtt{AD}R).
\end{aligned}$$

Therefore, $\mathcal{F}^{n\times n}$ is an integro-differential algebra over $\mathbb{K}$. $\square$

Now we define the algebra of integro-differential operators, which can also be extended to the vector case as shown in above proposition.

Definition 1.5([17, 18]). Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra over $\mathbb{K}$ and $\Phi \subseteq \mathcal{F}^*$ (dual of $\mathcal{F}$). The *integro-differential operators* $\mathcal{F}[\mathtt{D}, \mathtt{A}]$ are defined as the $\mathbb{K}$-algebra generated by the symbols $\mathtt{D}$ and $\mathtt{A}$, the functions $f \in \mathcal{F}$ and the characters (functionals) $\mathtt{E}_c \in \Phi$, modulo the Noetherian and confluent rewrite system given in Table 1.

Table 1: Rewrite rules for integro-differential operators

$fg \to f \cdot g$	$\mathtt{D}f \to f\mathtt{D} + f'$	$\mathtt{A}f\mathtt{A} \to (\mathtt{A}f)\mathtt{A} - \mathtt{A}(\mathtt{A}f)$
$\chi\phi \to \phi$	$\mathtt{D}\phi \to 0$	$\mathtt{A}f\mathtt{D} \to f - \mathtt{A}f' - \mathtt{E}(f)\mathtt{E}$
$\phi f \to \phi(f)\phi$	$\mathtt{DA} \to 1$	$\mathtt{A}f\phi \to (\mathtt{A}f)\phi$

Now, given any IVP for the system of DAEs can be represented in operators as

$$\begin{aligned} Tu &= f, \\ \mathtt{E}u &= 0, \end{aligned} \tag{1.7}$$

where $T = A\mathtt{D} + B \in \mathcal{F}^{n\times n}[\mathtt{D}]$ is the matrix differential operator, $f \in \mathcal{F}^n$ is the vector forcing function and $\mathtt{E}$ is the evaluation operator such that $\mathtt{E}u = u(a)$. We want to find the matrix Green's operator $G \in \mathcal{F}^{n\times n}[\mathtt{D}, \mathtt{A}]$ such that $Gf = u$ and $\mathtt{E}G = 0$.

2. A New Symbolic Algorithm

We want to find a solution not only for a particular system of the form (1.7) by fixing f on its right hand side; but also a generic expression for different vector forcing functions f to produce the corresponding solutions. Therefore, we propose a symbolic algorithm that transform the given system of DAEs and initial conditions into operator based notations on suitable spaces.

In the following proposition, we provide an algorithm to test the regularity of a given IVP for a system of DAEs.

Proposition 2.1([3, 5]). *Given a matrix differential operator $T = AD + B \in \mathcal{F}^{n\times n}[D]$, the system*

$$Tu = f$$

has a solution if the matrix

$$\delta A + B$$

is regular for all nonzero δ.

For shake of completeness, we provide a sketch of proof as follows.

Proof. Taking Laplace transformation to the system $Tu = f$, we get

$$A\delta\mathcal{L}[u] - Au(0) + B\mathcal{L}[u] = \mathcal{L}[f],$$

where $\mathcal{L}[\cdot]$ is the Laplace transform of the argument. We have

$$\mathcal{L}[u](\delta A + B) = \mathcal{L}[f] + Au(0).$$

Now $\mathcal{L}[u]$ is determined uniquely if $\delta A + B$ is regular as

$$\mathcal{L}[u] = (\delta A + B)^{-1}(\mathcal{L}[f] + Au(0)),$$

and hence the solution

$$u = \mathcal{L}^{-1}\left[(\delta A + B)^{-1}(\mathcal{L}[f] + Au(0))\right].$$

where $\mathcal{L}^{-1}[\cdot]$ is the inverse Laplace transform of the argument. $\square$

Dai discussed in [3, Chapter 1] that the regularity of a systems of DAEs is equivalent to the existence of a unique solution.

Example 2.2([5]). Consider a standard system of DAEs, for a body of mass m at position u_1 and velocity u_2 with affected force F,

$$\begin{aligned} u_1' - u_2 &= 0, \\ mu_2' &= F. \end{aligned} \tag{2.1}$$

In matrix notations, we have $(A\mathtt{D} + B)u = f$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 \\ F \end{pmatrix}.$$

The system (2.1) has a solution if $\delta A + B$ is regular, i.e., $m\delta^2 \neq 0$. Therefore, the given system is regular if and only if $m \neq 0$.

The following Lemma 2.3 is one of the essential steps for our algorithm. This lemma gives the variation of parameters formula for an IVP for higher-order scalar linear differential equations over integro-differential algebras.

Lemma 2.3([2, 17]). *Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra. For a monic scalar differential operator $L = \mathtt{D}^m + a_{m-1}\mathtt{D}^{m-1} + \cdots + a_0 \in \mathcal{F}[\mathtt{D}]$ of order m with fundamental system $v_1, \ldots, v_m$, the right inverse operator of L is given by*

$$L^{*} = \sum_{i=1}^{m} v_i \mathtt{A} w^{-1} w_i \in \mathcal{F}[\mathtt{D}, \mathtt{A}], \tag{2.2}$$

where w is the determinant of the Wronskian matrix W for $v_1, \ldots, v_m$ and w_i the determinant of the matrix W_i obtained from W by replacing the i-th column by m-th unit vector.

In other words, an initial value problem

$$Lv = g, \qquad (2.3)$$
$$\mathtt{E}\, v = \mathtt{E}\, \mathtt{D}\, v = \cdots = \mathtt{E}\, \mathtt{D}^{m-1} v = 0$$

has the unique solution

$$v = L^{\circledast} g = \sum_{i=1}^{m} v_i \mathtt{A} w^{-1} w_i g.$$

Before presenting the proposed algorithm, we find a *canonical form* of the given DAEs system using the *shuffle algorithm* [12, 5] that transforms the given system into another equivalent simpler form which produces the solution of the given system.

Consider the system $A\mathtt{D}u + Bu = f$ with initial conditions. The augmented matrix of the given DAEs system is

$$\begin{pmatrix} A & \vdots & B & \vdots & f \end{pmatrix} \qquad (2.4)$$

Using the Gauss elimination technique, we can transform the matrix (2.4) into the form

$$\begin{pmatrix} A_1 & \vdots & B_1 & \vdots & \widehat{f}_1 \\ 0 & \vdots & B_2 & \vdots & \widehat{f}_2 \end{pmatrix},$$

where A_1 has full row rank. Now we have the system

$$\begin{pmatrix} A_1 \\ 0 \end{pmatrix} \mathtt{D}u + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u = \begin{pmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{pmatrix}. \qquad (2.5)$$

By differentiating the second row of the system (2.5), we get

$$\tilde{A}\mathtt{D}u + \tilde{B}u = \tilde{f}, \qquad (2.6)$$

where $\tilde{A} = \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ and $\tilde{f} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = \begin{pmatrix} \widehat{f}_1 \\ \mathtt{D}\widehat{f}_2 \end{pmatrix}$. If $\tilde{A}$ is regular, we are done, otherwise repeat the procedure until we get a regular $\tilde{A}$.

Remark 2.4. The process of obtaining a regular $\tilde{A}$ in shuffle algorithm will terminate after a finite number of iterations, for a regular IVP for DAEs.

The matrix Green's operator and Green's function of a given system of DAEs with initial conditions with the help of the canonical form are computed in the following theorem.

Thoerem 2.5. *Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra. Suppose $\tilde{T} = \tilde{A}\mathtt{D} + \tilde{B} \in \mathcal{F}^{n\times n}[\mathtt{D}]$ is a canonical form of $T = A\mathtt{D} + B$ such that $\tilde{T}u = \tilde{f}$ with* *initial conditions; and* $\{v_1,\dots,v_n\}$ *is a fundamental system for* $L=\det(\tilde{T})$*. Then the regular IVP for system of DAEs*

$$\begin{aligned} Tu &= f,\\ Eu &= 0, \end{aligned}$$

has the unique solution

$$u=\begin{pmatrix}\sum_{i=1}^{n}(-1)^{i+1}\mathfrak{T}_i^1 L^{*}\tilde{f}_i\\ \vdots\\ \sum_{i=1}^{n}(-1)^{i+n}\mathfrak{T}_i^n L^{*}\tilde{f}_i\end{pmatrix}, \tag{2.7}$$

where $\mathfrak{T}_i^j$ *is the determinant of* $\tilde{T}$ *after removing* i*-th row and* j*-th column;* L^{*} *is the right inverse of* L*; and* $\tilde{f}=(\tilde{f}_1,\dots,\tilde{f}_n)^T$*. The matrix Green's operator is*

$$G=\begin{pmatrix}(-1)^{1+1}\mathfrak{T}_1^1L^{*} & \cdots & (-1)^{n+1}\mathfrak{T}_n^1L^{*}\\ \vdots & \ddots & \vdots\\ (-1)^{1+n}\mathfrak{T}_1^nL^{*} & \cdots & (-1)^{n+n}\mathfrak{T}_n^nL^{*}\end{pmatrix} \tag{2.8}$$

such that $G\tilde{f}=u$ *and* $\mathtt{E}\,G=0$*.*

Proof. Since the coefficients matrix $\tilde{A}$ of $\tilde{T}$ is regular, from Proposition 2.1, the system $\tilde{T}u=\tilde{f}$ is regular, hence $Tu=f$ is regular. Using the concept of the generalized Moore-Penrose inverse to the canonical system $\tilde{T}$, we compute the solution u by incorporating the initial conditions. Since, the matrix $\tilde{T}$ is a square matrix, the generalized Moore-Penrose inverse of $\tilde{T}$ is the inverse of $\tilde{T}$. Suppose $L=\det(\tilde{T})$. Then u is computed as

$$u=\begin{pmatrix}(-1)^{1+1}\mathfrak{T}_1^1 & \cdots & (-1)^{n+1}\mathfrak{T}_n^1\\ \vdots & \ddots & \vdots\\ (-1)^{1+n}\mathfrak{T}_1^n & \cdots & (-1)^{n+n}\mathfrak{T}_n^n\end{pmatrix}\frac{1}{L}\tilde{f}. \tag{2.9}$$

Now, the problem of solving system of $\tilde{T}u=\tilde{f}$ with initial conditions is reduced to an IVP for a scalar higher-order linear differential equation $Ly_i=\tilde{f}_i$ with initial conditions. Using Lemma 2.3, we compute the solution of scalar equation as $y_i=L^{*}\tilde{f}_i$. Therefore, from equation (2.9), the solution is

$$u=\begin{pmatrix}(-1)^{1+1}\mathfrak{T}_1^1 & \cdots & (-1)^{n+1}\mathfrak{T}_n^1\\ \vdots & \ddots & \vdots\\ (-1)^{1+n}\mathfrak{T}_1^n & \cdots & (-1)^{n+n}\mathfrak{T}_n^n\end{pmatrix}\begin{pmatrix}L^{*}\tilde{f}_1\\ \vdots\\ L^{*}\tilde{f}_n\end{pmatrix}.$$

After simplification, we get

$$u=\begin{pmatrix}\sum_{i=1}^{n}(-1)^{i+1}\mathfrak{T}_i^1 L^{*}\tilde{f}_i\\ \vdots\\ \sum_{i=1}^{n}(-1)^{i+n}\mathfrak{T}_i^n L^{*}\tilde{f}_i\end{pmatrix}. \tag{2.10}$$

Again, from equation (2.9), we have

$$u = \begin{pmatrix} (-1)^{1+1}\mathfrak{I}_1^1 & \cdots & (-1)^{n+1}\mathfrak{I}_n^1 \\ \vdots & \ddots & \vdots \\ (-1)^{1+n}\mathfrak{I}_1^n & \cdots & (-1)^{n+n}\mathfrak{I}_n^n \end{pmatrix} L^{\circledast} \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_n \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{1+1}\mathfrak{I}_1^1 L^{\circledast} & \cdots & (-1)^{n+1}\mathfrak{I}_n^1 L^{\circledast} \\ \vdots & \ddots & \vdots \\ (-1)^{1+n}\mathfrak{I}_1^n L^{\circledast} & \cdots & (-1)^{n+n}\mathfrak{I}_n^n L^{\circledast} \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_n \end{pmatrix},$$

which gives the Green's operator G such that $u = G\tilde{f}$ and $\mathtt{E}\, G = 0$, i.e., $\mathtt{E}\, u = 0$ as follows

$$G = \begin{pmatrix} (-1)^{1+1}\mathfrak{I}_1^1 L^{\circledast} & \cdots & (-1)^{n+1}\mathfrak{I}_n^1 L^{\circledast} \\ \vdots & \ddots & \vdots \\ (-1)^{1+n}\mathfrak{I}_1^n L^{\circledast} & \cdots & (-1)^{n+n}\mathfrak{I}_n^n L^{\circledast} \end{pmatrix} \tag{2.11}$$

We show that the solution in equation (2.10) and the Green's operator in equation (2.11) satisfy the given IVP. Indeed,

$$Tu = T \begin{pmatrix} \sum_{i=1}^n (-1)^{i+1}\mathfrak{I}_i^1 L^{\circledast} \tilde{f}_i \\ \vdots \\ \sum_{i=1}^n (-1)^{i+n}\mathfrak{I}_i^n L^{\circledast} \tilde{f}_i \end{pmatrix}$$

$$= T \begin{pmatrix} (-1)^{1+1}\mathfrak{I}_1^1 L^{\circledast} & \cdots & (-1)^{n+1}\mathfrak{I}_n^1 L^{\circledast} \\ \vdots & \ddots & \vdots \\ (-1)^{1+n}\mathfrak{I}_1^n L^{\circledast} & \cdots & (-1)^{n+n}\mathfrak{I}_n^n L^{\circledast} \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_n \end{pmatrix}$$

$$= T \begin{pmatrix} (-1)^{1+1}\mathfrak{I}_1^1 & \cdots & (-1)^{n+1}\mathfrak{I}_n^1 \\ \vdots & \ddots & \vdots \\ (-1)^{1+n}\mathfrak{I}_1^n & \cdots & (-1)^{n+n}\mathfrak{I}_n^n \end{pmatrix} \begin{pmatrix} L^{\circledast}\tilde{f}_1 \\ \vdots \\ L^{\circledast}\tilde{f}_n \end{pmatrix}$$

$$= \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = f,$$

and

$$\mathtt{E}u = \mathtt{E} \begin{pmatrix} \sum_{i=1}^n (-1)^{i+1}\mathfrak{I}_i^1 L^{\circledast} \tilde{f}_i \\ \vdots \\ \sum_{i=1}^n (-1)^{i+n}\mathfrak{I}_i^n L^{\circledast} \tilde{f}_i \end{pmatrix} = \begin{pmatrix} \mathtt{E}\sum_{i=1}^n (-1)^{i+1}\mathfrak{I}_i^1 L^{\circledast} \tilde{f}_i \\ \vdots \\ \mathtt{E}\sum_{i=1}^n (-1)^{i+n}\mathfrak{I}_i^n L^{\circledast} \tilde{f}_i \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^n (-1)^{i+1}\mathtt{E}\mathfrak{I}_i^1 \mathtt{E}L^{\circledast}\mathtt{E}\tilde{f}_i \\ \vdots \\ \sum_{i=1}^n (-1)^{i+n}\mathtt{E}\mathfrak{I}_i^n \mathtt{E}L^{\circledast}\mathtt{E}\tilde{f}_i \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0,$$

for $\mathtt{E}$ is multiplicative and $\mathtt{E}L^{*} = 0$. □

Remark 2.6. For better understanding to reader, we explain the Theorem 2.4 for $n = 3$. We have

$$T = A\mathtt{D} + B \in \mathcal{F}^{3\times 3}[\mathtt{D}], \tag{2.12}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in \mathcal{F}^{3\times 3}$ and $f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$.

Suppose the canonical form of the given differential operator (2.12) is
$\tilde{T} = \tilde{A}\mathtt{D} + \tilde{B}$, where $\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix}, \tilde{B} = \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{pmatrix}$ and $\tilde{f} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix}$.

Let $L = \det(\tilde{T}) = \widehat{a}_3\mathtt{D}^3 + \widehat{a}_2\mathtt{D}^2 + \widehat{a}_1\mathtt{D} + \widehat{a}_0 \in \mathcal{F}[\mathtt{D}]$ be a scalar differential operator with fundamental system $\{v_1, v_2, v_3\}$. From Lemma 2.3, we have the right inverse operator of L as follows

$$L^{*} = v_1\mathtt{A}(v_2v_3' - v_2'v_3)w^{-1} + v_2\mathtt{A}(v_3v_1' - v_3'v_1)w^{-1} + v_3\mathtt{A}(v_1v_2' - v_1'v_1)w^{-1}, \tag{2.13}$$

where w is the determinant of Wronskian matrix of the fundamental system $\{v_1, v_2, v_3\}$, i.e. $w = \begin{vmatrix} v_1 & v_2 & v_3 \\ v_1' & v_2' & v_3' \\ v_1'' & v_2'' & v_3'' \end{vmatrix}$.

Now, from Theorem 2.4 the Green's function is

$$u = \begin{pmatrix} \mathcal{T}_1^1 L^{*}\tilde{f}_1 - \mathcal{T}_2^1 L^{*}\tilde{f}_2 + \mathcal{T}_3^1 L^{*}\tilde{f}_3 \\ -\mathcal{T}_1^2 L^{*}\tilde{f}_1 + \mathcal{T}_2^2 L^{*}\tilde{f}_2 - \mathcal{T}_3^2 L^{*}\tilde{f}_3 \\ \mathcal{T}_1^3 L^{*}\tilde{f}_1 - \mathcal{T}_2^3 L^{*}\tilde{f}_2 + \mathcal{T}_3^3 L^{*}\tilde{f}_3 \end{pmatrix}, \tag{2.14}$$

and Green's operator is

$$G = \begin{pmatrix} \mathcal{T}_1^1 L^{*} & -\mathcal{T}_2^1 L^{*} & \mathcal{T}_3^1 L^{*} \\ -\mathcal{T}_1^2 L^{*} & \mathcal{T}_2^2 L^{*} & -\mathcal{T}_3^2 L^{*} \\ \mathcal{T}_1^3 L^{*} & -\mathcal{T}_2^3 L^{*} & \mathcal{T}_3^3 L^{*} \end{pmatrix}, \tag{2.15}$$

where L^{*} is the right inverse of L as in (2.13) and $\mathcal{T}_i^j$ is the determinant of $\tilde{T}$ after removing i-th row and j-th column. Indeed,

$$\mathcal{T}_1^1 = \begin{vmatrix} \tilde{a}_{22}\mathtt{D} + \tilde{b}_{22} & \tilde{a}_{23}\mathtt{D} + \tilde{b}_{23} \\ \tilde{a}_{32}\mathtt{D} + \tilde{b}_{32} & \tilde{a}_{33}\mathtt{D} + \tilde{b}_{33} \end{vmatrix}; \quad \mathcal{T}_1^2 = \begin{vmatrix} \tilde{a}_{21}\mathtt{D} + \tilde{b}_{21} & \tilde{a}_{23}\mathtt{D} + \tilde{b}_{23} \\ \tilde{a}_{31}\mathtt{D} + \tilde{b}_{31} & \tilde{a}_{33}\mathtt{D} + \tilde{b}_{33} \end{vmatrix};$$

$$\mathcal{T}_1^3 = \begin{vmatrix} \tilde{a}_{21}\mathtt{D} + \tilde{b}_{21} & \tilde{a}_{22}\mathtt{D} + \tilde{b}_{22} \\ \tilde{a}_{31}\mathtt{D} + \tilde{b}_{31} & \tilde{a}_{32}\mathtt{D} + \tilde{b}_{32} \end{vmatrix}; \quad \mathcal{T}_2^1 = \begin{vmatrix} \tilde{a}_{12}\mathtt{D} + \tilde{b}_{12} & \tilde{a}_{13}\mathtt{D} + \tilde{b}_{13} \\ \tilde{a}_{32}\mathtt{D} + \tilde{b}_{32} & \tilde{a}_{33}\mathtt{D} + \tilde{b}_{33} \end{vmatrix};$$

$$\mathcal{T}_2^2 = \begin{vmatrix} \tilde{a}_{11}\mathtt{D} + \tilde{b}_{11} & \tilde{a}_{13}\mathtt{D} + \tilde{b}_{13} \\ \tilde{a}_{31}\mathtt{D} + \tilde{b}_{31} & \tilde{a}_{33}\mathtt{D} + \tilde{b}_{33} \end{vmatrix}; \quad \mathcal{T}_2^3 = \begin{vmatrix} \tilde{a}_{11}\mathtt{D} + \tilde{b}_{11} & \tilde{a}_{12}\mathtt{D} + \tilde{b}_{12} \\ \tilde{a}_{31}\mathtt{D} + \tilde{b}_{31} & \tilde{a}_{32}\mathtt{D} + \tilde{b}_{32} \end{vmatrix};$$

$$\mathcal{T}_3^1 = \begin{vmatrix} \tilde{a}_{12}\mathtt{D} + \tilde{b}_{12} & \tilde{a}_{13}\mathtt{D} + \tilde{b}_{13} \\ \tilde{a}_{22}\mathtt{D} + \tilde{b}_{22} & \tilde{a}_{32}\mathtt{D} + \tilde{b}_{32} \end{vmatrix}; \quad \mathcal{T}_3^2 = \begin{vmatrix} \tilde{a}_{11}\mathtt{D} + \tilde{b}_{11} & \tilde{a}_{13}\mathtt{D} + \tilde{b}_{13} \\ \tilde{a}_{21}\mathtt{D} + \tilde{b}_{21} & \tilde{a}_{23}\mathtt{D} + \tilde{b}_{23} \end{vmatrix};$$

$$\mathcal{T}_3^3 = \begin{vmatrix} \tilde{a}_{11}\mathtt{D} + \tilde{b}_{11} & \tilde{a}_{12}\mathtt{D} + \tilde{b}_{12} \\ \tilde{a}_{21}\mathtt{D} + \tilde{b}_{21} & \tilde{a}_{22}\mathtt{D} + \tilde{b}_{22} \end{vmatrix}.$$

Remark 2.7. One can extend the algorithm in Theorem 2.4 to the higher-order linear DAEs systems, i.e., the IVP

$$\begin{aligned} Tu &= f, \\ \Gamma u &= 0, \end{aligned}$$

has the unique solution as given in Theorem 2.4, where $T = A_l\mathtt{D}^l + A_{l-1}\mathtt{D}^{l-1} + \cdots + A_0 \in \mathcal{F}^{n\times n}[\mathtt{D}]$ is a matrix differential operator and $\Gamma = \{B_1, \ldots, B_k\}$, $k < nl$, is a set of initial condition operators such that $\Gamma u = B_i u = \mathtt{E}\mathtt{D}^i u$, for $i = 1, \ldots, k$, here n is the size of coefficient matrices $A_l \neq 0, \ldots, A_0$ and l is the highest order of differential system. The number, k, of initial conditions depend on the number of differential equations in the given system of DAEs.

2.1. Inhomogeneous initial conditions

In this section, we provide computations for an IVP for the system of DAEs with inhomogeneous conditions, i.e the solution of

$$\begin{aligned} Tu &= f, \\ \mathtt{E}u &= \alpha, \end{aligned} \tag{2.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)^T$. Before finding the exact solution, we first check the consistency of the initial conditions. The following proposition provides the algorithm to check the consistency of the given inhomogeneous initial conditions.

Proposition 2.8. *Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra. Suppose $\tilde{T} = \tilde{A}\mathtt{D} + \tilde{B} \in \mathcal{F}^{n\times n}[\mathtt{D}]$ is a canonical form of a regular matrix differential operator $T = A\mathtt{D} + B$ such that $\tilde{T}u = \tilde{f}$ with inhomogeneous initial conditions. If the inhomogeneous initial condition $\mathtt{E}u = \alpha$ is consistent, then*

$$UU_a^{-1}\alpha \in Ker(T), \tag{2.2}$$

where U is the fundamental matrix of $\tilde{T}$ and U_a is value of U at the initial point a.

Proof. Suppose the system $\mathtt{E}u = \alpha$ is consistent, where $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the canonical system $\tilde{T}u = \tilde{f}$ with initial conditions $\mathtt{E}u = \alpha$ is regular, for the matrix differential operator T is regular. The solution of the canonical system can be decomposed into two cases, namely the solution, say u_p, of $\tilde{T}u = \tilde{f}$ with conditions $\mathtt{E}u = 0$ and the solution, say u_c, of $\tilde{T}u = 0$ with conditions $\mathtt{E}u = \alpha$. The solution of the first case is obtained from Theorem 2.4 as

$$u_p = \begin{pmatrix} \sum_{i=1}^{n} (-1)^{i+1} \mathcal{T}_i^1 L^* \tilde{f}_i \\ \vdots \\ \sum_{i=1}^{n} (-1)^{i+n} \mathcal{T}_i^n L^* \tilde{f}_i \end{pmatrix}.$$

The solution of the second case is depending only on the inhomogeneous initial data, this amounts to an *interpolation problem* with initial conditions. Suppose

$$u_c = c_1 v_1 + \cdots + c_n v_n \tag{2.3}$$

is the required solution of the second case, where $U = [v_1, \cdots, v_n]$ is the fundamental matrix of $\tilde{T}$ and $c_1, \ldots, c_n$ are the coefficients to be determined. From the given initial conditions with α, one can express the equation (2.3) as

$$U_a(c_1, \ldots, c_n)^T = (\alpha_1, \ldots, \alpha_n)^T,$$

since U_a is regular, we have

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = U_a^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}. \tag{2.4}$$

Therefore, from equations (2.3)-(2.4), the solution of the second case is

$$\begin{aligned} u_c &= (v_1, \ldots, v_n) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\ &= (v_1, \ldots, v_n) U_a^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \\ &= U U_a^{-1} \alpha. \end{aligned}$$

Now, the general solution is $u = u_p + u_c$ and it must satisfy the canonical system as well as the given system, hence we have

$$\begin{aligned} &Tu = T(u_p + u_c) = Tu_p + Tu_c = f + T(UU_a^{-1}\alpha), \\ &\mathtt{E}u = \mathtt{E}(u_p + u_c) = \mathtt{E}u_p + \mathtt{E}u_c = 0 + \mathtt{E}UU_a^{-1}\alpha = 0 + \alpha = \alpha. \end{aligned}$$

which gives that $T(UU_a^{-1}\alpha) = 0$, i.e., $UU_a^{-1}\alpha \in \operatorname{Ker}(T)$. □

The following theorem provides an algorithm to compute the general solution of a given IVP for DAEs.

Theorem 2.9. *Let* $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ *be an ordinary integro-differential algebra. Suppose* $\tilde{T} = \tilde{A}\mathtt{D} + \tilde{B} \in \mathcal{F}^{n\times n}[\mathtt{D}]$ *is a canonical form of* $T = A\mathtt{D} + B$ *such that* $\tilde{T}u = \tilde{f}$ *with consistent inhomogeneous initial conditions; and* $\{v_1, \ldots, v_n\}$ *is a fundamental system for* $L = \det(\tilde{T})$*. Then the regular IVP for the system of DAEs*

$$\begin{aligned} Tu &= f, \\ Eu &= \alpha, \end{aligned}$$

has the unique solution

$$u = u_c + u_p, \tag{2.5}$$

where $u_c = UU_a^{-1}\alpha$ *and* $u_p = \begin{pmatrix} \sum_{i=1}^{n}(-1)^{i+1}\mathcal{T}_i^1 L^{*}\tilde{f}_i \\ \vdots \\ \sum_{i=1}^{n}(-1)^{i+n}\mathcal{T}_i^n L^{*}\tilde{f}_i \end{pmatrix}$.

Proof. We prove by substituting into the system and initial conditions. Now,

$$\begin{aligned} Tu = Tu_c + Tu_p &= TUU_a^{-1}\alpha + T\begin{pmatrix} \sum_{i=1}^{n}(-1)^{i+1}\mathcal{T}_i^1 L^{*}\tilde{f}_i \\ \vdots \\ \sum_{i=1}^{n}(-1)^{i+n}\mathcal{T}_i^n L^{*}\tilde{f}_i \end{pmatrix} \\ &= 0 + T\begin{pmatrix} (-1)^{1+1}\mathcal{T}_1^1 L^{*} & \cdots & (-1)^{n+1}\mathcal{T}_n^1 L^{*} \\ \vdots & \ddots & \vdots \\ (-1)^{1+n}\mathcal{T}_1^n L^{*} & \cdots & (-1)^{n+n}\mathcal{T}_n^n L^{*} \end{pmatrix}\begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_n \end{pmatrix} \\ &= \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = f, \qquad \text{(by Theorem 2.4)}, \end{aligned}$$

for $UU_a^{-1}\alpha \in \operatorname{Ker}(T)$, and

$$\begin{aligned} \mathtt{E}u = \mathtt{E}u_c + \mathtt{E}u_p &= \mathtt{E}UU_a^{-1}\alpha + \mathtt{E}\begin{pmatrix} \sum_{i=1}^{n}(-1)^{i+1}\mathcal{T}_i^1 L^{*}\tilde{f}_i \\ \vdots \\ \sum_{i=1}^{n}(-1)^{i+n}\mathcal{T}_i^n L^{*}\tilde{f}_i \end{pmatrix} \\ &= \mathtt{E}U_aU_a^{-1}\alpha + \begin{pmatrix} \sum_{i=1}^{n}(-1)^{i+1}\mathtt{E}\mathcal{T}_i^1 \mathtt{E}L^{*}\mathtt{E}\tilde{f}_i \\ \vdots \\ \sum_{i=1}^{n}(-1)^{i+n}\mathtt{E}\mathcal{T}_i^n \mathtt{E}L^{*}\mathtt{E}\tilde{f}_i \end{pmatrix} \\ &= I\alpha + 0 = \alpha, \quad \text{(by Theorem 2.4)}, \end{aligned}$$

where I is the identity matrix of order n. ☐

2.2. Sample computations

The following Examples demonstrate the proposed algorithm to compute the matrix Green's operator and the vector Green's function of a given IVP, and the exact solution for a fixed vector forcing function f.

Example 2.10. Consider the following differential-algebraic equations.

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{2.1}$$

with initial condition $\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The canonical form of the system (2.1) is $\tilde{A}\mathtt{D}u + \tilde{B}u = \tilde{f}$, where

$$\tilde{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \tilde{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{f} = \begin{pmatrix} f_1 + 2f_2 \\ \mathtt{D}f_2 \end{pmatrix},$$

The matrix differential operator for the given system and the canonical form of (2.1) respectively, are

$$T = \begin{pmatrix} 1 + \mathtt{D} & -2 - \mathtt{D} \\ 0 & 1 \end{pmatrix}, \quad \tilde{T} = \begin{pmatrix} 1 + \mathtt{D} & -\mathtt{D} \\ 0 & \mathtt{D} \end{pmatrix}$$

Following the algorithm in Theorem 2.4, we have the matrix Green's operator of $\tilde{T}$ given by

$$G = \begin{pmatrix} e^{-x}\mathtt{A}e^{x} & e^{-x}\mathtt{A}e^{x} \\ 0 & \mathtt{A} \end{pmatrix},$$

and the vector Green's function is given by

$$u = G\tilde{f} = \begin{pmatrix} e^{-x}\int\limits_0^x e^{x}(f_1(x) + 2f_2(x))dx + e^{-x}\int\limits_0^x e^{x}\mathtt{D}f_2(x)dx \\ f_2(x) \end{pmatrix}. \tag{2.2}$$

For simplicity, if $f = \begin{pmatrix} 0 \\ \sin x \end{pmatrix}$, then the exact solution is obtained from the Green's function (2.2) as

$$u = \begin{pmatrix} \frac{1}{2}e^{-x} - \frac{1}{2}\cos x + \frac{3}{2}\sin x \\ \sin x \end{pmatrix}. \tag{2.3}$$

One can easily check that $Tu = f$ and $\mathtt{E}u = 0$.

Consider the inhomogeneous initial conditions $u(0) = \alpha$ with given system (2.1). From Proposition 2.5, the initial condition is consistent if $UU_0^{-1}\alpha \in \mathrm{Ker}(T)$, where

$$U = \begin{pmatrix} e^{-x} & 0 \\ 0 & 1 \end{pmatrix}, U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1X \end{pmatrix}, \text{ and } \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Now

$$UU_0^{-1}\alpha = \begin{pmatrix} e^{-x} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} e^{-x}\alpha_1 \\ \alpha_2 \end{pmatrix}$$

and

$$T(UU_0^{-1}\alpha) = \begin{pmatrix} 1 + \mathtt{D} & -2 - \mathtt{D} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-x}\alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -2\alpha_2 \\ \alpha_2 \end{pmatrix},$$

which gives $\alpha_2 = 0$ for $UU_0^{-1}\alpha \in \mathrm{Ker}(T)$, and hence the consistent initial condition is $\mathtt{E}u = (\alpha_1, 0)^T, \alpha_1 \in \mathbb{R}$. The solution u_c of the IVP $Tu = 0, \mathtt{E}u = (\alpha_1, 0)^T$ computed as in Theorem 2.6 is

$$u_c = UU_0^{-1}\alpha = \begin{pmatrix} e^{-x}\alpha_1 \\ 0 \end{pmatrix},$$

and the solution u_p of the IVP $Tu = f, \mathtt{E}u = 0$ computed as in equation (2.3) is

$$u_p = \begin{pmatrix} \frac{1}{2}e^{-x} - \frac{1}{2}\cos x + \frac{3}{2}\sin x \\ \sin x \end{pmatrix}.$$

The exact solution of the regular system

$$\begin{pmatrix} 1+\mathtt{D} & -2-\mathtt{D} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \sin x \end{pmatrix},$$
$$\mathtt{E}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}; \qquad \alpha_1 \in \mathbb{R}$$

is computed from Theorem 2.6 as $u = u_c + u_p$, i.e.

$$u = \begin{pmatrix} \frac{1}{2}e^{-x} - \frac{1}{2}\cos x + \frac{3}{2}\sin x + \alpha_1 e^{-x} \\ \sin x \end{pmatrix}.$$

Example 2.11. Consider an IVP with homogeneous initial conditions at zero

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} u' + \begin{pmatrix} 1 & -2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} u = \begin{pmatrix} e^x \\ \cos x \\ \sin x \end{pmatrix}. \tag{2.4}$$

We compute the exact solution of the given system similar to Example 2.7 as follows: The matrix differential operator of the IVP (2.4) and forcing function f are

$$T = \begin{pmatrix} 1+\mathtt{D} & -2-\mathtt{D} & 4+2\mathtt{D} \\ 0 & -1+\mathtt{D} & -1-2\mathtt{D} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

where $f_1 = e^x, f_2 = \cos x, f_3 = \sin x$. The canonical form of the given system is

$$\tilde{T} = \begin{pmatrix} 1+\mathtt{D} & -3 & \mathtt{D} \\ 0 & -1+\mathtt{D} & -\mathtt{D} \\ 0 & 0 & \mathtt{D} \end{pmatrix} \quad \text{and} \quad \tilde{f} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix} = \begin{pmatrix} f_1 - 5f_3 + f_2 \\ f_2 - f_3 \\ \mathtt{D}f_3 \end{pmatrix}.$$

Suppose $L = \det(\tilde{T}) = \mathtt{D}^3 - \mathtt{D}$. Then the fundamental system of L is $\{1, e^x, e^{-x}\}$ and the determinant of Wronskian matrix $w = 2$. From Lemma 2.3 or equation (2.13), the right inverse of L is

$$L^{*} = \frac{1}{2}e^{-x}\mathtt{A}e^x + \frac{1}{2}e^x\mathtt{A}e^{-x} - \mathtt{A}.$$

The Green's operator of the given system (2.4) obtained as in Theorem 2.4 or equation (2.15) is

$$G = \begin{pmatrix} e^{-x}\mathtt{A}e^x & \frac{3}{2}e^{-x}\mathtt{A}e^x + \frac{3}{2}e^x\mathtt{A}e^{-x} & \frac{-5}{2}e^{-x}\mathtt{A}e^x + \frac{3}{2}e^x\mathtt{A}e^{-x} \\ 0 & e^x\mathtt{A}e^{-x} & e^x\mathtt{A}e^{-x} \\ 0 & 0 & \mathtt{A} \end{pmatrix},$$

and the Green's function is $G\tilde{f} = u = (u_1, u_2, u_3)^T$, where

$$\begin{aligned}
u_1 &= e^{-x}\int_0^x e^x(f_1+f_2-5f_3)\,dx - \frac{3}{2}e^{-x}\int_0^x e^x(f_2-f_3)\,dx \\
&\quad + \frac{3}{2}e^x\int_0^x e^{-x}(f_2-f_3)\,dx - \frac{5}{2}e^{-x}\int_0^x e^x\mathtt{D}f_3\,dx + \frac{3}{2}e^x\int_0^x e^{-x}\mathtt{D}f_3\,dx \\
u_2 &= e^x\int_0^x e^{-x}(f_2-f_3)\,dx + e^x\int_0^x e^{-x}\mathtt{D}f_3\,dx \\
u_3 &= f_3.
\end{aligned}$$

Now the exact solution of the given IVP (2.4) for $f = (e^x, \cos x, \sin x)^T$ is obtained from the Green's function

$$u = \begin{pmatrix} \frac{5}{4}e^x - \frac{3}{2}e^{-x} - \frac{1}{2}\cos x - \sin x \\ \frac{1}{2}e^x - \frac{1}{2}\cos x + \frac{3}{2}\sin x \\ \sin x \end{pmatrix}. \tag{2.5}$$

Here $Tu = f$ and $\mathtt{E}u = 0$.

The consistent inhomogeneous initial condition is calculated using the algorithm presented in Proposition 2.5 as follows.

The fundamental matrix U of $\tilde{T}$ and U_0 are given by

$$U = \begin{pmatrix} e^x & e^{-x} & 0 \\ \frac{2}{3}e^x & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U_0 = \begin{pmatrix} 1 & 1 & 0 \\ \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From Proposition 2.5, we know that an initial condition $\mathtt{E}u = \alpha$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$, is consistent if $UU_0^{-1}\alpha \in \mathrm{Ker}(T)$. Hence, we have

$$T(UU_0^{-1}\alpha) = \begin{pmatrix} 1+\mathtt{D} & -2-\mathtt{D} & 4+2\mathtt{D} \\ 0 & -1+\mathtt{D} & 1-\mathtt{D} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-x}\alpha_1 + \frac{3}{2}(e^x - e^{-x})\alpha_2 \\ e^x\alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 4\alpha_3 \\ \alpha_3 \\ \alpha_3 \end{pmatrix},$$

which shows that $\alpha_3 = 0$ for $UU_0^{-1}\alpha \in \mathrm{Ker}(T)$. Therefore, the consistent initial condition is $\mathtt{E}u = (\alpha_1, \alpha_2, 0)^T$, $\alpha_1, \alpha_2 \in \mathbb{R}$. The solution u_c of IVP $Tu = 0, \mathtt{E}u = (\alpha_1, \alpha_2, 0)^T$ is obtained from Theorem 2.6 as

$$u_c = UU_0^{-1}\alpha = \begin{pmatrix} e^{-x}\alpha_1 + \frac{3}{2}(e^x - e^{-x})\alpha_2 \\ e^x\alpha_2 \\ 0 \end{pmatrix},$$

and the solution u_p of IVP $Tu = f, \mathtt{E}u = 0$ is obtained from equation (2.5)

$$u_p = \begin{pmatrix} \frac{5}{4}e^x - \frac{3}{2}e^{-x} - \frac{1}{2}\cos x - \sin x \\ \frac{1}{2}e^x - \frac{1}{2}\cos x + \frac{3}{2}\sin x \\ \sin x \end{pmatrix}.$$

Now, the exact solution of IVP for DAEs

$$\begin{pmatrix} 1+\mathtt{D} & -2-\mathtt{D} & 4+2\mathtt{D} \\ 0 & -1+\mathtt{D} & 1-\mathtt{D} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} e^x \\ \cos x \\ \sin x \end{pmatrix},$$
$$\begin{pmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix}, \qquad \alpha_1, \alpha_2 \in \mathbb{R},$$

calculated as in Theorem 2.6 is $u = u_c + u_p$, i.e.

$$u = \begin{pmatrix} \frac{5}{4}e^x - \frac{3}{2}e^{-x} - \frac{1}{2}\cos x - \sin x + e^{-x}\alpha_1 + (-\frac{3}{2}e^{-x} + \frac{3}{2}e^x)\alpha_2 \\ \frac{1}{2}e^x - \frac{1}{2}\cos x + \frac{3}{2}\sin x + e^x\alpha_2 \\ \sin x \end{pmatrix}.$$

Example 2.12. Consider a system of DAEs as given below,

$$\begin{aligned} &u_1' + (\lambda - 2)u_1 = f_1, \\ &u_2' + \lambda u_2 = f_2, \\ &u_1 + u_2 + \lambda u_3 = f_3, \end{aligned}$$

with initial conditions at zero.

Matrix differential operator of the given system is

$$T = \begin{pmatrix} \lambda - 2 + \mathtt{D} & 0 & 1 \\ 0 & \lambda + \mathtt{D} & 0 \\ 1 & 1 & \lambda \end{pmatrix}$$

If $\lambda = 0$, then the canonical form of the given system is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \mathtt{D}u + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u = \begin{pmatrix} \mathtt{D}f_3 - f_2 \\ f_2 \\ -\frac{1}{2}\mathtt{D}^2 f_3 + \frac{1}{2}\mathtt{D}f_1 + \mathtt{D}f_3 + \frac{1}{2}\mathtt{D}f_2 \end{pmatrix}.$$

Exact solution of the given system, if $f = (0, e^x, \sin x)^T$ for simplicity, is

$$u = \begin{pmatrix} 1 + \sin x - e^x \\ e^x - 1 \\ 2 + 2\sin x - e^x - \cos x \end{pmatrix}.$$

If $\lambda = 1$, then the canonical form of the given system is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \mathtt{D}u + \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} u = \begin{pmatrix} f_1 + f_3 \\ f_2 \\ \mathtt{D}f_3 \end{pmatrix}.$$

The exact solution of the given system for $f = (0, e^x, \sin x)^T$ is

$$u = \begin{pmatrix} \frac{1}{6}e^{-x} - \frac{1}{2}e^x + \frac{2}{15}e^{2x} + \frac{1}{5}\cos x + \frac{2}{5}\sin x \\ \frac{1}{2}e^{-x}\left(e^{2x} - 1\right) \\ \frac{1}{3}e^{-x} - \frac{2}{15}e^{2x} - \frac{1}{5}\cos x + \frac{3}{5}\sin x \end{pmatrix}.$$

It is observed that $Tu = f$ and $\mathtt{E}u = 0$ in both cases $\lambda = 0, 1$.

3. Proposed algorithm in Maple

In this section, we discuss Maple implementation, `daeSolve`, for the proposed algorithm by creating different data types with the help of the Maple package `IntDiffOp` implemented by Anja Korporal et al. [11]. The data type `MatrixDiffOperator(A,B)` is created to generate the matrix differential operator T of a given system, where `A` and `B` are the coefficient matrices of a given system. The function `CanonicalMatrixSystem(A,B,f)` produces the canonical form of a given DAEs, where `f` is a given vector forcing function and `ExactSolution(A,B,f)` generates the exact solution of a given system. The maple package `daeSolve` is available at http://www.srinivasaraothota.webs.com/research with examples.

In the following example, sample computations using the Maple implementation of the proposed algorithm is presented.

Example 3.1. Recall Example 2.7, for the sample computations as follows.

```
> with(IntDiffOp):with(daeSolve): # Loading packages.
> A:=Matrix([[1,-1],[0,0]]): # The coefficient matrix of T = AD + B.
> B:=Matrix([[1,-2],[0,1]]): # The coefficient matrix of T = AD + B.
> f:=Matrix([[0], [sin(x)]]): # The vector forcing function such that Tu = f.
> T:=MatrixDiffOperator(A,B); # Matrix differential operator T = AD + B.
```

$$T := \begin{bmatrix} 1 + \mathtt{D} & -2 - \mathtt{D} \\ 0 & 1 \end{bmatrix}$$

```
> CanonicaMatrixDiffSystem(A,B,f);
```

Produces $\tilde{T}$ and $\tilde{f}$ of the canonical form $\tilde{T}u = \tilde{f}$ of the given system.

$$\begin{bmatrix} 1 + \mathtt{D} & -\mathtt{D} \\ 0 & \mathtt{D} \end{bmatrix}, \begin{bmatrix} 2\sin(x) \\ \cos(x) \end{bmatrix}$$

```
> u := ExactSolution(A,B,f);
```

Exact solution of the given IVP $Tu = f, \mathtt{E}u = 0$.

$$u := \begin{bmatrix} -\frac{1}{2}e^{-x}\left(-1 + e^{x}\cos(x) - 3e^{x}\sin(x)\right) \\ \sin(x) \end{bmatrix}$$

```
> ApplyMatrixOperator(T,u);
```

Checking $Tu = f$.

$$\begin{bmatrix} 0 \\ \sin(x) \end{bmatrix}$$

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