KYUNGPOOK Math. J. 56(2016), 1069-1083
http://dx.doi.org/10.5666/KMJ.2016.56.4.1069
pISSN 1225-6951 eISSN 0454-8124

Some Characterizations of Modules via Essentially Small Submodules

Le Van Thuyet
Department of Mathematics, Hue University, 3 Le Loi, Hue city, Vietnam
e-mail: lvthuyet@hueuni.edu.vn

Phan Hong Tin*
Department of Mathematics, Hue University's College of Education, 32 Le Loi, Hue city, Vietnam
e-mail: phtin@hueic.edu.vn

Abstract. In this paper, the structure of e-local modules and classes of modules via essentially small are investigated. We show that the following conditions are equivalent for a module M:

(1) M is e-local;

(2) $\mathrm{Rad}_e(M)$ is a maximal submodule of M and every proper essential submodule of M is contained in a maximal submodule;

(3) M has a unique essential maximal submodule and every proper essential submodule of M is contained in a maximal submodule.

1. Introduction

Throughout this paper, R will be an associative ring with identity and all modules are unitary R-module. We write M_R (resp., ${}_RM$) to indicate that M is a right (resp., left) R-module. All modules are right unital unless stated otherwise. If N is a submodule of M, we denote by $N \leq M$. Moreover, we write $N \leq^e M$, $N \leq^{\oplus} M$ and $N \ll M$ to indicate that N is an essential submodule, a direct summand and a small submodule of M, respectively. If X is a subset of a right R-module M, the

* Corresponding Author.
Received March 12, 2015; accepted May 11, 2016.
2010 Mathematics Subject Classification: 16D10, 16D40, 16D60, 16D70.
Key words and phrases: e-local module; e-noncosingular module; $\mathcal{T}$-e-noncosingular module.
This work was supported by National Foundation for Science and Technology Development of Vietnam and Vietnam Institute for Advanced Study in Mathematics.

right annihilator of X in R is denoted by $r_R(X)$ or simply $r(X)$ if no confusion appears.

Recently, some authors have studied generalizations of semiperfect rings and perfect rings via projectivity of modules and small submodules of modules see [7, 11, 16, 18, 19]... Following [19], a submodule N of M is called δ-small in M (denote $N \ll_\delta$ M) if $M = N + L$ and M/L singular then $L = M$. In [7], the author extends the definition of lifting and supplemented modules to what he calls δ-lifting and δ-supplemented. This extension is made by replacing in the definitions the concept of small submodule by the corresponding one of δ-small submodule. Most properties of lifting and supplemented modules are adapted to this new setting.

A submodule N of M is called e-small (essentially small) in M, denote $N \ll_e M$, if $M = N + L$ and $L \leq^e M$ then $L = M$ ([20]). In [12], the authors were introduced a class of all e-lifting modules. A module M is called e-lifting if for any $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll_e M$. Some homology properties of e-lifting modules class were obtained. It proved that $\mathrm{Rad}_e(M)$ is a Noetherian (Artinian) module if only if M has ACC(reps. DCC) on e-small submodules.

In [19], the author denoted

$$\delta(M) = Rej_M(\wp) = \bigcap\{N \leq M | M/N \in \wp\} = \sum\{N \leq M | N \ll_\delta M\}$$

where $\wp$ is the class of all singular simple modules. Similarly, there is the concept of modules via e-small submodules ([20]). Call $\wp_0$ the class of all essential maximal submodules of M.

$$\mathrm{Rad}_e(M) = \bigcap\{N \leq M \mid N \in \wp_0\} = \sum\{N \leq M \mid N \ll_e M\}.$$

Note that $Rad(M) \leq \delta(M) \leq \mathrm{Rad}_e(M)$. If $\delta(M) \ll_\delta M$ and $\delta(M)$ is a maximal submodule of M, M is called a δ-local module ([4]). In [15], the author studied δ-local modules and established some properties of finitely generated amply δ-supplemented modules. A necessary and sufficient condition is provided for a module to be δ-local module. In this paper, we continue studying class of e-supplemented modules and introduce the concept of e-local modules. A module M is called e-local if $\mathrm{Rad}_e(M)$ is a maximal submodule of M and $\mathrm{Rad}_e(M) \ll_e M$. We show that $M = N \oplus K$ is an e-local module if and only if either N is an e-local module and K is semisimple, or K is an e-local module and N is semisimple.

Recall that the singular submodule of a module M is the set

$$Z(M) = \{m \in M \mid r(m) \leq^e R\}.$$

In [6], the author introduced the notions of singular modules and nonsigular modules. A module M is called singular (resp., nonsingular) if $Z(M) = M$ (resp., $Z(M) = 0$). In [13], the author defined the notion of dual singular submodules, that is $\overline{Z}(M) = \bigcap\{\mathrm{Ker}\, g | g : M \to N, N \text{ is a small module}\}$. M is called cosingular (resp., noncosingular) module if $\overline{Z}(M) = 0$ (resp., $\overline{Z}(M) = M$). A generalization of cosingular and noncosingular, which is δ-cosingular and δ-noncosingular (respectively) were introduced and studied in [10].

In [8], the authors introduce the notion of $\mathcal{T}$-noncosingular modules as the notion of dual $\mathcal{K}$-nonsingular modules and generalizations of noncosingular modules. It turns out that some results about $\mathcal{K}$-nonsingular modules hold for dual $\mathcal{T}$-noncosingular modules. The structure of finitely generated $\mathcal{T}$-noncosingular $\mathbb{Z}$-modules is described, and a necessary and sufficient condition is provided for a direct sum of $\mathcal{T}$-noncosingular modules to be $\mathcal{T}$-noncosingular. Rings for which all right modules are $\mathcal{T}$-noncosingular are shown to be precisely right V-rings. A module M is called $\mathcal{T}$-noncosingular relative to N if, for every nonzero homomorphism $f : M \to N$, $\operatorname{Im} f$ is not small in N. M is called $\mathcal{T}$-noncosingular if M is $\mathcal{T}$-noncosingular relative to M. In this paper, we introduce to a special case of $\mathcal{T}$-noncosingular modules which are $\mathcal{T}$-e-noncosingular modules. A module M is called $\mathcal{T}$-e-noncosingular relative to N if, for every nonzero homomorphism $f : M \to N$, $\operatorname{Im} f$ is not e-small in N. M is called $\mathcal{T}$-e-noncosingular if M is $\mathcal{T}$-e-noncosingular relative to M. Some properties of this class of modules and the relation to other kinds of modules are shown in section 3. We show that every right R-module is $\mathcal{T}$-e-noncosingular if and only if every right R-module is e-noncosingular, if and only if for any right R-module M, $\operatorname{Rad}_e(M) = 0$. Furthermore, $\mathcal{T}$-e-noncosingular modules and e-lifting modules are dual Baer modules.

2. e-local Modules

Recall that a submodule N of M is said to be e-small in M (denoted by $N \ll_e M$), if $N + L = M$ with $L \leq^e M$ implies $L = M$.

The following lemma is proved in [20]:

Lemma 2.1. *Let M be a module. Then*

(1) *If $N \ll_e M$ and $K \leq N$, then $K \ll_e M$ and $N/K \ll_e M/K$.*

(2) *Let $N \ll_e M$ and $M = X + N$. Then $M = X \oplus Y$ for some a semisimple submodule Y of M.*

(3) *Let $N, K \leq M$. Then $N + K \ll_e M$ if and only if $N \ll_e M$ and $K \ll_e M$.*

(4) *If $K \ll_e M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_e N$. In particular, if $K \ll_e M \leq N$, then $K \ll_e N$.*

(5) *Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is e-small in $M_1 \oplus M_2$ if and only if $K_1 \ll_e M_1$ and $K_2 \ll_e M_2$.*

Lemma 2.2. *Let M be an R-module and $x \in M$. The following conditions are equivalent:*

(1) *$x \in \operatorname{Rad}_e(M)$;*

(2) *$xR \ll_e M$.*

Proof. It is clear and omit the proof. □

Corrolary 2.3. *If* $M = \bigoplus_{i\in I} M_i$*, then* $\mathrm{Rad}_e(M) = \bigoplus_{i\in I} \mathrm{Rad}_e(M_i)$.

Proof. It is clear $\bigoplus_{i\in I} \mathrm{Rad}_e(M_i) \leq \mathrm{Rad}_e(M)$. For every $j \in I$, call $\pi_j : M \to M_j$ the canonical projection. If $x \in \mathrm{Rad}_e(M)$, then $xR \ll_e M$. It follows that $\pi_j(xR) \ll_e M_j$ or $\pi_j(x) \in \mathrm{Rad}_e(M_j)$. This gives $x \in \bigoplus_{i\in I} \mathrm{Rad}_e(M_i)$. □

Lemma 2.4. *Let* M *be a module. The following are equivalent:*

(1) $M \ll_e M$*;*

(2) M *is a semisimple module;*

(3) *Any submodule of* M *is e-small in* M*.*

Proof. (1) ⇒ (2). Let A and B be submodules of M with $A \oplus B \leq^e M$. As $M = M + (A \oplus B)$ and $M \ll_e M$, then $M = A \oplus B$. It follows that M is a semisimple module.

(2) ⇒ (1) and (2) ⇔ (3) are obvious. □

Recall that a module M is called local if the sum of all proper submodules of M is also a proper submodule of M. We call M an e-local module if $\mathrm{Rad}_e(M)$ is a maximal submodule of M and $\mathrm{Rad}_e(M) \ll_e M$.

Let N, L be submodules of M. L is called an e-supplement of N in M if $M = N + L$ and $N \cap L$ is e-small in L. A module M is called e-supplemented if every submodule of M has an e-supplement in M [12].

Lemma 2.5. *Any e-local module is e-supplemented.*

Proof. Let M be an e-local module and N be a proper submodule of M. Since $\mathrm{Rad}_e(M)$ is a maximal submodule of M, either $N \leq \mathrm{Rad}_e(M)$ or $\mathrm{Rad}_e(M) + N = M$. If $N \leq \mathrm{Rad}_e(M)$ then M is an e-supplement of N in M. Now suppose $N + \mathrm{Rad}_e(M) = M$. It follows that $N \oplus Y = M$ for some semisimple submodule Y of M. Clearly, Y is an e-supplement of N in M. Thus M is e-supplemented. □

Remark 2.6. The following statements hold

(1) Every simple module is local.

(2) Every semisimple module M is not e-local, since $\mathrm{Rad}_e(M) = M$.

We next give some characterizations of e-local modules with semisimple property. Furthermore, the relationship between of e-local modules and local modules are considered.

Proposition 2.7. *Every local module is either simple or e-local.*

Proof. Assume that L is a local module and not simple. It is well-known that $\mathrm{Rad}(L)$ is the unique maximal submodule of L, $Rad(L) \ll L$ and $Rad(L) \leq^e L$.

Suppose that $\mathrm{Rad}_e(L) \neq \mathrm{Rad}(L)$. Call $x \in \mathrm{Rad}_e(L)$ and $x \notin \mathrm{Rad}(L)$. Then $xR \ll_e L$ by Lemma 2.2. Since $xR + \mathrm{Rad}(L) = L$ and $\mathrm{Rad}(L) \ll L$, then we have $xR = L$. Hence, $L \ll_e L$. By Lemma 2.4, L is semisimple. So, $\mathrm{Rad}(L) = 0$. Let H be a proper submodule of M. Since $\mathrm{Rad}(L)$ is an only maximal submodule of M, $H \leq \mathrm{Rad}(L)$. Hence, $H = 0$. It follows that M is simple, a contradiction. Thus, $\mathrm{Rad}_e(L) \leq \mathrm{Rad}(L)$. On the other hand, since $\mathrm{Rad}(L) \ll L$, we have $\mathrm{Rad}(L) \leq \mathrm{Rad}_e(L)$. Thus $\mathrm{Rad}(L) = \mathrm{Rad}_e(L)$ is a maximal submodule of L and e-small in L. □

Proposition 2.8. *The following conditions are equivalent for an e-local module M:*

(1) *M is local;*

(2) *M is an indecomposable module.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1). Note that $\mathrm{Rad}_e(M)$ is a maximal submodule of M. Let L be a proper submodule of M. Suppose that $L \nleq \mathrm{Rad}_e(M)$. Then $L + \mathrm{Rad}_e(M) = M$. Since $\mathrm{Rad}_e(M) \ll_e M$, there is a decomposition $M = L \oplus L'$ with L' semisimple. But M is indecomposable. Thus $L = M$ or $L = 0$. But $L \nleq \mathrm{Rad}_e(M)$ and so $L = M$, a contradiction. It follows that $L \leq \mathrm{Rad}_e(M)$. Consequently, M is a local module. □

Theorem 2.9. *Let $M = N \oplus K$ be a module. The following statements are equivalent:*

(1) *M is e-local;*

(2) *Either (a) N is e-local and K is semisimple, or (b) K is e-local and N is semisimple.*

Proof. By Corollary 2.3, we have $\mathrm{Rad}_e(M) = \mathrm{Rad}_e(N) \oplus \mathrm{Rad}_e(K)$.

(1) ⇒ (2). Since $\mathrm{Rad}_e(M)$ is a maximal submodule of M, we have

$$\mathrm{Rad}_e(N) = N \text{ or } \mathrm{Rad}_e(K) = K.$$

Assume that $\mathrm{Rad}_e(N) = N$. If X is a submodule of K with $\mathrm{Rad}_e(K) \leq X$, then $\mathrm{Rad}_e(M) \leq N \oplus X$. So $X = \mathrm{Rad}_e(K)$ or $X = K$. Therefore $\mathrm{Rad}_e(K)$ is a maximal submodule of K. Moreover, $\mathrm{Rad}_e(K)$ is e-small in K and $N \ll_e N$. Thus K is e-local and N is semisimple by Lemma 2.4.

Similarly, if $\mathrm{Rad}_e(K) = K$, then we also have N is e-local and K is semisimple.

(2) ⇒ (1). Assume that K is e-local and N is semisimple. Then $N \ll_e N$ and $Rad_e(N) = N$ by Lemma 2.4. So $\mathrm{Rad}_e(M) = N \oplus \mathrm{Rad}_e(K) \ll_e M$. Let $L \leq M$ be a submodule such that $\mathrm{Rad}_e(M) \leq L$. It follows that $\mathrm{Rad}_e(K) \leq K \cap L$. As $\mathrm{Rad}_e(K)$ is a maximal submodule of K, we have $K \cap L = \mathrm{Rad}_e(K)$ or $K \cap L = K$. Note that$L = N \oplus (K \cap L)$. This gives that $L = \mathrm{Rad}_e(M)$ or $L = M$. Therefore $Rad_e(M)$ is a maximal submodule of M. Consequently, M is an e-local module. □

Corollary 2.10. *A direct sum of two e-local modules is never e-local.*

Proof. Let $M = L_1 \oplus L_2$ be a module with e-local modules L_1 and L_2. Suppose that M is e-local. By Theorem 2.9, one of the L_i $(i = 1, 2)$ is semisimple. It follows that $Rad_e(L_1) = L_1$ or $\mathrm{Rad}_e(L_2) = L_2$, a contradiction. □

Example 2.11.

(1) Let M be a simple singular module. Then M is δ-local but it is not e-local. For example, $M = \mathbb{Z}/p\mathbb{Z}$, p is a prime number. Then M is a $\mathbb{Z}$-module simple and singular.

(2) Let N be an e-local projective module and K, a non-projective semisimple module. By Theorem 2.9 and [15, Proposition 2.17], $N \oplus K$ is an e-local module but it is not δ-local.

(3) Let $R = \mathbb{Z}, M = \mathbb{Z}/24\mathbb{Z}$. Then, $\mathrm{Rad}(M) = \delta(M) = 6M, \mathrm{Rad}_e(M) = 2M$. So, M is an e-local module but it is neither local nor δ-local.

(4) Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then R is δ-local but it is not local ([15, 2.5]). Moreover, R is an e-local module by projectivity of R.

Proposition 2.12. *A module M is e-local if and only if $M = L \oplus N$ such that L is a cyclic e-local module and N is a semisimple module.*

Proof. $(\Rightarrow)$. Assume that M is an e-local module. Then $Rad_e(M)$ is a maximal submodule of M. Call $x \in M$ and $x \notin \mathrm{Rad}_e(M)$. By maximality of $\mathrm{Rad}_e(M)$, then $M = \mathrm{Rad}_e(M) + xR$. Furthermore, $\mathrm{Rad}_e(M) \ll_e M$, there exists a nonzero semisimple submodule X of M such that $M = X \oplus xR$. It follows that $\mathrm{Rad}_e(X) = X$ and so X is not e-local. We deduce that xR is e-local by Theorem 2.9.

$(\Leftarrow)$. By Theorem 2.9. □

Theorem 2.13. *The following conditions are equivalent for a module M:*

(1) *M is an e-local module;*

(2) $\mathrm{Rad}_e(M)$ *is a maximal submodule of M and every proper essential submodule of M is contained in a maximal submodule;*

(3) *M has a unique essential maximal submodule and every proper essential submodule of M is contained in a maximal submodule.*

Proof. (1) $\Leftrightarrow$ (2) is clear.

(1) $\Rightarrow$ (3). Since M is e-local, M is not semisimple. Assume that there is a nonzero submodule $X \leq M$ such that $\mathrm{Rad}_e(M) \cap X = 0$. Since $\mathrm{Rad}_e(M)$ is a maximal submodule of M, $M = \mathrm{Rad}_e(M) \oplus X$. This gives that X is a simple module. As $\mathrm{Rad}_e(M) \ll_e M$, there exists a semisimple submodule $L \leq M$ such that $M = L \oplus X$. We deduce that M is a semisimple module, a contradiction. It follows that $\mathrm{Rad}_e(M)$ is essential in M. Now suppose that M contains an essential

maximal submodule N such that $N \nleq \mathrm{Rad}_e(M)$. Then $M = \mathrm{Rad}_e(M) + N$. Since $\mathrm{Rad}_e(M) \ll_e M$, there exists a semisimple submodule E of M such that $M = E \oplus N$. But N is essential in M, we have $E = 0$ and so $N = M$, a contradiction. Consequently, $\mathrm{Rad}_e(M)$ is the only essential maximal submodule of M.

(3) $\Rightarrow$ (1). Assume that every proper essential submodule M is contained in a maximal submodule and K is the only essential maximal submodule of M. If $x \in M \setminus K$, then $M = xR + K$ by maximality of K. By our assumption $K \leq^e M$, xR is not e-small in M. This gives that $x \notin \mathrm{Rad}_e(M)$. We deduce that $\mathrm{Rad}_e(M) \leq K$. Let Y be a proper essential submodule M, then $Y \leq K$ and $Y + K = K \neq M$. It follows that $K \ll_e M$, i.e. $K \leq \mathrm{Rad}_e(M)$. Thus $\mathrm{Rad}_e(M) = K \ll_e M$. □

Following [12], a module M is called e-supplemented if every submodule of M has an e-supplement in M. A module M is called amply e-supplemented if for any submodules A, B of M with $M = A + B$, there exists an e-supplement P of A such that $P \leq B$. In this case, we say that A has ample e-supplements in M.

Proposition 2.14. *Let M be an e-local module. If N is a submodule of M, then N is either e-small in M or there exists a semisimple submodule X of M such that $M = N \oplus X$.*

Proof. Let N be a submodule of M. Assume N is not e-small in M. Then $N \nleq \mathrm{Rad}_e(M)$. By maximality of $\mathrm{Rad}_e(M)$, we have $N + \mathrm{Rad}_e(M) = M$. As $\mathrm{Rad}_e(M) \ll_e M$, $M = N \oplus X$ for some a semisimple submodule X of M. □

Lemma 2.15. *Let N be a maximal submodule of a module M. If K is an e-supplement of N in M, then K is either e-local or semisimple.*

Proof. By assumption, we have $N + K = M$ and $N \cap K \ll_e K$. Therefore $N \cap K \leq \mathrm{Rad}_e(K)$. As $M/N \simeq K/(N \cap K)$, $N \cap K$ is a maximal submodule of K. It follows that $\mathrm{Rad}_e(K) = N \cap K$ or $\mathrm{Rad}_e(K) = K$. If $\mathrm{Rad}_e(K) = N \cap K$, then K is an e-local module. Assume that $\mathrm{Rad}_e(K) = K$. For any $x \in K \setminus (N \cap K)$, we have $xR + (N \cap K) = K$. Furthermore, we have $xR \ll_e K$ by Lemma 2.2 and $N \cap K \ll_e K$. Thus $K \ll_e K$ by Lemma 2.1. By Lemma 2.4, K is a semisimple module. □

Lemma 2.16. *Let $L_1, L_2, .., L_n$ be submodules of M such that either L_i is e-local or L_i is semisimple. Assume that N is a submodule of M and $N + L_1 + ... + L_n$ has an e-supplement K in M. Then, there exists a subset I of $\{1, ..., n\}$ such that $K + \sum_{i \in I} X_i$ is an e-supplement of N in M, where $X_i = L_i$ or X_i is a semisimple direct summand of L_i.*

Proof. If $n = 1$ then $N + (K + L_1) = M$ and $K \cap (N + L_1) \ll_e K$. Call $H = (N + K) \cap L_1$. Assume that $H \ll_e L_1$. We have

$$N \cap (K + L_1) \leq [(N + L_1) \cap K] + [(N + K) \cap L_1 \ll_e K + L_1].$$

It follows that $K + L_1$ is an e-supplement of N in M.

If $H \not\ll_e L_1$ then L_1 is not semisimple by Lemma 2.4. By hypothesis, L_1 is e-local. From Proposition 2.14, there exists a semisimple submodule $X_1 \leq L_1$ such that $H \oplus X_1 = L_1$. Hence $N + (K + X_1) = M$. We have that

$$N \cap (K + X_1) \leq (N + K) \cap X_1 + (N + X_1) \cap K,$$

$$(N + K) \cap X_1 \ll_e X_1, (N + X_1) \cap K \leq (N + L_1) \cap K \ll_e K$$

and obtain that $N \cap (K + X_1) \ll_e K + X_1$. This gives that $K + X_1$ is an e-supplement of N in M.

Assume that $n > 1$. By induction on n, there exist a subset $\mathcal{J}$ of $\{2, ..., n\}$ and $X_j \leq L_j, j \in \mathcal{J}$ such that $K + \sum_{j\in\mathcal{J}} X_j$ is an e-supplement of $N + L_1$ in M, which either $X_j = L_j$ or X_j is a semisimple direct summand of L_j for all $j \in \mathcal{J}$. Then, there exists a submodule X_1 of L_1 such that $K + \sum_{j\in\mathcal{J}} X_j + X_1$ is an e-supplement of N in M and either $X_1 = L_1$ or X_1 is a semisimple direct summand of L_1. □

Proposition 2.17. *Let M be a finitely generated module. The following conditions are equivalent:*

(1) *M is an amply e-supplemented module;*

(2) *Every maximal submodue of M has ample e-supplement in M;*

(3) *If L, N are submodules of M and $M = L + N$ then $M = N + L_1 + ... + L_n$, where n is positive integer number, either L_i is e-local or L_i is semisimple.*

Proof. (1) ⇒ (2). It is clear.

(2) ⇒ (3). Let N, L be submodules of M and $M = N + L$. Call Γ a class of all submodules X of M such that $X \leq L$ and $X = X_1 + ... + X_k$, where either X_i is e-local or X_i is semisimple. Assume that $M \neq N + A$ for all $A \in \Gamma$. By [15, Lemma 3.5], there exists a submodule $U \leq M$ such that $N \leq U$ and U is a maximal submodule of M satisfying $M \neq U + A$ for all $A \in \Gamma$. Since M is finitely generated and $U \neq M$, there exists a maximal submodule $K \leq M$ such that $U \leq K$. So $K + L = M$. By hypothesis, there exists a submodule $E \leq L$ such that E is an e-supplement of K in M. Following Lemma 2.15, either E is e-local or E is semisimple. It is easy to see that $U \neq U + E$. Otherwise, we have $E \leq U \leq K$ and $K = K + E = M$. It follows $M = U + E + F$, $F \in \Gamma$. So $E + F \in \Gamma$, a contradiction.

(3) ⇒ (1). By Lemma 2.16. □

Lemma 2.18. *Let N, L be submodules of M such that $M = N + L$. If L is an e-supplemented module then L contains an e-supplement of N in M.*

Proof. By hypothesis, there exists a submodule K of L such that $(N \cap L) + K = L$ and $(N \cap L) \cap K \ll_e K$. Then $N + K = M$ and $N \cap K \ll_e K$. So K is an e-supplement of N in M. □

Proposition 2.19. *Let M be a module. If every cyclic submodule of M is e-supplemented then every maximal submodule of M has ample e-supplement.*

Proof. Assume that N is a maximal submodule of M. Let L be a submodule of M such that $M = N + L$. There exists x in L satifying $x \notin N$ and $xR + N = M$. Following Lemma 2.18, xR containt an e-supplement of N in M. □

Corollary 2.20. *If M is a finitely generated module and every cyclic submodule of M is e-supplemented then M is an e-supplemented module.*

Proof. By Proposition 2.17 and Proposition 2.19. □

3. $\mathcal{T}$-e-noncosingular Modules

Let M, N be right R-modules. We call M $\mathcal{T}$-e-noncosingular relative to N if $\operatorname{Im} f$ is not e-small in N for any nonzero homomorphism $f : M \to N$. M is called $\mathcal{T}$-e-noncosingular if M is $\mathcal{T}$-e-noncosingular relative to M. The ring R is called right (left) $\mathcal{T}$-e-noncosingular if the right (left) module R_R ($_RR$) is $\mathcal{T}$-e-noncosingular, respectively.

We denote

$$\nabla_e[M, N] = \{f : M \to N | \operatorname{Im} f \ll_e N\}.$$

It is easily to check that M is $\mathcal{T}$-e-noncosingular relative to N if and only if $\nabla_e[M, N] = 0$.

Proposition 3.1. *Let M, N be right R-modules and K is a direct summand of M. If $\nabla_e[M, N] = 0$ then $\nabla_e[K, N] = 0$.*

Proof. Assume that $M = K \oplus L$ and $\varphi \in \nabla_e[K, N]$. Then $\operatorname{Im} \varphi \ll_e N$. We consider the homomorphism $\varphi \oplus 0_L : M \to N$ defined by $(\varphi \oplus 0_L)(k + l) = \varphi(k)$ for all $k \in K, l \in L$. So $\operatorname{Im}(\varphi \oplus 0_L) = \operatorname{Im} \varphi \ll_e N$. As $\nabla_e[M, N] = 0$, $\varphi \oplus 0_K = 0$ and hence $\varphi = 0$. □

Proposition 3.2. *Let M, N be right R-modules. If $\nabla_e[M, N] = 0$ then $\nabla_e[M, P] = 0$ for all submodule P of N.*

Proof. Assume that $P \leq N$ and $\varphi \in \nabla_e[M, P]$. Then $\operatorname{Im} \varphi \ll_e P$. It follows that $\operatorname{Im} \varphi \ll_e N$. Since $\nabla_e[M, N] = 0$, $\varphi = 0$. □

Corollary 3.3. *Every direct summand of a $\mathcal{T}$-e-noncosingular module is also a $\mathcal{T}$-e-noncosingular module.*

Proof. It is followed from Proposition 3.1. □

Proposition 3.4. *Let $M = \oplus_{i\in I} M_i$, $N = \oplus_{j\in J} N_j$ be right R-modules, where I, J are non-empty sets. Then $\nabla_e[M, N] = 0$ if only if $\nabla_e[M_i, N_j] = 0$ for all $i \in I, j \in J$.*

Proof. Assume that $\nabla_e[M_i, N_j] = 0$ for all $i \in I, j \in J$. Let $f \in \nabla_e[M, N_j]$ and the conlusion $\iota_i : M_i \to M$. Since $\operatorname{Im} f \ll_e N_j$, $\operatorname{Im} f\iota_i \ll_e N_j$ for all $i \in I$. Hence $f\iota_i = 0$ for all $i \in I$. It follows that $f = 0$. Now, let $\varphi \in \nabla_e[M, N]$ and the projection $\pi_j : N \to N_j$. Set $\varphi_j = \pi_j\varphi : M \to N_j$. Since $\operatorname{Im} \varphi \ll_e N$, $\operatorname{Im} \varphi_j \ll_e N_j$ for all $i \in I$. By hypothesis, $\varphi_j = 0$. It follows that $\varphi = 0$.

The converse is followed by Lemma 3.1 and Lemma 3.2. □

Corollary 3.5. *Let $M = \oplus_{i\in I}M_i$, $N = \oplus_{j\in J}N_j$ be right R-modules, where I, J are non-empty sets. Then M is $\mathcal{T}$-e-noncosingular relative to N if only if M_i is $\mathcal{T}$-e-noncosingular relative to N_j for all $i \in I, j \in J$.*

Corrllary 3.6. *Let $(M_i)_{i\in I}$ be a family of modules. Then $M = \oplus_{i\in I}M_i$ is a $\mathcal{T}$-e-noncosingular if and only if M_i is $\mathcal{T}$-e-noncosingular relative to M_j for all $i, j \in I$.*

Let M be a module. We call M an e-small module if M is e-small in injective envelope of M. We denote

$$\overline{Z}_e(M) = \bigcap\{\operatorname{Ker} g | g : M \to N, N \text{ is } e\text{-small module}\}.$$

If $\overline{Z}_e(M) = M$, then M is called an e-noncosingular module.

Proposition 3.7. *The following conditions are equivalent for a ring R:*

(1) *Every right R-module is $\mathcal{T}$-e-noncosingular;*

(2) *Every right R-module is e-noncosingular;*

(3) *For any right R-module M,* $\operatorname{Rad}_e(M) = 0$.

Proof. (1) ⇒ (2). Let $N \ll_e E(N)$. We will prove $N = 0$. We consider the homomorphism $f : M \oplus N \to E(N)$ given by $f(m + n) = n$ for all $m \in M, n \in N$. Then $\operatorname{Im} f = N \ll_e E(N)$. We have that $M \oplus N \oplus E(N)$ is an $\mathcal{T}$-e-noncosingular module and obtain that $M \oplus N$ is $\mathcal{T}$-e-noncosingular relative to $E(N)$. This gives $f = 0$. It is easily to check that $N = 0$. Furthermore, for any R-module M, $\overline{Z}_e(M) = \bigcap\{\operatorname{Ker} g | g : M \to 0\} = M$, i.e., M is e-noncosingular.

(2) ⇒ (3). Assume that N is an e-small submodule of M. Call $\pi : M \oplus N \to N$ the projection. By hypothesis, $M \oplus N$ is e-noncosingular. We have that $\overline{Z}_e(M \oplus N) = M \oplus N$ and obtain that $f = 0$. Thus $N = 0$.

(3) ⇒ (1). It is clear. □

Now, we denote:

$$Z_{e-M}(N) = \bigcap_{\varphi\in\nabla_e[M,N]} \operatorname{Ker}\varphi$$

Proposition 3.8. *Let M be a module. Then the following conditions hold:*

(1) $\overline{Z}_e(M) \leq Z_{e-M}(N)$.

(2) $Z_{e-M}(N)$ *is a fully invariant submodule of M.*

(3) $\nabla_e[M, N] = 0$ *if only if* $M = Z_{e-M}(N)$.

(4) *If $M = \oplus_{i\in I}M_i$ then $\overline{Z}_{e-M}(N) \leq \oplus_{i\in I}\overline{Z}_{e-M_i}(N)$.*

Proof. (1) By definition, we get

$$\overline{Z}_e(M) \leq \bigcap\{\operatorname{Ker} g : M \to N | N = \operatorname{Im} f, f \in \nabla_e[M,N]\} = Z_{e-M}(N).$$

(2) Assume $f \in \operatorname{End}(M)$ and $\varphi \in \operatorname{Hom}(M,N)$ such that $\operatorname{Im}\varphi \ll_e N$. Therefore $\operatorname{Im}\varphi f \leq \operatorname{Im}\varphi$. So $\operatorname{Im}\varphi f \ll_e N$. For all $x \in Z_{e-M}(N)$, $\varphi(x) = 0$ implies $\varphi f(x) = 0$. Thus $f(x) \in Z_{e-M}(N)$, i.e., $Z_{e-M}(N)$ is fully invariant.

(3) It is clear.

(4) As $Z_{e-M}(N)$ is fully invariant, $Z_{e-M}(N) = \oplus_{i\in I}(Z_{e-M}(N) \cap M_i)$. We will prove that $Z_{e-M}(N) \cap M_i \subset Z_{e-M_i}(N)$. Let $x_i \in Z_{e-M}(N) \cap M_i$ and $\varphi_i : M_i \to N$ such that $\operatorname{Im}\varphi_i \ll_e N$. Then $\psi_i : M \to M$ extends φ_i ($\psi_i|_{M_j} = 0$ for all $j \neq i$). This gives $\operatorname{Im}\psi_i \ll_e N$. Thus $\psi_i(x_i) = \varphi_i(x_i) = 0$ and hence $x_i \in Z_{e-M_i}(N)$. □

Corollary 3.9. *Let M and N be modules. Then M is $\mathcal{T}$-e-noncosingular relative to N if and only if $Z_{e-M}(N) = M$.* □

Remark 3.10. It is clearly to see that $Z_{e-M}(M) \leq \overline{Z}_{\mathcal{T}}(M) = \bigcap\{\operatorname{Ker}\varphi | \varphi \in \operatorname{End}(M), \operatorname{Im}\varphi \ll M\}$. So, if M is a $\mathcal{T}$-e-noncosingular then M is a $\mathcal{T}$-noncosingular module. The converse is not true in general.

Example 3.11.

(1) $\mathbb{Z}$-module $\mathbb{Z}$ is $\mathcal{T}$-e-noncosingular.

(2) If $M_{\mathbb{Z}} = \mathbb{Z}_6$ then $Rad(M) = 0$ and $Z_{e-M}(M) = 0$. It follows that M is $\mathcal{T}$-noncosingular but not $\mathcal{T}$-e-noncosingular.

(3) Let R be a proper Dedekind domain and P be a nonzero prime ideal of R. Consider module $M = R(P^\infty) \oplus R/P$. Then M is not a $\mathcal{T}$-noncosingular module (see Example 2.12, [9]). So M is not a $\mathcal{T}$-e-noncosingular module.

(4) As $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_2) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Q}) = 0$, $\mathbb{Q}_{\mathbb{Z}}$ is $\mathcal{T}$-e-noncosingular relative to $\mathbb{Z}_2$ and $\mathbb{Z}_2$ is $\mathcal{T}$-e-noncosingular relative to $\mathbb{Q}$. Hence $(\mathbb{Q} \oplus \mathbb{Z}_2)_{\mathbb{Z}}$ is $\mathcal{T}$-e-noncosingular by Lemma 3.6.

Proposition 3.12. *Let M be an R-module which $S = \operatorname{End}(M)$ is Von Neumann regular and $T(M) = \{N \leq M | \operatorname{Rad}_e(N) = N\}$. If $T(M) = 0$ then M is $\mathcal{T}$-e-noncosingular.*

Proof. Let $f \in \operatorname{End}(M)$ such that $\operatorname{Im} f \ll_e M$. Then $\operatorname{Im} f \leq \operatorname{Rad}_e(M)$. Since S is regular, there exists $g \in S$ such that $f = fgf$. Hence fg is an idempotent and $M = \operatorname{Im} fg \oplus \operatorname{Ker} fg$. Since $\operatorname{Im} fg \leq \operatorname{Im} f \leq \operatorname{Rad}_e(M)$, $\operatorname{Rad}_e(M) = \operatorname{Rad}_e(\operatorname{Im} fg) \oplus \operatorname{Rad}_e(\operatorname{Ker} fg)$. So, $\operatorname{Im} fg \cap \operatorname{Rad}_e(M) = \operatorname{Im} fg = \operatorname{Rad}_e(\operatorname{Im} fg) \oplus (\operatorname{Im} fg \cap \operatorname{Rad}_e(\operatorname{Ker} fg))$. It follows $\operatorname{Im} fg = \operatorname{Rad}_e(\operatorname{Im} fg)$. Therefore $\operatorname{Im} fg \in T(M)$. We have $fg = 0$ and $f = 0$. □

Note that if $\mathrm{Rad}_e(M) = 0$ then M is a $\mathcal{T}$-e-noncosingular module. But the converse is not true in general. For example, let $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}_2$ in Example 3.11. Then M is $\mathcal{T}$-e-noncosingular. However, we have

$$\mathrm{Rad}_e(\mathbb{Q} \oplus \mathbb{Z}_2) = \mathrm{Rad}_e(\mathbb{Q}) \oplus \mathrm{Rad}_e(\mathbb{Z}_2) = 0 \oplus \mathbb{Z}_2 \neq 0.$$

Proposition 3.13. *Let $M = xR$ be a cyclic module such that $r(x)$ is an ideal of R. Then M is $\mathcal{T}$-e-noncosingular if and only if $\mathrm{Rad}_e(M) = 0$.*

Proof. Assume that M is $\mathcal{T}$-e-noncosingular and $\mathrm{Rad}_e(M) \neq 0$. There exists $a \in R$ such that $xa \neq 0$ and $xa \in \mathrm{Rad}_e(M)$. Call f an endomorphism of M with $f(xr) = xar$ for all $r \in R$. We have $\mathrm{Im}\, f \leq \mathrm{Rad}_e(M)$ and $f \neq 0$. But $\mathrm{Rad}_e(M) \ll_e M$, a contradiction. The converse is clear. □

Corollary 3.14. *A ring R is right $\mathcal{T}$-e-noncosingular if and only if $\mathrm{Rad}_e(R_R) = 0$.*

Example 3.15.

(1) Consider $\mathbb{Z}_6$ as a ring. We have $J(\mathbb{Z}_6) = 0$, $\mathrm{Rad}_e(\mathbb{Z}_6) = \mathbb{Z}_6$. So $\mathbb{Z}_6$ is $\mathcal{T}$-noncosingular but is not $\mathcal{T}$-e-noncosingular.

(2) Let R be a discrete valuation ring with maximal ideal m. Then R is not $\mathcal{T}$-noncosingular following Example 4.7,[14]. So R is not $\mathcal{T}$-e-noncosingular.

For $N \leq M, I \leq S = \mathrm{End}(M)$, denote $N \trianglelefteq M$ means that N is a fully invariant submodule of M and $E_M(I) = \sum\limits_{\phi \in I} \mathrm{Im}\,\phi$; $D_S(N) = \{\phi \mid \mathrm{Im}\,\phi \leq N\}$.

Lemma 3.16. *Let $N \leq M, I \leq S, P \trianglelefteq M, L \trianglelefteq S$. Then:*

(1) $E_M(D_S(E_M(I))) = E_M(I)$;

(2) $D_S(E_M(D_S(N))) = D_S(N)$;

(3) $E_M(L) \trianglelefteq M$;

(4) $D_S(P) \trianglelefteq S$.

Proof. (1) $E_M(D_S(E_M(I))) = \sum\limits_{\phi \in D_S(E_M(I))} \mathrm{Im}\,\phi \leq E_M(I)$. Conversely, for all $\varphi \in I, \mathrm{Im}\,\varphi \leq E_I(M)$. So $\varphi \in D_S(E_M(I)) = \{\phi | \mathrm{Im}\,\phi \leq E_M(I)\}$.

(2) $E_M(D_S(N)) \leq N$ implies $D_S(E_M(D_S(N))) \leq D_S(N)$. Conversely, for all $\varphi, \mathrm{Im}\,\varphi \leq N, \mathrm{Im}\,\varphi \leq \sum\limits_{\mathrm{Im}\,\phi \leq N} \mathrm{Im}\,\phi = E_M(D_S(N))$. So $D_S(N) \leq D_S(E_M(D_S(N)))$.

(3) Let $f : M \to M$, $f(E_M(L)) = \sum\limits_{\phi \in L} f(\mathrm{Im}\,\phi) = \sum\limits_{\phi \in L} \mathrm{Im}\,\phi f$. Since $L \trianglelefteq S, \phi f \in L$. So $\sum\limits_{\phi \in L} \mathrm{Im}\,\phi f \leq \sum\limits_{\psi \in L} \mathrm{Im}\,\psi = E_M(L)$.

(4) For all $\psi \in S$, $\phi \in D_S(P)$. We have $\psi\phi(M) \leq \psi(P) \leq P$ and $\phi\psi(M) \leq \phi(M) \leq P$. So $\psi\phi \in D_S(P)$ and $\phi\psi \in D_S(P)$. □

Proposition 3.17. *Let M be an R-module. M is $\mathcal{T}$-e-noncosingular if and only if for all $I \leq S, E_M(I) = eM \oplus L$, in which $L \ll_e M, e^2 = e \in S$ implies $I \cap (1-e)S = 0$.*

Proof. ($\Rightarrow$). Assume $I \leq S, E_M(I) = eM \oplus L$, in which $L \ll_e M, e^2 = e \in S$. We have $E_M(I \cap (1-e)S) \leq E_M(I) \cap E_M(1-e)S \leq E_M(I) \cap (1-e)M = (eM \oplus L) \cap (1-e)M \leq (1-e)L$. Since $L \ll_e M$, $(1-e)L \ll_e M$. Hence $E_M(I \cap (1-e)S) \ll_e M$. M is $\mathcal{T}$-e-noncosingular, so $I \cap (1-e)S = 0$.

($\Leftarrow$). Let $\phi \in S, \operatorname{Im}\phi \ll_e M$. We have $E_M(\phi S) = \sum\limits_{\psi \in S} \operatorname{Im}\phi\psi = \phi(\sum\limits_{\psi \in S} \operatorname{Im}\psi) = \phi(M) \ll_e M$. By hypothesis, $I \cap S = 0$. Hence $I = 0$, i.e., $\phi = 0$. □

Corollary 3.18. *M is a $\mathcal{T}$-e-noncosingular module if and only if for all $I \leq S, E_M(I) \ll_e M$ implies that $I = 0$.*

Now, we call M an e-$\mathcal{K}$- module if for all $N \leq M$, $D_S(N) = 0$ implies $N \ll_e M$.

Proposition 3.19. *M is an e-$\mathcal{K}$-module if and only if, for all $N \leq M, E_M(D_S(N))$ is a direct summand of M implies that $N = E_M(D_S(N)) \oplus L$ with $L \ll_e M$.*

Proof. Assume that $N \leq M$ and $E_M(D_S(N)) \leq^{\oplus} M$. Then $E_M(D_S(N)) = eM, e^2 = e \in S$. Clearly, $eM = E_M(D_S(N)) \leq N$. On the other hand, $D_S(eM) \cap D_S((1-e)M \cap N) = 0$ and $D_S((1-e)M \cap N) \leq D_S(N) = D_S(eM)$. Hence $D_S((1-e)M \cap N) = 0$. Since M is an e-$\mathcal{K}$-module, we have $(1-e)M \cap N \ll_e M$. Thus $N = E_M(D_S(N)) \oplus ((1-e)M \cap N)$ and $(1-e)M \cap N \ll_e M$.

Conversely, assume $N \leq M$ and $D_S(N) = 0$. Then $E_M(D_S(N)) = 0$. By hypothesis, $N = E_M(D_S(N)) \oplus L$ with $L \ll_e M$. Thus $N = L \ll_e M$. □

Recalled that a module M is e-lifting if for all submodule N of M, there exists decompsiton $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll_e B$ ([12]). A module M is called dual Baer if for all $N \leq M$, there exists an idempotent $e \in S = \operatorname{End}(M)$ such that $D_S(N) = eS$([8]).

Lemma 3.20. *A dual Baer e-$\mathcal{K}$-module is e-lifting.*

Proof. Assume M is a dual Baer and e-$\mathcal{K}$-module. Let N be a submodule of M. There exists an idempotent $e \in S = \operatorname{End}(M)$ such that $D_S(N) = eS$. We have $eM = E_M(eS) \leq N$. Hence $N = eM \oplus ((1-e)M \cap N)$. For all $\phi \in D_S((1-e)M \cap N)$, $\operatorname{Im}\phi \leq N$. It follows $\phi \in D_S(N) = eS$. Since $\phi(M) \leq (1-e)M \cap eM = 0$, then $\phi = 0$, i.e., $D_S((1-e)M \cap N) = 0$. Since M is an e-$\mathcal{K}$-module, $(1-e)M \cap N \ll_e M$. Thus M is e-lifting. □

Theorem 3.21. *A $\mathcal{T}$-e-noncosingular e-lifting module is dual Baer.*

Proof. Assume that M is a $\mathcal{T}$-e-noncosingular e-lifting module and $N \leq M$. Then $N = eM \oplus B$ which $e^2 = E \in S, B = (1-e)M \cap N \ll_e M$. Hence $eS \leq D_S(eM) \leq D_S(N)$. If there exists $\phi \in D_S(N) \backslash eS$, then $(1-e)\phi = eS \cap D_S(N)$. We obtain that $(1-e)\phi M \leq N$ and $(1-e)\phi M \leq (1-e)M$. So $(1-e)\phi M \leq N \cap (1-e)M = B \ll_e M$.

Since M is $\mathcal{T}$-e-noncosingular, which follows $(1-e)\phi = 0$, i.e., $\phi = e\phi \in eS$. This is a contradition. Thus $D_S(N) = eS$, i.e., M is dual Baer. □

Lemma 3.22. *Let M be a $\mathcal{T}$-e-noncosingular module and X, a fully invariant submodule of M and $X = N \oplus B$ with $B \ll_e M$. If N is a direct summand of M then N is a fully invariant submodule of M.*

Proof. Assume $M = N \oplus P$ and $\phi \in \text{End}(M)$. Set $\psi = \pi_P \phi|_N \pi_N$. If there exists $x \in N$ such that $\phi(x) \notin N$, then $\psi(x) \neq 0$. Since X is a fully invariant submodule of M, $\phi(N) \leq \phi(X) \leq X$. So

$$\phi(M) = \pi_P \phi|_N \pi_N(M) = \pi_P \phi|_N(N) \leq \pi_P(X) = X \cap P.$$

Then $X \cap P \cong B$. It follows $X \cap P \ll_e M$. As M is $\mathcal{T}$-e-noncosingular, $\psi = 0$, a contradiction. Thus $\phi(N) \leq N$. □

Proposition 3.23. *Let M be a $\mathcal{T}$-e-noncosingular module. The following conditions are equivalent:*

(1) *For every fully invariant submodule N of M, there exists a direct summand B of M such that $N/B \ll_e M/B$;*

(2) *For every fully invariant submodule N of M, there exists a fully invariant direct summand B of M such that $N/B \ll_e M/B$.*

Proof. (2) $\Rightarrow$ (1) is clear. It suffices to prove (1) $\Rightarrow$ (2). Assume $X \trianglelefteq M$. By (1), we have $X = N \oplus B$, $B \ll_e M$ and N is a direct summand of M. By Lemma 3.22, N is a fully invariant submodule of M. Thus (2) holds. □

Acknowledgments. The authors would like to thank the referee for the very helpful comments and suggestions.

References

[1] I. Al-Khazzi and P. F. Smith, *Modules with chain conditions on superfluous submodules*, Comm. Algebra, **19(8)**(1991), 2331–2351.

[2] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Springer -Verlag - New York, 1974.

[3] P. Aydogdu, *Rings over which every module has a flat δ-cover*, Turkish J. Math., **37**(2013), 182–194.

[4] E. Büyükaşik and C. Lomp, *When δ-semiperfect rings are semiperfect*, Turkish J. Math., **34**(2010), 317–324.

[5] N. Er, E. Nil Orhan, *Lifting modules with indecomposable decompositions*, Comm. Algebra, **36(2)**(2008), 395–404.

[6] R. E. Jonhson, *Structure theory of faithful rings*, Trans. Amer. Math. Soc., **84**(1957), 508–522.

[7] M. T. Kosan, *δ-lifting and δ-supplemented modules*, Algebra Colloq., **14(1)**(2007), 53–60.

[8] D. Keskin Tütüncü and R. Tribak, *On dual Baer modules*, Glassgow Math. J., **52**(2010), 261–269.

[9] D. Keskin Tütüncü and R. Tribak, *On $\mathcal{T}$-noncosingular modules*, Bull. Aust. Math. Soc., **80**(2009), 462–471.

[10] A. C. Özcan, *The torsion theory cogenerated by δ-M-small modules and GCO-modules*, Comm. Algebra, **35**(2007), 623–633.

[11] T.C. Quynh, On pseudo semi-projective modules, Turkish J. Math., **37(1)**(2013), 27–36.

[12] T. C. Quynh and P. H. Tin, *Somes Properties of e-supplemented and e-lifting modules*, Vietnam J. Math., **41(3)**(2013), 303–312.

[13] Y. Talebi and N. Vanaja, *The torsion theory cogenerated by M-small modules*, Comm. Algebra, **30(3)**(2002), 1449–1460.

[14] R. Tribak, *Some results on $\mathcal{T}$-noncosingular modules*, Turkish J. Math., **38**(2014), 29–39.

[15] R. Tribak, *On δ-local modules and amply δ-supplemented modules*, J. Algebra Appl., **12(2)**(2013), 1250144, 14 pp.

[16] Y. Wang, *δ-small submodules and δ-supplemented Modules*, Int. J. Math. Math. Sci. (2007), 58–132

[17] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading 1991.

[18] M. Yousif and Y. Zhou, *Semiregular, Semiperfect and perfect rings relative to an ideal*, Rocky Mountain J. Math., **32(4)**(2002) 1651–1671.

[19] Y. Zhou, *Generalizations of perfect, semiperfect, and semiregular rings*, Algebra Colloq., **7**(2000), 305–318.

[20] D. X. Zhou, X. R. Zhang, *Small-essential submodules and Morita duality*, Southeast Asian Bull. Math., **35**(2011), 1051–1062.