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# On the Diameter, Girth and Coloring of the Strong Zero-Divisor Graph of Near-rings

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Abstract. In this paper, we study a directed simple graph $\Gamma_s(N)$ for a near-ring $N$, where the set $V^*(N)$ of vertices is the set of all left $N$-subsets of $N$ with nonzero left annihilators and for any two distinct vertices $I, J \in V^*(N)$, $I$ is adjacent to $J$ if and only if $IJ = 0$. Here, we deal with the diameter, girth and coloring of the graph $\Gamma_s(N)$. Moreover, we prove a sufficient condition for occurrence of a regular element of the near-ring $N$ in the left annihilator of some vertex in the strong zero-divisor graph $\Gamma_s(N)$.

## 1. Introduction

In this paper by a near-ring $N$, we mean a zero symmetric (right) near-ring not necessarily containing 1. A subset $I$ of $N$ is left(right)$N$-subset of $N$ if $NI \subseteq I(IN \subseteq I)$ and $I$ is invariant if it is both left as well as right $N$-subset of $N$. If $I$ is a left $N$-subset of $N$, then the ideal $l(I) = \{x \in N \mid xI = 0\}$ is the left annihilator of $I$. The set $Z_l = \{n \in N \mid \text{for some } x \in N\backslash\{0\}, nx = 0\}$ [12] is the set of left zero-divisors of $N$. We consider the strong zero-divisor graph $\Gamma_s(N)$, where the set $V^*(N)$ of vertices is the set of all left $N$-subsets of $N$ with nonzero left annihilators and for any two distinct vertices $I, J \in V^*(N)$, $I$ is adjacent to $J$ if and only if $IJ = 0$. If $I$ and $J$ are singleton sets, then the strong graph $\Gamma_s(N)$ reduced to the graph $\Gamma(N)$ of $N$ where $x(\neq 0) \in N$ is adjacent to $y(\neq 0) \in N$ if and only if $xy = 0$.

The concept of zero-divisor graph of a commutative ring was first introduced by Beck in [5]. Beck [5] was mainly interested in the coloring of the ring. This notion was redefined in [3] and they proved that such a graph is always connected and its diameter is always less than or equal to 3. Anderson and Mulay in [4] studied diameter and girth of zero-divisor graph of a commutative ring. The notion of zero-divisor graph was extended to a non-commutative ring [1] and various properties of

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diameter and girth were established. In [10], Redmond has generalised the notion of zero-divisor graph. For an ideal $I$ of a commutative ring $R$, Redmond [10] defined an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\}$ where distinct vertices $x$ and $y$ are adjacent if and only if $xy \in I$. Behboodhi [6] studied annihilator ideal graph dealing with the annihilators of ideals of a commutative ring.

In this paper, we study the graph theoretic aspect of a near-ring $N$ which is a less symmetric algebraic structure with + and ., where both operations are non-commutative. An element $d \in N$ is distributive if $d(n_1 + n_2) = dn_1 + dn_2$ for any $n_1, n_2 \in N$ and $N_d$ denotes the set of all distributive elements of $N$. If $N = N_d$ and $(N, +) =< N_d >$, then $N$ is distributive and distributively generated, respectively. For a distributive near-ring $N$ with 1, the graph $\Gamma(N)$ is the zero-divisor graph of a non-commutative ring $N$.

For basic definitions and results related to near-ring, we would like to mention Pilz [9].

Recall that a graph $G$ is connected if there is a path between any two distinct vertices and is complete if every two vertices are adjacent. The distance between two distinct vertices $x$ and $y$ of $G$ is the length of the shortest path from $x$ to $y$ and is denoted by $d(x, y)$. If no such path exists, then $d(x, y) = \infty$. The diameter of the graph $G$ is the $\sup\{d(x, y)|x$ and $y$ are distinct vertices of $G\}$ and is denoted by $diamG$. The girth of $G$ is the length of distance of the shortest cycle in $G$, denoted by $gr(G)$. If no such cycle, then $gr(G) = \infty$.

A left $N$-subset $I$ of $N$ is nilpotent if there exists a positive integer $n$ such that $I^n = 0$ and $I^{n-1} \neq 0$. The near-ring $N$ is strongly semi-prime if it has no nonzero nilpotent invariant subsets. The notion of simple graph excludes the loops which is compatible to the strongly semi-prime character of the near-ring. The graph that we dealt here is a connected one and has diameter 3 or less, the proof of which follows in alike way to that of the theorem 2.3 [3]. It is due to the proposition 1.3.2 [8], if a graph $G$ has a cycle, then the $gr(G)$ is less than $2diamG + 1$. In this paper, we study diameter and girth of the strong zero-divisor graphs of near-rings. Anderson [3] has conjectured that if a zero-divisor graph had a cycle, then its girth was 3 or 4. Haevey Mudd and Jamson gave an elegant proof to the conjecture of Anderson[3]. We establish a sufficient condition for diameter 3 for the graph $\Gamma_s(N)$ of the near-ring $N$. Existence of a cycle in the strong zero-divisor graph deserves exclusive interest. We prove that in a strongly semi-prime near-ring, if $\Gamma_s(N)$ has a cycle with an invariant vertex, then $gr(\Gamma_s(N)) \leq 4$.

Moreover, in this paper, we deal with coloring of $\Gamma_s(N)$. The minimal numbers of colors so that no two adjacent elements of the graph $G$ have same color is the chromatic number of $G$ and is denoted by $\chi(G)$. A clique of $G$ is the maximal connected subgraph of it. The number of vertices in the largest clique in the graph $G$ is the $cliqueG$. Beck, [5] conjectured that $\chi(\Gamma(R)) = clique(\Gamma(R))$. But D.D.Anderson and M.Nasser [2] gave the counter example such as $R = Z_4[[x, y, z]/(x^2 - 2, y^2 - 2, z^2, 2x, 2y, 2z, xy, xz, yz - 2)]$ for which $\chi(\Gamma(R)) = 5$ and $clique(\Gamma(R)) = 4$. Beck [5]has proved characterisation of rings with finite chromatic number and showed that such rings have the ascending chain condition(acc) on annihilators. We here deal with the strong zero-divisor graph $\Gamma_s(N)$ having finite chromatic number. A left $N$-subset(ideal) $I$ of $N$ is essential in $N$ if for any non zero left $N$-subset(ideal) $A$ of $N$, $I \cap A \neq 0$. We prove that chromatic number of such a graph showing alike relation with the numbers of maximal annihilator ideals as well as with that of essential annihilator ideals of the near-ring. Also we deal with the strong zero-divisor graph $\Gamma_s(N)$ having bipartite character, i.e., the set of vertices of $\Gamma_s(N)$ can be decomposed into two disjoint parts such that every edge joints a vertex of one part to that of the other part. We establish that if $\Gamma_s(N)$ is bipartite where $N$ is strongly semi-prime without unity, then $N$ has exactly two invariant subsets $I_1$ and $I_2$ (say) provided $l(I_1)$ and $l(I_2)$ are essential. In addition to it we show that if $\Gamma_s(N)$ is bipartite with nonzero nilpotent invariant subsets in $N$, then $\Gamma_s(N)$ is a star graph.

The following are some examples of strong zero-divisor graphs.

**Example 1.1.**

(1) $\Gamma_s(Z_4) \cong \Gamma(Z_4)$



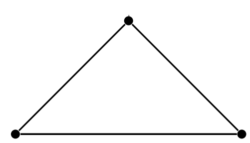
$\Gamma_s(Z_4)$ or $\Gamma(Z_4)$

(2) $\Gamma_s(Z_2 \times Z_2) \cong \Gamma(Z_2 \times Z_2) \cong \Gamma_s(Z_6)$ $(Z_2 \times Z_2 \ncong Z_6)$



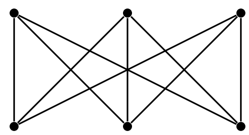
$\Gamma_s(Z_2 \times Z_2)$, $\Gamma(Z_2 \times Z_2)$ or $\Gamma_s(Z_6)$

(3) $\Gamma_s(\frac{Z_3[x]}{\langle x^2 \rangle}) \cong \Gamma_s(\frac{Z_2[x,y]}{\langle x^2,y^2,xy \rangle})$ but $\frac{Z_3[x]}{x^2} \ncong \frac{Z_{2[x,y]}}{\langle x^2,y^2,xy \rangle}$



$\Gamma_s(\frac{Z_3[x]}{\langle x^2 \rangle})$ or $\Gamma_s(\frac{Z_2[x,y]}{\langle x^2,y^2,xy \rangle})$

(4) $\Gamma_s(\frac{Z_4[x]}{\langle x^2\rangle}) \ncong \Gamma_s(\frac{Z_2[x,y]}{\langle x^2,xy,y^2\rangle})$ $(\frac{Z_4[x]}{\langle x^2\rangle} \cong \frac{Z_2[x,y]}{\langle x^2,xy,y^2\rangle})$



$\Gamma_s(\frac{Z_4[x]}{\langle x^2\rangle})$

## 2. Main Results

In this section, we present results regarding diameter and girths of $\Gamma_s(N)$ in contrast to $\Gamma(N)$ in some cases. We note that the vertex 0 is adjacent to every other vertices which we exclude here for obvious reason.

A vertex $I$ of $\Gamma_s(N)$ is an invariant vertex if it is an invariant $N$ subset of the near-ring $N$. The right annihilator of a left $N$-subset $I$ of $N$ is $r(I) = \{x \in N \mid Ix = 0\}$ which is a right $N$-subset of $N$, need not coincide to $l(I)$ in general. However in a strongly semi-prime near-ring $N$ for an invariant subset $I$, $Il(I) = 0$ as $(Il(I))^2 = I(l(I)I)l(I) = 0$ giving thereby $l(I) \subseteq r(I)$. Similarly $r(I) \subseteq l(I)$. Thus we state the following lemma.

**Lemma 2.1.** *Let $N$ be a strongly semi-prime near-ring. Then for an invariant subset $I$ of $N$, $l(I) = r(I)$.*

For a subset $I$ of $N$, $l(I) \neq 0$ may not imply $l(I + J) \neq 0$ for any subset $J$ of $N$. Below we present when it occurs.

**Lemma 2.2.** *Let $N$ be a near-ring such that the left annihilators are distributively generated. If $I$ be a left $N$-subset with $l(I) \neq 0$ and $J \subseteq l(I)$ is a nilpotent left $N$-subset of $N$, then $l(I + J) \neq 0$.*

*Proof.* Since $l(I) \neq 0$, there exists an $x(\neq 0) \in N$ such that $xI = 0$. Now $J$ is nilpotent gives a positive integer $m$ such that $xJ^m = 0$ and $xJ^{m-1} \neq 0$. Again $xJ^{m-1}J = xJ^m = 0$ and $xJ^{m-1}I = xJ^{m-2}JI = 0$. Thus $xJ^{m-1}(I + J) = 0$ giving thereby $xJ^{m-1} \subseteq l(I + J)$. Thus $l(I + J) \neq 0$. □

Thus in this lemma, we see that the nilpotency of $J \subseteq l(I)$ leads us to $l(I+J) \neq 0$. In the next, we present diameter of the strong zero-divisor graph $\Gamma_s(N)$, where $N$ is a strongly semi-prime near-ring.

**Theorem 2.3.** *Let $N$ be a strongly semi-prime near-ring such that the left annihilators are distributively generated. If there exists a nilpotent vertex $J$ and an invariant vertex $I$ such that $l(I + J) = 0$, then $diam(\Gamma_s(N)) = 3$.*

*Proof.* We give the proof in two steps such as

(i) Step1:

Suppose $d(I, J) = 2$. Let $M \in V^*(N)$ be such that, $I \longrightarrow M \longrightarrow J$ is a directed path. Then $IM = 0$ and $MJ = 0$ which gives that $M \in r(I) = l(I)$. Now, $M(I + J) = 0$ gives that $M(\neq 0) \subseteq l(I + J)$ . Thus $l(I + J) \neq 0$, a contradiction.

(ii) Step2:

CaseI: If $IJ \neq 0$, consider $M = l(I)$, $N = l(J)$. Claim: $I \longrightarrow M = l(I) \longrightarrow N = l(J) \longrightarrow J$ is a directed path. It is enough to show that $l(I)l(J) = 0$. Suppose there exists an $x \in l(I), y \in l(J)$ such that $xy \neq 0$. Now $x \in l(I) = r(I)$ gives $Ixy = 0$. Thus $xy \in r(I) = l(I)$ gives $xyI = 0$. Again $y \in l(J)$ gives $xyJ = 0$. Now we get $xy(I + J) = 0$ which gives $(0 \neq)xy \in l(I + J)$, a contradiction.

CaseII: If $IJ = 0$, then $(I+J)^2 \subseteq I^2+J^2$. And $l(I+J)^2 = 0$, as $x \in l(I+J)^2$ gives $x(I+J) \subseteq l(I+J)$ giving thereby $x \in l(I+J) = 0$. Since $J$ is nilpotent, $qJ$ is also so where $q \in l(I)$ with $qJ^2 \neq 0$. Now $qJ \subseteq l(I)$ gives $l(I + qJ) \neq 0$ [Lemma 2.2]. Again $I + qJ \neq J$, otherwise $I \subseteq J$ implies $l(I + J) = l(J) \neq 0$, a contradiction. Hence $I + qJ$,$J$ are distinct and $I + J = I + qJ + J$ which gives $l(I + qJ + J) = 0$ and $(I + qJ)J \neq 0$. Hence $d(I + qJ, J) = 3$[caseI]. □

**Theorem 2.4.** *Let $N$ be a strongly semi-prime near-ring such that the left annihilators are distributively generated. If $I$ is an invariant $N$-subset of $N$ containing a non nilpotent subset $I_1$ with maximal left annihilators, then $d(I, J) \neq 2$ for any $J \subseteq l(I_1)$ with $l(I_1) \cap l(J) = 0$.*

*Proof.* Let $y \in l(I_1 + J)$, $y = \sum \pm d_i$ where $l(I_1 + J) =< S >$, $d_i \in S$, a set of distributive elements of $N$. Now $d_i(I_1 + J) = 0$ gives $(d_iI_1 + d_iJ)i_1 = 0$ for each $i_1 \in I_1$. Thus $d_iI_1i_1 = 0$ as $J \subseteq l(I_1)$ giving thereby $d_i \in l(I_1) = l(I_1i_1)$, since $l(I_1)$ is maximal. Now we get $d_iJ = 0$ which gives $d_i \in l(I_1) \cap l(J)$ for each $i$. Thus $y = 0$ which gives $l(I_1 + J) = 0$ giving thereby $l(I + J) = 0$. Hence $d(I, J) \neq 2$. [Theorem 2.3(i)] □

**Theorem 2.5.** *Let $P_1 = l(I_1)$ and $P_2 = l(I_2)$ be two prime ideals of $N$ such that $P_1 \cap P_2 = 0$, where $I_1$ and $I_2$ are invariant subsets of $N$. Then $I_1I_2 = (0) = I_2I_1$.*

*Proof.* For $I_1I_2 \neq 0$, we get $I_1 \nsubseteq l(I_2) = P_2$ and $I_2 \nsubseteq r(I_1) = l(I_1) = P_1$. Now $P_1I_1 \subseteq P_2$ gives $P_1 \subseteq P_2$ as $I_1 \nsubseteq P_2 = l(I_2)$ giving thereby $P_1 \cap P_2 = P_1 \neq 0$, a contradiction. Similarly, $I_2I_1 = 0$ □

**Definition 2.6.** Invariant associated of a near-ring $N$ denoted by $I - Ass(N)$ is the collection of $l(I_i)$'s, where each $l(I_i)$ is a prime ideal with invariant $N$-subset $I_i$ such that $l(I_i) \cap l(I_j) = 0$ for $i \neq j$.

**Corollary 2.7.** *If in a strongly semi-prime near-ring $N$, $|(I - AssN)| \geq 3$, then $gr(\Gamma_s(N)) = 3$.*

*Proof.* Let $I - Ass(N) = \{P_1, P_2, P_3\}$, then $P_1 = l(I_1), P_2 = l(I_2)$ and $P_3 = l(I_3)$ for some invariant subsets $I_1, I_2$ and $I_3$ respectively. Then $I_1I_2 = 0, I_2I_3 = 0$ and

$I_3I_1 = 0$ [theorem 2.5]. Hence $I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow I_1$ is a cycle of length 3. Thus $gr(\Gamma_s(N)) = 3$. □

**Theorem 2.8.** *If* $|I - AssN| \geq 5$, *then* $\Gamma_s(N)$ *is not a planner graph.*

*Proof.* Let $I - AssN = \{P_1, P_2, P_3, P_4, P_5\}$ where $P_i = l(I_i)$(say), $1 \leq i \leq 5$. Here $I_iI_j = 0$ for $i \neq j$ [theorem 2.5]. Thus the graph $\Gamma_s(N)$ contains Kuratowski's first graph. Hence $\Gamma_s(N)$ is not planner. □

Next we determine the girth of the graph $\Gamma_s(N)$ of a stongly semi-prime near-ring $N$ if it has a cycle with at least one invariant vertex.

**Theorem 2.9.** *Let $N$ be a strongly semi-prime near-ring. If $\Gamma_s(N)$ contains a cycle with an invariant vertex in it, then $gr(\Gamma_s(N)) \leq 4$.*

*Proof.* Assume $n = gr(\Gamma_s(N))$ is 5, 6 or 7. Let $I_1 \longrightarrow I_2 \longrightarrow I_3 ..... \longrightarrow I_n \longrightarrow I_1$...(i) be a cycle with minimal length $n$. Let $I_i$ be an invariant vertex. Now consider the subgraph $\Gamma_s^{/}(N)$ of $\Gamma_s(N)$ spanned by the vertices $I_1, I_2, ...., I_iI_{i+2}$. If $I_iI_{i+2} \neq I_k$ for any $k, 1 \leq k \leq n$, then $I_{i-1} \longrightarrow I_i \longrightarrow I_{i+1} \longrightarrow I_iI_{i+2} \longrightarrow I_{i-1}, (i \geq 2)$ is a cycle of length 4. Let $I_iI_{i+2} = I_k$ for some $k$. Now we show the following.

(i) $I_iI_{i+2} \neq I_{i+1}$. If $I_iI_{i+2} = I_{i+1}$, then $(I_iI_{i+2})I_{i-1} = I_{i+1}I_{i-1}$. Now $I_{i+1}I_{i-1} = (I_iI_{i+2})I_{i-1} \subseteq I_iI_{i-1} = 0$, which gives $I_{i+1}I_{i-1} = 0$. Thus $I_{i-1} \longrightarrow I_i \longrightarrow I_{i+1} \longrightarrow I_{i-1}$ is a cyclic, a contradiction to (i).

(ii) $I_iI_{i+2} \neq I_{i-1}$. For otherwise, $I_{i+1}(I_iI_{i+2}) = I_{i+1}I_{i-1}$ which gives $I_{i+1}I_{i-1} = (I_{i+1}I_i)I_{i+2} = 0$. Thus $I_{i-1} \longrightarrow I_i \longrightarrow I_{i+1} \longrightarrow I_{i-1}$ is a cycle, a contradiction to (i).

(iii) $I_iI_{i+2} \neq I_{i+3}$. If $I_iI_{i+2} = I_{i+3}$, then we get $I_{i+3}I_{i+1} = (I_iI_{i+2})I_{i+1} \subseteq I_iI_{i-1} = 0$, gives the cycle $I_{i+1} \longrightarrow I_{i+2} \longrightarrow I_{i+3} \longrightarrow I_{i+1}$, a contradiction.

Now $I_iI_{i+2}$ is adjacent to three distinct vertices $I_{i-1}, I_{i+1}$ and $I_{i+3}$. Thus there exists an extra edge in $\Gamma_s^{/}(N)$ which is not in the original cycle. Hence there must exist a smaller cycle $\Gamma_s^{/}(N)$, a contradiction. □

Now we present coloring of the strong zero-divisor graph $\Gamma_s(N)$ of $N$.

**Theorem 2.10.** *Let $N$ be a strongly semi-prime near-ring. If $N$ has $k$ number of maximal ideals of the form $l(I_i)$ where $I_i$'s are invariant subsets such that $l(I_i) \cap l(I_j) = 0$ for $i \neq j, 1 \leq i, j \leq k$, then $\chi(\Gamma_s(N)) \leq k + 1$.*

*Proof.* First we give $k$ distinct colors to $I_i$'s and an extra color to 0. Here $I_iI_j = 0$ for $i \neq j$ [Theorem 2.6]. Now we color the invariant vertices. If $I(\neq 0)$ be an arbitrary invariant vertex, we give to $I$ the color which is given to $I_n^{th}$ vertex, where $n$ is the minimal $\{i | l(I) \nsubseteq l(I_i)\}$. Let $I$ and $J$ be two invariant vertices such that same color of $I_k$ is given to them. Then $l(I) \nsubseteq l(I_k)$ and $l(J) \nsubseteq l(I_k)$. If $IJ = 0$, then $I \subseteq l(J) \nsubseteq l(I_k)$ and $J \subseteq r(I) = l(I) \nsubseteq l(I_k)$ which leads to $IJ \nsubseteq l(I_k)$, a contradiction. Next we show that these $k + 1$ colors are enough to color the whole graph. Let $I(\neq 0)$ be a left $N$-subset of $N$ and $I \in V^*(N)$.

Consider $Il(J)(\neq 0)$ with some $J \in V^*(N)$. If $Il(J) = 0$ for any $J \in V^*(N)$, then $Il(I_n) = 0$ for all n, $1 \leq n \leq k$. Thus $l(I_n) \subseteq l(I)$ gives $l(I_{_n}) = l(I)$ for all $n$, a contradiction. Now we give the color to $I$ which is given to the invariant vertex $Il(J)$. Here $IIl(J) \neq 0$, for otherwise $I \subseteq l(Il(J)) = r(Il(J))$ which gives $Il(J)I = 0$, giving thereby $(Il(J))^2 = 0$, a contradiction. Suppose $I$ and $I^/$ has the color of $I_k$(say). Then we get some $J, J^/ \in V^*(N)$ such that $Il(J)$ and $I^/l(J^/)$ are given the color of $I_k$. Now $l(Il(J)) \nsubseteq l(I_k)$ and $l(I^/l(J^{/})) \nsubseteq l(I_k)$. If $(Il(J))(I^/l(J^/)) = 0$, then $Il(J) \subseteq l(I^/l(J^/)) \nsubseteq l(I_k)$ and $I^/l(J^/) \subseteq r(Il(J)) = l(Il(J)) \nsubseteq l(I_k)$ which implies that $(Il(J))(I^/l(J^/)) \nsubseteq l(I_k)$, a contradiction. Now we show that $I$ and $I^/$ are not adjacent. If $II^/ = 0$, then $II^/l(J^/) = 0$ gives $(I^/l(J^/))I = 0$. Thus $(I^/l(J^/))(Il(J)) = 0$ gives $(Il(J))^2 = (I^/l(J^/))^2 = 0$, a contradiction. □

**Example 2.11.** Consider $Z_6 = \{0, 1, 2, 3, 4, 5\}$ which is a near-ring with respect to the tables given below. The only left $N$ subsets are $I_1 = \{0, 3\}$, $I_2 = \{0, 2, 4\}$ and $I_3 = \{0, 2, 3.4\}$ which are invariant also and $l(I_1) = I_2$ and $l(I_2) = I_1$ are two maximal ideals of the annihilator ideal form. Here the chromatic number $\chi(\Gamma_s(Z_6))$ is $2 + 1 = 3$, i.e., $\chi(\Gamma_s(Z_6))$ is equal to $p + 1$, where $p$ is the number of maximal ideals of the form of left annihilator.

**Table**

| + | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 | 5 |

| · | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 2 | 4 | 0 | 2 | 4 |
| 3 | 0 | 3 | 0 | 3 | 0 | 3 |
| 4 | 0 | 4 | 2 | 0 | 4 | 2 |
| 5 | 0 | 5 | 4 | 3 | 2 | 1 |

In the results below, we deal with the essentiality of annihilator ideals in a near-ring $N$ to determine the chromatic number of $\Gamma_s(N)$.

**Theorem 2.12.** *Let $N$ be a near-ring with unity, then the following two are equivalent.*

(i) *If for a left $N$-subset $I$ of $N$, $l(I)$ is essential, then $I = 0$.*

(ii) *$N$ is strongly semi-prime.*

*Proof.*

(a) $(i) \Rightarrow (ii)$ Suppose $J$ is an invariant $N$-subset of $N$ such that $J^2 = 0$. Let $A$ be a nonzero ideal of $N$. If $AJ = 0$ then $A = A \cap l(J) \neq 0$. If $AJ \neq 0$, then $AJ(\neq 0) \subseteq A \cap l(J)$. Thus in either cases $l(J)$ is essential. Hence $J = 0$.

(b) $(ii) \Rightarrow (i)$ Let $I$ be a left $N$-subset such that $l(I)$ is essential. Let $J = l(I) \cap IN$. Now $J^2 \subseteq l(I)IN = 0$. Thus $J = 0$, i.e., $l(I) \cap IN = 0$ which gives $IN = 0$ as $l(I)$ is essential. Hence $I = 0$. □

**Example 2.13.** Consider the ring $Z_6 = \{0, 1, 2, 3, 4, 5\}$ which is strongly semi-prime with unity. Here $I_1 = l(I_2) = \{0, 3\}$ and $I_2 = l(I_1) = \{0, 2, 4\}$ are the only nonzero ideals and $Z_6 = Ann(0)$ is the only essential ideal.

**Example 2.14.** $Z_4 = \{0, 1, 2, 3\}$ is a ring with unity. Here $Z_4$ is not strongly semi-prime as for $I = \{0, 2\}$, $I^2 = 0$ and $l(I)$ is an essential ideal of $Z_4$

**Theorem 2.15**: Let $N$ be a near-ring and $x \in N$ be such that every vertex $v \in \Gamma(N)$ is adjacent to $x$. Then $l(x)$ is an essential ideal of $N$.

**Theorem 2.16.** *Let $N$ be a strongly semi-prime near-ring. If $\Gamma(N)$ has no infinite clique, then the near-ring $N$ satisfies the acc on essential left $N$-subsets.*

*Proof.* Let $I_1 < I_2 < I_3 <$ ......be an ascending chain for left $N$-subsets, where each $I_i$'s are essential in $N$. Suppose $I_i < I_{i+1}$. Now $I_i \cap l(I_i) < I_{i+1} \cap l(I_i)$. Here $I_i \cap l(I_i) \neq 0$ and $I_{i+1} \cap l(I_i) \neq 0$. Also $I_i \cap l(I_i) \neq I_{i+1} \cap l(I_i)$ for otherwise $(I_i \cap l(I_i))^2 = (I_{i+1} \cap l(I_i))(I_i \cap l(I_i)) \subseteq l(I_i)I_i = 0$, a contradiction. Now consider an element $x_n \in I_n \cap l(I_{n-1})$ such that $x_n \notin I_{n-1} \cap l(I_{n-1})$. Here for $i \neq j$ (suppose $i > j$), $x_i x_j \in (I_i \cap l(I_{i-1}))(I_j \cap l(I_{j-1})) \subseteq l(I_{i-1})I_j = 0$. Thus we get an infinite clique in $N$, a contradiction. □

**Theorem 2.17.** *Let $N$ be a strongly semi-prime near-ring without unity. If $\Gamma_s(N)$ has no infinite clique, then $N$ satisfies the acc on invariant subsets having essential left annihilators.*

*Proof.* Let $I_1 < I_2 < I_3$.... be an ascending chain of invariant subsets with essential left annihilators. Suppose $I_i \subsetneqq I_{i+1}$. Let $x_{i+1}(\neq 0) \in I_{i+1} \setminus I_i$. Now consider $J_{i+1} = l(I_{i+1}) \cap \langle x_{i+1} \rangle \neq 0$, where $\langle x_{i+1} \rangle$ is the ideal generated by $x_{i+1}$. Here $J_i J_j = 0$ for $i < j$, a contradiction. □

**Theorem 2.18.** *Let $N$ be a strongly semi-prime near-ring without unity and $l(I_1), l(I_2), ...., l(I_n)$ are the only essential $N$-subsets of $N$ with each $I_i$ is an ideal. Then $\chi(\Gamma_s(N)) \leq n + 1$.*

*Proof.* We give $n$ distinct colors to $l(I_i)$'s. Here $I_i I_{i+1} = 0$ since for otherwise $(l(I_i) \cap I_i I_{i+1}) \neq 0$. Now $(l(I_i) \cap I_i I_{i+1})^2 \subseteq l(I_i) I_i I_{i+1} = 0$ which gives $l(I_i) \cap I_i I_{i+1} = 0$, a contradiction. Now let $I$ be an arbitrary vertex.

(i) CaseI: If $I_i \subseteq I$ for some $i$, then give the color of $I_k$ to $I$ if $k$ is the $max\{i | I_i \subset I\}$. Here $I$ and $I_k$ are not adjacent since for otherwise $I \subseteq l(I_k)$ together with $I_k \subseteq I$ gives that $(I_k)^2 = 0$, a contradiction.

(ii) CaseII: If $I_i \nsubseteq I$ for any $i$, then there exists an $x \in I_i$ such that $x \notin I$. Now consider the ideal generated by $x$ denoted $< x >$ which is clearly non zero. Thus $l(I_i) \cap < x > \neq 0$. But $(l(I_i) \cap < x >)^2 \subseteq l(I_i) I_i = 0$, a contradiction.

Suppose two distinct vertices $I$ and $J$ are given the same color of $I_k$(say). Here $IJ \neq 0$ for otherwise $I \subseteq l(J)$ which leads $I_k \subset I \subseteq l(J)$. Thus we get $I_k^2 = 0$ as $I_k \subset J$, a contradiction. □

Now we mention the following notes:

(i) Note 1: In a near-ring $N$, $\chi(\Gamma_s(N)) = 2$ if and only if for any two nonzero $I, J \in V^*(N)$, $IJ \neq 0$ whenever $I \neq 0$, $J \neq 0$. For, suppose there exists $I \neq 0$ and $J \neq 0$ such that $IJ = 0$. Then $\{0, I, J\}$ is a clique. Thus clique$(\Gamma_s(n)) > \chi(\Gamma_s(N))$, a contradiction.

(ii) Note 2: In a strongly semi-prime near-ring without unity, every essential ideal of the form $l(I_i)$ with invariant $I_i$ is maximal. For suppose $l(I_i)$ is not maximal, there exists a proper ideal $K$ of $N$ such that $l(I_i) \subset K \subset N$. Now consider the ideal $J$ generated by $I_i x(\neq 0)$ for some $x(\neq 0) \in K$. Here $l(I_i) \cap J \neq 0$ but $(l(I_i) \cap J)^2 = 0$, a contradiction.

**Example 2.19.** Consider the set $Z_{(p^\infty)}$ of all rational numbers of the form $\frac{m}{p^k}$ such that $0 \leq \frac{m}{p^k} < 1$, where $p$ is a fixed prime number, $n$ runs through all nonnegative integers. Then $Z(p^\infty)$ is a ring with respect to addition modulo 1 and multiplication defined as $ab = 0$ for all $a, b \in Z(p^\infty)$. It is to be noted that each subgroup of $Z(p^\infty)$ is an ideal of it and the only proper ideals of $Z(p^\infty)$ are of the form $I_{k-1} = \{0, \frac{1}{p^{k-1}}, \frac{2}{p^{k-2}}, ...., \frac{p^{k-1}-1}{p^k-1}\}$ for each positive integer $k$. Thus the ideals are in a chain $0 < I_1 < I_2 < .....$ and each $I_i$'s are essential $Z_{p^\infty}$ is a reduced ring without unity. But here $l(I_{k-1}) = 0$ for all $k$ which are not essential. Here $I_i I_j \neq 0$ for any $i, j$ and $\chi(\Gamma_s(Z_{p^\infty})) = 2$.

**Example 2.20.** Consider the set $M(N) = \begin{pmatrix} Z_2 & N \\ 0 & Z_2 \end{pmatrix}$ which is the set of elements of the form $\{\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & n \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\}$, where $n \in N$. Here $M(N)$ is a near-ring with respect to ordinary addition and multiplication($\overline{x}n = xn, \overline{x} \in Z_2$) with unity which is not strongly semi-prime as

$\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. If $N$ is not finite, then $M(N)$ has infinite invariant sets $I_i (i = 1, 2, 3, ...)$ such that $l(I_i) = \{\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mid n \in N\}$ is essential and $\chi(\Gamma_s(M(N))) = \infty$.

**Theorem 2.21.** *Let $\Gamma_s(N)$ be a bipartite graph with two non-empty parts $V_1$ and $V_2$. Then*

(i) *If $N$ is strongly semi-prime without unity, then $N$ has exactly two invariant $N$ subsets $I_1$ and $I_2$, where $l(I_1)$ and $l(I_2)$ are essential.*

(ii) *If $N$ is not strongly semi-prime, then $\Gamma_s(N)$ is a star graph with more than one vertices.*

*Proof.*

(i) Let $I_1, I_2$ and $I_3$ be three distinct invariant $N$-subsets of $N$ such that $l(I_i)$'s are essential. Now $J_1 = l(I_1) \cap I_2 \neq 0$, $J_2 = l(I_3) \cap I_2 \neq 0$ and $J_3 = l(I_2) \cap I_3 \neq 0$. Here $J_3 \neq J_1$ for otherwise $(l(I_2) \cap I_3)^2 = (l(I_2) \cap I_3)(l(I_1) \cap I_2) = 0$, a contradiction. Thus $(l(I_2) \cap I_3)(l(I_1) \cap I_2) = 0$. Similarly $(l(I_1) \cap I_2)(l(I_3) \cap I_1) = 0$ and $(l(I_3) \cap I_1)(l(I_2) \cap I_3) = 0$. Thus $J_1 \longrightarrow J_2 \longrightarrow J_3 \longrightarrow J_1$ is a cycle, a contradiction.

(ii) Suppose $N$ is not strongly semi-prime and let $I \neq 0$ be an invariant $N$ subset such that $I^2 = 0$. Assume that $I \in V_1$. We show that $V_1 = \{I\}$. Here either $l(I)$ is not essential or $I$ is minimal. Suppose $l(I)$ is essential and there exists an $I_1(\neq 0)$ such that $I_1 \subsetneq I$. Now $l(I) \cap I_1 \neq 0$ and $(l(I) \cap I_1)I = 0$ gives $l(I) \cap I_1 \in V_2$ and $II_1 = 0$ gives $I_1 \in V_2$. But $(l(I) \cap I_1)I = 0$, a contradiction. Now we consider the following cases.

(a) CaseI: If $l(I)$ is essential. Suppose there exists a $P \in V_1 \setminus \{I\}$. If $IP = 0$ , then $P \in V_2$, a contradiction. Hence $IP \neq 0$. Since $\Gamma_s(N)$ is connected, there exists a $K \in V_2$ such that $PK = 0$. Now $l(I) \cap IP \neq 0$ and $I(l(I) \cap IP) = 0$ gives that $l(I) \cap IP \in V_2$. But $(l(I) \cap IP)K = 0$, a contradiction.

(b) CaseII: If $l(I)$ is not essential. Now suppose $I$ is minimal. Then $I \cap P = P$ which gives that $(I \cap P)I = I^2 = 0$. Thus $I \cap P \in V_2$. But $(I \cap P)K = 0$, a contradiction. If $I$ is not minimal, then $IP \subset I$ which gives $(IP \cap I)K = (IP)K = 0$. Thus $IP \cap I \in V_2$, a contradiction to $(IP \cap I)K = 0$. □

**Theorem 2.22.** ([7]) *A strongly semi-prime near-ring $N$ satisfying the acc on left annihilators has no nonzero nil left $N$-subsets in it.*

In the example 2.11, we see that every essential left ideal is essential as left $N$-subgroup also. We call such a near-ring a near-ring with total essential character. Moreover for near-ring with the a.c.c on annihilators, we would like to refer [11].

**Theorem 2.23.** ([7]) *If a strongly semi-prime near-ring $N$ is with total essential character, then $N$ satisfies the dcc (descending chain condition) on left annihilators.*

**Theorem 2.24***: Let $N$ be a strongly semi-prime near -ring with the acc on left annihilators satisfying total essential character and the left annihilators are distributively generated. Let $I$ be a vertex of $\Gamma_s(N)$ such that every other vertex is adjacent to $I$. Then $l(I)$ contains a left non-zero divisor.*

*Proof.* Here $l(I)$ is essential. Consider $I_1(\neq 0) \subseteq l(I)$ such that $I_1$ is non nilpotent and $l(I_1)$ is as large as possible. If $l(I_1) = 0$, we stop. If not, there exists a left $N$-subset $X(\neq 0)$ such that $XI_1 = 0$. But $X \cap l(I) \neq 0$ as $l(I)$ is essential. Consider $a_1(\neq 0) \in X \cap l(I)$ such that $l(Na_1)$ is as large as possible. Now $Na_1 \subseteq$

$X \cap l(I)$. If $l(Na_1) = 0$, we stop. Suppose $l(Na_1) \neq 0$. Now $Na_1 I_1 \subseteq l(I)$ and $Na_1 + I_1 \subseteq l(I)$. If $l(Na_1 + I) = 0$, then we stop. If not, then $l(Na_1 + I_1) \cap l(I) \neq 0$. Again $l(Na_1 + I_1) = l(Na_1) \cap l(I_1)$[theorem 2.4] gives $l(Na_1 + I_1) \cap l(I) = l(Na_1) \cap l(I_1) \cap l(I) \neq 0$. Now we consider $a_2(\neq 0) \in l(Na_1) \cap l(I_1) \cap l(I)$ with $a_2$ non nilpotent and $l(Na_2)$ is as large as possible. If $l(Na_2 + Na_1 + I_1) = 0$, we stop. If not, proceeding in the same way, we get $l(I_1) \supseteq l(I_1) \cap l(Na_1) \supseteq l(I_1) \cap l(Na_1) \cap l(Na_2) \supseteq .......$ which is stationary. Hence we get a positive integer $t$ such that $l(I_1) \cap l(Na_1) \cap .... \cap l(Na_t) = l(I_1) \cap l(Na_1) \cap .... \cap l(Na_{t+1})$. Now $l(I_1)+l(Na_1)+....+l(Na_t) = l(I_1)+l(Na_1)+....+l(Na_{t+1})$=$l(I_1+Na_1+....+Na_t) \cap l(Na_{t+1}) \subseteq l(Na_{t+1})$. Now $Na_{t+1} \subseteq l(I_1+Na_1+....+Na_t)$ gives $Na_{t+1} \subseteq l(Na_{t+1})$ which gives $(Na_{t+1})^2 = 0$, a contradiction. Thus $l(I_1 + Na_1 + .... + Na_t) \cap l(I) = 0$ giving thereby $l(I_1 + Na_1 + .... + Na_t) = 0$ as $l(I)$ is essential. □

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