

## The Structure of Walled Signed Brauer Algebras

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ABSTRACT. In this paper, a new class of diagram algebras which are subalgebras of signed brauer algebras, called the Walled Signed Brauer algebras denoted by  $\vec{D}_{r,s}(x)$ , where  $r, s \in \mathbb{N}$  and  $x$  is an indeterminate are introduced. A presentation of walled signed Brauer algebras in terms of generators and relations is given. The cellularity of a walled signed Brauer algebra is established. Finally,  $\vec{D}_{r,s}(x)$ , is quasi-hereditary if either the characteristic of a field, say  $p$ ,  $p = 0$  or  $p > \max(r, s)$  and either  $x \neq 0$  or  $x = 0$  and  $r \neq s$ .

### 1. Introduction

The walled Brauer algebras  $B_{r,s}(\delta)$  [4, section 2] are defined as subalgebras of Brauer algebras  $B_{r+s}(\delta)$ . It was introduced independently by Turaev [16], Koike [10] and Benkart et al.[1] which was partially motivated by Schur- Weyl duality between walled Brauer algebras  $B_{r,s}(\delta)$  and general linear group  $GL_\delta(\mathbb{C})$  arising from mutually commuting actions on the mixed tensor space  $V^{\otimes r} \otimes W^{\otimes s}$ , where  $V$  is the natural representation of  $GL_\delta(\mathbb{C})$  and  $W := V^*$ , the dual of the natural representation of  $GL_\delta(\mathbb{C})$ . Cox et al. [4] and Brundan and Stroppel [3] have also studied walled Brauer algebras.

Brauer and walled Brauer algebras arose in invariant theory. Brauer algebras [2] have a basis consisting of undirected graphs. This motivated Parvathi and Kamaraj [14] to define a new class of diagram algebras which are known as signed Brauer algebras denoted by  $\vec{D}_f(x)$ , having a basis consisting of signed diagrams. These algebras contain Brauer algebras  $D_f(x)$  and the group algebras  $k(x)\vec{S}_f$ , where  $\vec{S}_f$  is isomorphic to the hyperoctahedral group  $(\mathbb{Z}_2 \wr S_f)$ , as subalgebras in a natural fashion where as the Brauer algebras contain the group algebra of symmetric group. The flip map gives an isomorphism between the group algebra of symmetric group  $S_{r+s}$  and the walled Brauer algebra  $B_{r,s}(\delta)$ . The structure of signed Brauer algebras

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over  $k(x)$ , where  $x$  is an indeterminate, has been studied in [14].

These works **motivated** us to define a new class of diagram algebras over  $k(x)$  which are subalgebras of signed Brauer algebras. These new algebras are called Walled signed Brauer algebras and denoted by  $\vec{D}_{r,s}(x)$ , where  $r, s \in \mathbb{N}$  and  $x$  is an indeterminate.

In section 3, we define the walled signed Brauer algebras and give a presentation of walled signed Brauer algebras in terms of generators and relations. In section 4, the cellularity of these algebras are established and we proved that  $\vec{D}_{r,s}(x)$ , are quasi-hereditary if either characteristic of a field, say  $p$ ,  $p = 0$  or  $p > \max(r, s)$  and either  $x \neq 0$  or  $x = 0$  and  $r \neq s$ .

## 2. Preliminaries

In this section, we collect some preliminary results that we need for the development of the paper.

**Definition 2.1.**([15]) A partition of a non-negative integer  $k$  is a sequence of non-negative integers  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$  such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$  and  $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_r = k$ . The non-zero  $\lambda_i$ 's are called the parts of  $\lambda$  and the number of non-zero parts is called the length of  $\lambda$ . The notation  $\lambda \vdash k$  denotes that  $\lambda$  is a partition of  $k$ .

**Definition 2.2.**([9]) A partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$  of  $k$  is said to be  $p$ -regular if either  $p > 0$  and there is no  $1 \leq i \leq r$  such that  $\lambda_i = \lambda_{i+1} = \dots = \lambda_{i+p}$  or  $p = 0$ .

**Definition 2.3.**([7]) A bi-partition of a non-negative integer  $n$  is an ordered pair  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  of partitions  $\lambda^{(1)}$  and  $\lambda^{(2)}$  such that  $|\lambda^{(1)}| + |\lambda^{(2)}| = n$ . For every bipartition  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ , we associate a Young subgroup  $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda^{(1)}} \times \mathfrak{S}_{\lambda^{(2)}}$

**Definition 2.4.**([9]) A bi-partition  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  of  $n$  is said to be  $p$ -regular if the partitions  $\lambda^{(1)}$  and  $\lambda^{(2)}$  are  $p$ -regular.

**Definition 2.5.**([7]) Suppose that  $\mathbf{a} = (a_1, a_2)$  is an 2-tuple of integers  $a_1$  and  $a_2$  such that  $0 \leq a_1, a_2 \leq n$ .

Let  $U_{\mathbf{a}}^+ = U_{a_1} U_{a_2}$ ; where  $U_{a,k} = \prod_{m=1}^{a_k} (L_m - Q_k)$  for  $1 \leq k \leq 2$

Here  $Q_1 = 1, Q_2 = -1$  and  $L_m = s_{m-1} \dots s_1 s_0 s_1 \dots s_{m-1}$ .

**Definition 2.6.**([7]) Suppose that  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  is a bi-partition of  $n$  and define  $\mathbf{a} = (a_1, a_2)$  by  $a_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$ . Let  $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$  and set  $m_\lambda = U_{\mathbf{a}}^+ x_\lambda$ .

**Definition 2.7.**([7]) Suppose that  $\lambda$  is a bi-partition of  $n$  and that  $s$  and  $t$  are row standard  $\lambda$ -tableaux. Let  $m_{st} = T_{d(s)}^* m_\lambda T_{d(t)}$ .

**Definition 2.8.**([7]) Suppose that  $\lambda$  is a bi-partition of  $n$ .

1. Let  $N^\lambda$  be the  $R$ - module spanned by  $\{m_{st}/s$  and  $t$  are standard  $\mu$ - tableaux for some bi-partition  $\mu$  of  $n$  with  $\mu \trianglerighteq \lambda\}$ .
2. Let  $\overline{N^\lambda}$  be the  $R$ - module spanned by  $\{m_{st}/s$  and  $t$  are standard  $\mu$ - tableaux for some bi-partition  $\mu$  of  $n$  with  $\mu \triangleright \lambda\}$ .

**Proposition 2.9.**([7]) *Suppose that  $\lambda$  is a bi-partition of  $n$ . Then  $N^\lambda$  and  $\overline{N^\lambda}$  are two-sided ideals of an algebra  $\mathcal{H}$ , where  $\mathcal{H}$  is the Iwahori-Hecke algebra of type  $B$  and  $\mathcal{H} \cong k(\mathbb{Z}_2 \wr \Sigma_n)$ .*

**Theorem 2.10.** [7] The algebra  $\mathcal{H}$  is a free  $R$ -module with basis  $\mathcal{M} = \{m_{st}/s$  and  $t$  are standard  $\lambda$ -tableaux for some bi-partition of  $n\}$ . Moreover,  $\mathcal{M}$  is a cellular basis of  $\mathcal{H}$ .

**Definition 2.11.**([7]) Suppose that  $\lambda$  is a bi-partition of  $n$ . Let  $z_\lambda = (\overline{N^\lambda} + m_\lambda) / \overline{N^\lambda}$ . The **Specht module**  $S^\lambda$  of  $k(\mathbb{Z}_2 \wr \Sigma_n)$  is the submodule of  $\mathcal{H} / \overline{N^\lambda}$  given by  $S^\lambda = z_\lambda \mathcal{H}$ . Also  $S^\lambda$  is a free  $R$ - module with basis  $\{z_\lambda T_{d(t)} \mid t \text{ is a standard } \lambda\text{-tableaux}\}$ .

**Definition 2.12.**([8]) Let  $A$  be an associative algebra over the field  $K$ . The associative algebra  $A$  is called a **cellular algebra** with cell datum  $(\Lambda, M, C, i)$  if following conditions are satisfied:

1. The finite set  $\Lambda$  is partially ordered. Associated with each  $\lambda \in \Lambda$  there is a finite set  $M(\lambda)$ . The algebra  $A$  has an  $K$ - basis  $C_{S,T}^\lambda$  where  $(S, T)$  runs through all elements of  $M(\lambda) \times M(\lambda)$  for all  $\lambda \in \Lambda$ .
2. The map  $i$  is an  $K$ - linear anti-automorphism of  $A$  with  $i^2 = id$  which sends  $C_{S,T}^\lambda$  to  $C_{T,S}^\lambda$ .
3. For each  $\lambda \in \Lambda$  and  $S, T \in M(\lambda)$  and each  $a \in A$ , the product  $aC_{S,T}^\lambda$  can be written as  $\left(\sum_{U \in M(\lambda)} r_a(U, S)C_{U,T}^\lambda\right) + r'$ , where  $r'$  is a linear combination of basis elements with upper index  $\mu$  strictly smaller than  $\lambda$ , and where the coefficients  $r_a(U, S) \in K$  do not depend on  $T$ .

**Definition 2.13.**([11]) Let  $A$  be an algebra over a Noetherian commutative integral domain  $R$ . Assume there is an involution  $i$  on  $A$ . A two sided ideal  $J$  in  $A$  is called a cell ideal if and only if  $i(J) = J$  and there exists a left ideal  $\Delta \subset J$  such that  $\Delta$  is finitely generated and free over  $R$  and there is an isomorphism of  $A$ -bimodules  $\alpha : J \simeq \Delta \otimes_R i(\Delta)$  (where  $i(\Delta) \subset J$  is the  $i$ -image of  $\Delta$ ) making the following diagram commutative.

The algebra  $A$  (with the involution  $i$ ) is called cellular if and only if there is an  $R$ -module decomposition  $A = J'_1 \oplus J'_2 \oplus \dots \oplus J'_n$  (for some  $n$ ) with  $i(J'_j) = J'_j$  for some  $j$  and such that setting  $J_j = \bigoplus_{i=1}^j J'_i$  gives a chain of two-sided ideals of  $A$ :  $0 = J_0 \subset J_1 \subset J_2 \subset \dots \subset J_n = A$  (each of them fixed by  $i$ ) and for

each  $j(j = 1, \dots, n)$  the quotient  $J'_j = J_j/J_{j-1}$  is a cell ideal (with respect to the involution induced by  $i$  on the quotient) of  $A/J_{j-1}$ .

**Definition 2.14.** (Inflating algebras along free modules) ([12]) Given a  $k$ -algebra  $B$ , a  $k$ -vector space  $V$ , and a bilinear form

$\phi : V \otimes V \rightarrow B$  with values in  $B$ , we define an associative algebra (possibly without unit)  $A(B, V, \phi)$  as follows: As a  $k$ -vector space,  $A = V \otimes V \otimes B$ . The multiplication is defined on basis elements as follows:

$$(a \otimes b \otimes x)(c \otimes d \otimes y) = a \otimes d \otimes x \phi(b, c) y.$$

Assume that  $i$  is an involution on  $B$  with  $i(\phi(v, w)) = \phi(w, v)$  then we can define an involution  $j$  on  $A$  by putting

$$j(a \otimes b \otimes x) = b \otimes a \otimes i(x).$$

This definition makes  $A$  into a  $k$ -algebra (possibly without unit), and  $j$  is an involutory anti-automorphism of  $A$ . The algebra  $A(B, V, \phi)$  is an inflation of  $B$  along  $V$ .

**Proposition 2.15.** ([12]) *There exist an element  $b$  in  $B$  such that  $b+1(C)$  is a unit element in  $A$  if and only if  $b$  satisfies the following two equations.*

1. For all  $c$  in  $C$  there is an equality  $\delta(1, c) + \beta(b, c) = 0 = \delta(c, 1) + \gamma(c, b)$ .
2. For all  $d$  in  $B$  there are equalities  $(b-1)d = \gamma(1, d)$  and  $d(b-1) = \beta(d, 1)$ .

**Definition 2.16.** Inflating an algebra along another one ([12]) Suppose we are given an algebra  $B$  (may be without unit) and an algebra  $C$  (with unit). We define an algebra structure on  $A := B \oplus C$  which extends the given structures and which makes  $B$  into a two-sided ideal such that  $A/B$  becomes isomorphic to  $C$ . Multiplication is defined by fixing the eight summands of a multiplication map  $(B \oplus C) \otimes (B \oplus C) \rightarrow (B \oplus C)$ . In order to make  $B$  into an ideal we put the summands  $B \otimes B \rightarrow C$ ,  $C \otimes B \rightarrow C$  and  $B \otimes C \rightarrow C$  all to zero. The summands  $C \otimes C \rightarrow C$  and  $B \otimes B \rightarrow B$  are defined to be the given multiplication on  $C$  and  $B$ , respectively. Thus we have to choose three bilinear maps  $\delta : C \otimes C \rightarrow B$ ,  $\beta : B \otimes C \rightarrow B$  and  $\gamma : C \otimes B \rightarrow B$ . Then multiplication in  $A$  is defined by  $(b_1 + c_1)(b_2 + c_2) = b_1 b_2 + \beta(b_1, c_2) + \gamma(c_1, b_2) + \delta(c_1, c_2) + c_1 c_2$ .

This multiplication is associative if and only if the following conditions are satisfied:

1. The map  $\beta$  is a homomorphism of left  $B$ -module.
2. The map  $\gamma$  is a homomorphism of right  $B$ -module.
3. For all  $b$  in  $B$  and  $c_1, c_2$  in  $C$  there is an equality  $\beta(\beta(b, c_1), c_2) = b\delta(c_1, c_2) + \beta(b, c_1 c_2)$ .
4. For all  $b$  in  $B$  and  $c_1, c_2$  in  $C$  there is an equality  $\gamma(c_1, \gamma(c_2, b)) = \gamma(c_1 c_2, b) + \delta(c_1, c_2)b$ .
5. For all  $c_1, c_2, c_3$  in  $C$  there is an equality  $\delta(c_1 c_2, c_3) + \beta(\delta(c_1, c_2), c_3) = \delta(c_1, c_2 c_3) + \gamma(c_1, \delta(c_2, c_3))$ .

- 6. For all  $b_1, b_2$  in  $B$  and  $c$  in  $C$  there is an equality  $\beta(b_1, c)b_2 = b_1\gamma(c, b_2)$ .
- 7. For all  $c_1, c_2$  in  $C$  and  $b$  in  $B$  there is an equality  $\beta(\gamma(c_1, b), c_2) = \gamma(c_1, \beta(b, c_2))$ .

We call  $A$  an inflation of  $C$  along  $B$ . Moreover, an inductive application of this procedure to algebras  $C, B_1, B_2, \dots, B_n$  ensures that inflation pieces,  $B_i = V_i \otimes V_i \otimes B'_i$ , we define an iterated inflation  $A$  of  $C, B'_1, B'_2, \dots, B'_n$

**Proposition 2.17.**([12]) *An inflation of a cellular algebra is cellular again. In particular, an iterated inflation of  $n$  copies of  $R$  is cellular, with a cell chain of length  $n$ .*

**Theorem 2.18.**([12]) *Any cellular algebra over  $R$  is the iterated inflation of finitely many copies of  $R$ . Conversely, any iterated inflation of finitely many copies of  $R$  is cellular.*

**Definition 2.19.**([13]) Let  $A$  be a  $k$ -algebra. An ideal  $J$  in  $A$  is called a heredity ideal if  $J$  is idempotent,  $J(rad(A))J = 0$  and  $J$  is a projective left(or, right)  $A$ -module. The algebra  $A$  is called quasi-hereditary provided there is a finite chain  $0 = J_0 \subset J_1 \subset J_2 \subset \dots \subset J_n = A$  of ideals of  $A$  such that  $J_j/J_{j-1}$  is a heredity ideal in  $A/J_{j-1}$  for all  $j$ . Such a chain is then called a heredity chain of the quasi-hereditary algebra  $A$ .

### 3. Walled Signed Brauer Algebra

In this section, we define the Walled Signed Brauer Algebras and give a presentation of walled signed Brauer algebras in terms of generators and relations.

Fix an algebraically closed field  $k$  of characteristic  $p \geq 0$  and  $x$  an indeterminate. For  $r, s \in \mathbb{N}$ , the walled signed Brauer algebra  $\vec{D}_{r,s}(x)$  can be defined as a subalgebra of the signed Brauer algebra  $\vec{D}_{r+s}(x)$  in the following manner.

Recall that for  $n \in \mathbb{N}$ , the signed Brauer algebra  $\vec{D}_n(x)$  [14] is defined over the field  $k(x)$ , where  $k$  is any arbitrary field and  $x$  an indeterminate.

A graph is said to be a signed diagram if every edge is labeled by a plus sign or a minus sign and edges of a signed diagram are called signed edges. An edge labeled by a plus (resp., minus) sign will be called a positive (resp., negative) edge. A positive vertical (resp., horizontal) edge will be denoted by  $\downarrow$  (resp.,  $\rightarrow$ ) and a negative vertical (resp., horizontal) edge will be denoted by  $\uparrow$  (resp.,  $\leftarrow$ ).

Let  $\vec{V}_n$  be the set of all signed diagram  $\vec{b}$  with  $n$  signed edges and  $2n$  vertices, arranged in two rows of  $n$  vertices each. In these signed diagrams, each signed edge belongs to exactly two vertices, and each vertex belongs to exactly one signed edge.

The signed Brauer algebra  $\vec{D}_n$  is a vector space spanned by  $\vec{V}_n$  over  $k(x)$ .

The multiplication in  $\vec{D}_n$  is defined as follows: First, take the product of two undirected graphs  $a, b$ ; where  $\vec{a}, \vec{b}$  are signed diagrams as in [14]; that is, draw  $\vec{b}$  below  $\vec{a}$ , and connect  $i^{th}$  upper vertex of  $\vec{b}$  with the  $i^{th}$  lower vertex of  $\vec{a}$ . Then

$a.b = x^d c$ , where  $d$  is the number of loops in  $a.b$ , and  $c$  is the undirected graph. A new edge obtained in the product  $\vec{a}.\vec{b}$  is labeled by a plus sign or a minus sign according as the number of negative edges obtained from  $\vec{a}$  and  $\vec{b}$  to form this edge is even or odd.

A loop  $\beta$  in  $\vec{a}.\vec{b}$  is said to be positive ( resp., negative) if the number of negative edges obtained from  $\vec{a}$  and  $\vec{b}$  to form this loop is even(resp., odd). A positive (resp., negative) loop  $\beta$  in  $\vec{a}.\vec{b}$  is replaced by the variable  $x^2$  (resp.,  $x$ ) in  $\vec{a}.\vec{b}$ .

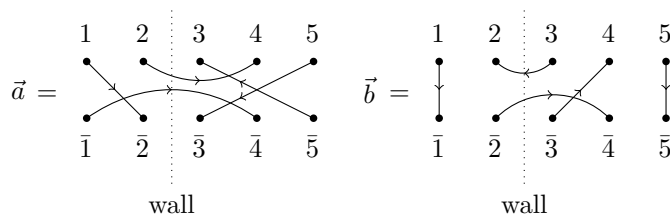
Now,  $\vec{c}$  is the signed diagram where each edge is labeled as above and  $\vec{a}.\vec{b} = x^d \vec{c}$ ,  $d$  is the number of loops in  $\vec{c}$ . Then  $\vec{a} \cdot \vec{b} = x^{(2d_1+d_2)} \vec{c}$ , where  $d_1$ (resp.,  $d_2$ ) is the number of positive (resp., negative) loops in  $\vec{a} \cdot \vec{b}$ .

It is usual to represent basis elements graphically by means of diagrams with  $n$  upper vertices numbered 1 to  $n$  from left to right; and  $n$  lower vertices numbered  $\bar{1}$  to  $\bar{n}$  from left to right, where each vertex is connected to precisely one other by a signed edge. Edges connecting a upper vertex and a lower vertex are called propagating lines, and the remainder are called upper or lower horizontal arcs.

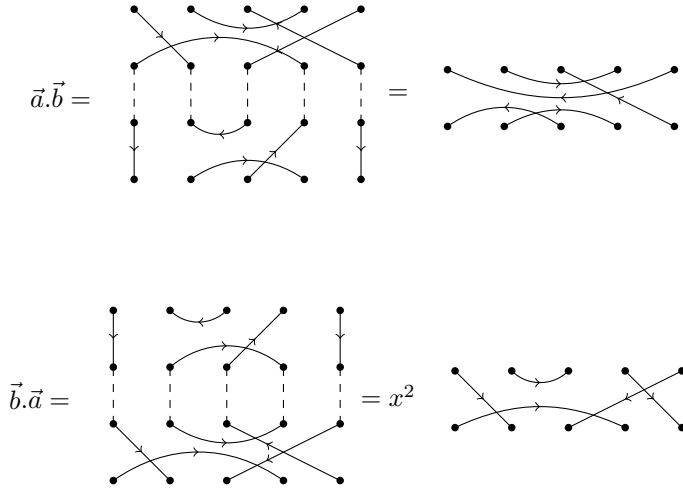
Partition the basis diagrams with the wall separating the first  $r$  upper vertices and the first  $r$  lower vertices from the reminder then the *Walled Signed Brauer Algebra*  $\vec{D}_{r,s}(x)$  is the subalgebra of  $\vec{D}_{r+s}(x)$  with basis those signed diagrams such that no propagating edge crosses the wall and every upper or lower horizontal arc does cross the wall. We call those diagrams as walled signed Brauer diagrams.

**Remark 3.1** If we allow vertical edges can cross the wall and allow horizontal edges may not cross the wall (that is, a vertex can be connected to any other vertex), then we obtain  $r + s$  signed diagram. The signed Brauer algebra  $\vec{D}_{r+s}(x)$  is spanned by all  $r + s$  signed diagrams with product defined as above. Thus walled signed Brauer diagram is a signed diagram and walled signed Brauer algebra  $\vec{D}_{r,s}(x)$  is a subalgebra of the signed Brauer algebra  $\vec{D}_{r+s}(x)$ .

For example,



$\vec{a}$  and  $\vec{b}$  are basis elements in  $\vec{D}_{2,3}(x)$  and the multiplication of  $\vec{a}$  and  $\vec{b}$  are given by



It is useful to compare  $\vec{D}_{r,s}(x)$  with the group algebra  $k(\mathbb{Z}_2 \wr S_{r+s})$  of the hyperoctahedral group  $(\mathbb{Z}_2 \wr S_{r+s})$ ; where  $S_{r+s}$  is a symmetric group of  $(r + s)$  symbols and  $\mathbb{Z}_2$  is a group consisting of two elements. The group algebra  $k(\mathbb{Z}_2 \wr S_{r+s})$  can be viewed diagrammatically with its signed diagram with no horizontal edge.

We define a map

$$f_{r,s} : k(\mathbb{Z}_2 \wr S_{r+s}) \longrightarrow \vec{D}_{r,s}(x)$$

by mapping a signed diagram with no horizontal edge to the walled signed Brauer diagram obtained by adding a wall between the  $r^{th}$  and  $(r + 1)^{th}$  vertices, then flipping the part of the diagram that is to the right of the wall in its horizontal axis without disconnecting any edges and without changing the sign also.

The map  $f_{r,s}$  is a vector space isomorphism.

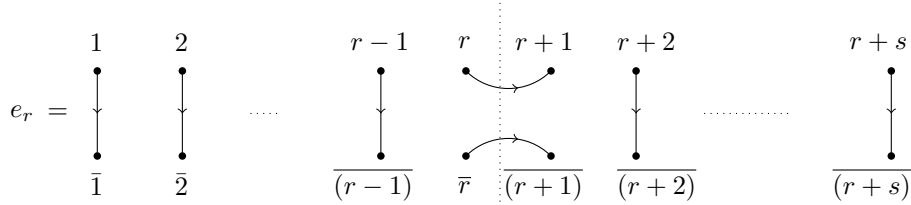
$$\dim(\vec{D}_{r,s}(x)) = \dim(K(\mathbb{Z}_2 \wr S_{r+s})) = 2^{r+s}(r + s)!$$

### 3.1. Generators, relations of the walled signed Brauer algebra

The algebra  $G_n := K(\mathbb{Z}_2 \wr S_{r+s})$  is generated by  $t = \vec{h}_1$  and the transpositions  $s_i := (i, i + 1)$  for  $i = 1, 2, \dots, (r + s - 1)$  subject to the relations (usual hyperoctahedral relations).

1.  $t^2 = 1$
2.  $s_i^2 = 1$ ;  $1 \leq i \leq r + s - 1$
3.  $ts_1ts_1 = s_1ts_1t$
4.  $s_i s_j = s_j s_i$ ;  $|i - j| \geq 2$
5.  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ ;  $1 \leq i \leq r + s - 1$
6.  $s_i t = t s_i$ ;  $2 \leq i \leq r + s - 1$

Let  $g_0 = f_{r,s}(t) = f_{r,s}(h_1) = h_1 = t$  and for  $i = 1, 2, \dots, (r - 1), (r + 1), \dots, (r + s - 1)$ ,  $g_i = f_{r,s}(s_i) = s_i$ .  
 So for  $i \neq r$  the diagram of  $g_i$  is the same as  $s_i$  (with the addition of the wall).  
 while  $g_r = e_r$  is the diagram,



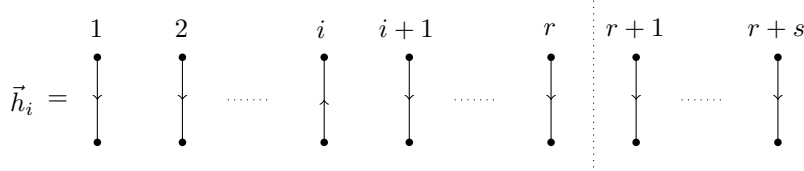
**Theorem 3.2.** *The walled signed Brauer algebra  $\vec{D}_{r,s}(x)$  is generated by the elements  $\vec{h}_1, \vec{h}_{r+1}, g_1, g_2, \dots, g_{r-1}, e_r, g_{r+1}, \dots, g_{r+s-1}$  and satisfying the following relations:*

1.  $g_i^2 = 1; \quad i = 1, 2, \dots, (r - 1), (r + 1), \dots, (r + s - 1).$
2.  $g_i g_j = g_j g_i \quad \text{if } |i - j| > 1$
3.  $g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i$
4.  $e_r g_i = g_i e_r \quad 1 \leq i \leq r - 2 \text{ or } r + 2 \leq i \leq r + s - 1$
5.  $e_r^2 = x^2 e_r$
6.  $e_r g_{r-1} e_r = e_r$
7.  $e_r g_{r+1} e_r = e_r$
8.  $g_{r-1} g_{r+1} e_r g_{r-1} g_{r+1} e_r = e_r g_{r-1} g_{r+1} e_r$
9.  $e_r g_{r-1} g_{r+1} e_r g_{r-1} g_{r+1} = e_r g_{r-1} g_{r+1} e_r$
10.  $\vec{h}_i^2 = 1; \quad i = 1, 2, \dots, (r + s).$
11.  $\vec{h}_1 g_i = g_i \vec{h}_1; \quad i \neq 1$
12.  $\vec{h}_1 g_1 \vec{h}_1 g_1 = g_1 \vec{h}_1 g_1 \vec{h}_1$
13.  $\vec{h}_{r+1} g_i = g_i \vec{h}_{r+1}; \quad i \neq r, (r + 1)$
14.  $\vec{h}_{r+1} g_{r+1} \vec{h}_{r+1} g_{r+1} = g_{r+1} \vec{h}_{r+1} g_{r+1} \vec{h}_{r+1}$
15.  $e_r \vec{h}_r e_r = x e_r \quad \text{and } e_r \vec{h}_{r+1} e_r = x e_r$
16.  $g_i \vec{h}_{i+1} = \vec{h}_i g_i; \quad i \neq r$
17.  $e_r \vec{h}_{r+1} = e_r \vec{h}_r$
18.  $\vec{h}_{r+1} e_r = \vec{h}_r e_r$

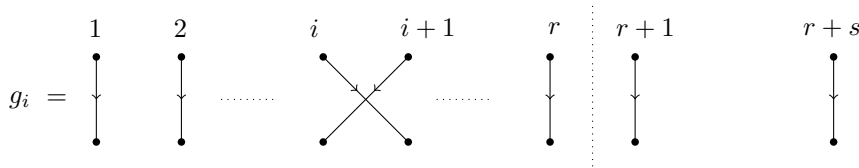


- 19.  $e_r \vec{h}_i = \vec{h}_i e_r; \quad i \neq r, (r + 1)$
- 20.  $e_r \vec{h}_r g_{r+1} e_r = e_r \vec{h}_{r+2} \quad \text{and} \quad e_r \vec{h}_{r+1} g_{r+1} e_r = e_r \vec{h}_{r+2}$

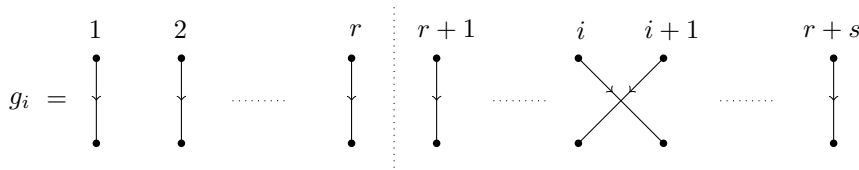
where,



For  $1 \leq i \leq r - 1$



For  $r + 1 \leq i \leq r + s - 1$



*Proof.* The walled Brauer algebra  $D_{r,s}(x^2)$  is generated by  $g_1, g_2, \dots, g_{r-1}, e_r, g_{r+1}, \dots, g_{r+s-1}$ , with the relation from (1) to (9), there exists a unique algebra homomorphism,

$\Phi' : D_{r,s}(x^2) \rightarrow A$ , such that  $\Phi'(g_1), \dots, \Phi'(g_{r-1}), \Phi'(e_r), \Phi'(g_{r+1}), \dots, \Phi'(g_{r+s-1})$  satisfy the relation from (1) to (9), where  $A$  is the free associated algebra over  $k$  and is generated by  $x_1, x_{r+1}, y_1, y_2, \dots, y_{r-1}, z_r, y_{r+1}, \dots, y_{r+s-1}$  satisfying the relations from (1) to (20), where  $\Phi'(g_i) = y_i, i = 1, 2, \dots, r - 1, r + 1, \dots, r + s - 1$  and  $\Phi'(e_r) = z_r$ .

Similarly the generators  $\vec{h}_1, \vec{h}_{r+1}, g_1, g_2, \dots, g_{r-1}, g_{r+1}, \dots, g_{r+s-1}$  satisfy the hyperoctahedral relations of the hyperoctahedral groups  $\mathbb{Z}_2 \wr \Sigma_r$  and  $\mathbb{Z}_2 \wr \Sigma_s$ , there exists a unique algebra homomorphism,

$\Phi'' : k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s) \rightarrow A$ , such that  $\Phi''(\vec{h}_1), \Phi''(\vec{h}_{r+1}), \Phi''(g_1), \dots, \Phi''(g_{r-1}), \Phi''(g_{r+1}), \dots, \Phi''(g_{r+s-1})$  satisfy the hyperoctahedral relations and  $\Phi''|_{k(\Sigma_r \times \Sigma_s)} = \Phi''|_{k(\Sigma_r \times \Sigma_s)}$  and  $\Phi''(\vec{h}_1) = x_1, \Phi''(\vec{h}_{r+1}) = x_{r+1}$ , and  $\Phi''(g_i) = y_i$

First we prove the following for a signed walled Brauer diagram  $d \in D_{r,s}(x^2)$

- (a)  $\Phi'(d).\Phi''(\vec{h}_i).\Phi''(\vec{h}_j) = \Phi'(d)$  whenever  $d\vec{h}_i\vec{h}_j = d$ ,  $1 \leq i \leq r$  and  $r + 1 \leq j \leq r + s$
- (b)  $\Phi''(\vec{h}_i).\Phi''(\vec{h}_j).\Phi'(d) = \Phi'(d)$  whenever  $\vec{h}_i\vec{h}_jd = d$ ,  $1 \leq i \leq r$  and  $r + 1 \leq j \leq r + s$
- (c)  $\Phi''(\vec{h}_s).\Phi'(d).\Phi''(\vec{h}_t) = \Phi'(d)$  whenever  $\vec{h}_sd\vec{h}_t = d$ , either  $1 \leq s, t \leq r$  or  $r + 1 \leq s, t \leq r + s$ .

For  $1 \leq i \leq r$  and  $r + 1 \leq j \leq r + s$ ,

Let  $e_{i,j} = g_{j-1}g_{j-2} \dots g_{r+1}g_i \dots g_{r-1}e_r g_{r-1} \dots g_i g_{r+1} \dots g_{j-1}$   
 then  $e_{r,r+1} = e_r$ .

$$\begin{aligned} e_{i,j}\vec{h}_i\vec{h}_j &= (g_{j-1} \dots g_{r+1}g_i \dots g_{r-1}e_r g_{r-1} \dots g_i g_{r+1} \dots g_{j-1})\vec{h}_i\vec{h}_j \\ &= g_{j-1} \dots g_{r+1}g_i \dots g_{r-1}e_r \vec{h}_r \vec{h}_{r+1} g_{r-1} \dots g_i g_{r+1} \dots g_{j-1}, \text{ by identity(15)} \\ &= g_{j-1} \dots g_{r+1}g_i \dots g_{r-1}e_r g_{r-1} \dots g_i g_{r+1} \dots g_{j-1}, \text{ by identity(17)} \\ &= e_{i,j}. \end{aligned}$$

Consider,  $\Phi'(d).\Phi''(\vec{h}_i).\Phi''(\vec{h}_j)$   
 $= \Phi'(g_{j-1} \dots g_{r+1}g_i \dots g_{r-1}e_r g_{r-1} \dots g_i g_{r+1} \dots g_{j-1})\Phi''(\vec{h}_i).\Phi''(\vec{h}_j)$   
 since  $\Phi'$  is an algebra homomorphism on  $D_{r,s}(x^2)$  and  $\Phi' |_{k(\Sigma_r \times \Sigma_s)} = \Phi'' |_{k(\Sigma_r \times \Sigma_s)}$ ,  
 $\Phi'(d).\Phi''(\vec{h}_i).\Phi''(\vec{h}_j) = \Phi'(g_{j-1} \dots g_{r+1}g_i \dots g_{r-1})\Phi'(e_r)\Phi''(g_{r-1} \dots g_i g_{r+1} \dots g_{j-1})$   
 $\Phi''(\vec{h}_i).\Phi''(\vec{h}_j)$   
 since  $\Phi''$  is an algebra homomorphism on  $k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s)$  and using the identity (15),

$$\begin{aligned} &\Phi'(d).\Phi''(\vec{h}_i).\Phi''(\vec{h}_j) \\ &= \Phi'(g_{j-1} \dots g_{r+1}g_i \dots g_{r-1})\Phi'(e_r)\Phi''(\vec{h}_r \vec{h}_{r+1} g_{r-1} \dots g_i g_{r+1} \dots g_{j-1}) \\ &= \Phi'(g_{j-1} \dots g_i \dots g_{r-1})\Phi'(e_r)\Phi''(\vec{h}_r)\Phi''(\vec{h}_{r+1})\Phi'(g_{r-1} \dots g_i g_{r+1} \dots g_{j-1}) \\ &= \Phi'(g_{j-1} \dots g_i \dots g_{r-1})\Phi'(e_r)\Phi'(g_{r-1} \dots g_i g_{r+1} \dots g_{j-1}), \text{ by identity(17)} \\ &= \Phi'(d). \end{aligned}$$

Hence (a) holds for  $d = e_{i,j}; 1 \leq i \leq r$  and  $r + 1 \leq j \leq r + s$ .

If  $d\vec{h}_i\vec{h}_j = d; 1 \leq i \leq r$  and  $r + 1 \leq j \leq r + s$   
 then  $d$  has a horizontal edge connecting the vertices  $i$  and  $j$ , we can write  $d$  as,  
 $d = d'e_{i,j}$

$$\begin{aligned} \Phi'(d)\Phi''(\vec{h}_i)\Phi''(\vec{h}_j) &= \Phi'(d')\Phi'(e_{i,j})\Phi''(\vec{h}_i)\Phi''(\vec{h}_j) \\ &= \Phi'(d')\Phi'(e_{i,j}), \text{ by the previous result,} \\ &= \Phi'(d) \end{aligned}$$

which proves (a).(b) can also be proved similarly.

For (c),  $d \in D_{r,s}(x^2)$  with  $\vec{h}_sd\vec{h}_t = d$   
 then  $d$  has an edge connecting the  $s^{th}$  vertex in the top row and  $t^{th}$  vertex in the bottom row.

Take  $1 \leq s, t \leq r$  and  $d$  has  $k$  horizontal edges,  $d$  can be written as,  $d = d_\sigma \cdot e_{i_1, j_1} e_{i_2, j_2} \dots e_{i_k, j_k}$ , for some  $\sigma \in \Sigma_r \times \Sigma_s$ , where  $1 \leq i_1 < i_2 < \dots < i_k \leq r$ ,  $r + 1 \leq j_1, j_2, \dots, j_k \leq r + s$  and  $j_i$ 's are all different and  $d_\sigma$  has an edge connecting the  $s^{th}$  vertex in the top row and  $t^{th}$  vertex in the bottom row [details in 21].

$$\begin{aligned} \vec{h}_s d \vec{h}_t &= \vec{h}_s (d_\sigma \cdot e_{i_1, j_1} e_{i_2, j_2} \dots e_{i_k, j_k}) \vec{h}_t \\ &= \vec{h}_s d_\sigma \vec{h}_t e_{i_1, j_1} e_{i_2, j_2} \dots e_{i_k, j_k}, \text{ since } t \neq i_p, j_q \text{ for } p, q = 1, 2, \dots, k \\ &= d, \text{ since } \vec{h}_s d_\sigma \vec{h}_t = d_\sigma \end{aligned}$$

$\Phi''(\vec{h}_s) \Phi'(d) \Phi''(\vec{h}_t) = \Phi''(\vec{h}_s) \Phi'(d_\sigma \cdot e_{i_1, j_1} e_{i_2, j_2} \dots e_{i_k, j_k}) \Phi''(\vec{h}_t)$  since  $\Phi'$  is an algebra homomorphism on  $D_{r,s}(x^2)$  and using the result,  $\Phi'(e_{i,j}) \Phi''(\vec{h}_k) = \Phi''(\vec{h}_k) \Phi'(e_{i,j})$  if  $k \neq i, j$  (This result has been proved below),

$$\begin{aligned} \Phi''(\vec{h}_s) \Phi'(d) \Phi''(\vec{h}_t) &= \Phi''(\vec{h}_s) \Phi'(d_\sigma) \Phi''(\vec{h}_t) \Phi'(e_{i_1, j_1}) \Phi'(e_{i_2, j_2}) \dots \Phi'(e_{i_k, j_k}) \\ &= \Phi'(d), \text{ since } \Phi''(\vec{h}_s) \Phi'(d_\sigma) \Phi''(\vec{h}_t) = \Phi'(d_\sigma), \end{aligned}$$

which proves (c).

Similarly, (c) holds for  $r + 1 \leq s, t \leq r + s$ .

Now, let  $\vec{d} \in \vec{D}_{r,s}(x)$  then there exists  $\vec{h}, \vec{h}' \in H$  such that  $\vec{d} = \vec{h} d \vec{h}'$ , where  $d$  is the underlying walled signed Brauer diagram of  $\vec{d}$  and  $H$  is a subgroup of  $\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s$  generated by  $h_i, 1 \leq i \leq r + s$ .

Suppose that there exists another elements  $\vec{h}'', \vec{h}''' \in H$  such that  $\vec{d} = \vec{h}'' d \vec{h}'''$ . i.e.,  $\vec{h} \vec{h}'' d \vec{h}''' \vec{h}' = d$ . By the observations made from (a) to (c),

$$\begin{aligned} \Phi''(\vec{h} \vec{h}'') \Phi'(d) \Phi''(\vec{h}''' \vec{h}') &= \Phi'(d). \\ \Phi''(\vec{h}'') \Phi'(d) \Phi''(\vec{h}''') &= \Phi''(\vec{h}'') \Phi'(d) \Phi''(\vec{h}'). \end{aligned}$$

So, define  $\Phi : \vec{D}_{r,s}(x) \rightarrow A$  by  $\Phi(\vec{d}) = \Phi''(\vec{h}) \Phi'(d) \Phi''(\vec{h}')$ .

It is immediate that  $\Phi = \Phi'$  on  $D_{r,s}(x^2)$ ,  $\Phi = \Phi''$  on  $k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s)$ . Extend it to the whole space  $\vec{D}_{r,s}(x)$  by linearity property.

First, we shall show that

$$(3.1.1) \quad \Phi(e_{i,j} \vec{h} e_{l,m}) = \Phi'(e_{i,j}) \Phi''(\vec{h}) \Phi'(e_{l,m})$$

where  $e_{i,j} = \prod_p e_{i_p, j_p}$ ,  $e_{l,m} = \prod_q e_{l_q, m_q}$ ,  $\vec{h} = \prod_n \vec{h}_{k_n}$ .

The following identities can be established immediately from the definition of the multiplication of walled signed Brauer diagrams.

For  $1 \leq i, l \leq r, r + 1 \leq j, m \leq r + s$ , and  $1 \leq k \leq r + s$ .

$$\begin{aligned}
(3.1.2) \quad \vec{h}_k e_{l,m} &= e_{l,m} \vec{h}_k \text{ if } k \neq l, m \\
e_{i,j} \vec{h}_k e_{l,m} &= e_{i,j} e_{l,m} \vec{h}_k, \text{ if } k \neq l, m \\
&= \vec{h}_k e_{i,j} e_{l,m}, \text{ if } k \neq i, j \\
&= e_{i,k} e_{l,m} \vec{h}_i, \text{ if } j = k, m \neq k, i \neq l \\
&= \vec{h}_m e_{i,j} e_{k,m}, \text{ if } k = l, i \neq k, j \neq m \\
&= x e_{i,j}, \text{ if either } i = k = l, j = m \text{ or } k = j = m, i = l \\
g_{i-1} e_{i,j} g_{i-1} &= g_{j-1} e_{i-1, j-1} g_{j-1}, \quad |i - j| \geq 2.
\end{aligned}$$

By using induction, we shall prove that  $\Phi$  preserves the relations (3.1.2).

Let  $1 \leq i, l \leq r$ ,  $r + 1 \leq j, m \leq r + s$ , and  $1 \leq k \leq r + s$ .

Assume that  $k \neq l, m$  then

$$\begin{aligned}
\Phi(\vec{h}_k) \Phi(e_{l,m}) &= \Phi''(\vec{h}_k) \Phi'(e_{l,m}) \\
&= \Phi''(\vec{h}_k) \Phi'(g_{l-1}) \Phi'(g_{m-1}) \Phi'(e_{l-1, m-1}) \Phi'(g_{m-1}) \Phi'(g_{l-1}), \text{ by (3.1.2)}.
\end{aligned}$$

case(1):  $k \neq l - 1, m - 1$

$$\begin{aligned}
\Phi(\vec{h}_k) \Phi(e_{l,m}) &= \Phi'(g_{l-1}) \Phi'(g_{m-1}) \Phi''(\vec{h}_k) \Phi'(e_{l-1, m-1}) \Phi'(g_{m-1}) \Phi'(g_{l-1}), \\
&= \Phi'(g_{l-1}) \Phi'(g_{m-1}) \Phi'(e_{l-1, m-1}) \Phi''(\vec{h}_k) \Phi'(g_{m-1}) \Phi'(g_{l-1}), \text{ by in-} \\
\text{duction,} & \\
&= \Phi'(e_{l,m}) \Phi''(\vec{h}_k) = \Phi(e_{l,m}) \Phi(\vec{h}_k).
\end{aligned}$$

case(2):  $k = l - 1$

$$\begin{aligned}
\Phi(\vec{h}_k) \Phi(e_{l,m}) &= \Phi'(g_{l-1}) \Phi''(\vec{h}_l) \Phi'(g_{m-1}) \Phi'(e_{l-1, m-1}) \Phi'(g_{m-1}) \Phi'(g_{l-1}), \\
&= \Phi'(g_{l-1}) \Phi'(g_{m-1}) \Phi''(\vec{h}_l) \Phi'(e_{l-1, m-1}) \Phi'(g_{m-1}) \Phi'(g_{l-1}), \\
&= \Phi'(g_{l-1}) \Phi'(g_{m-1}) \Phi'(e_{l-1, m-1}) \Phi''(\vec{h}_l) \Phi'(g_{m-1}) \Phi'(g_{l-1}), \text{ by in-} \\
\text{duction,} & \\
&= \Phi'(e_{l,m}) \Phi''(\vec{h}_k) = \Phi(e_{l,m}) \Phi(\vec{h}_k).
\end{aligned}$$

Similarly, the result holds for  $k = m - 1$ .

From this result, we get,

$$\begin{aligned}
\Phi(e_{i,j}) \Phi(\vec{h}_k) \Phi(e_{l,m}) &= \Phi(e_{i,j}) \Phi(e_{l,m}) \Phi(\vec{h}_k), \text{ if } k \neq l, m \\
\Phi(e_{i,j}) \Phi(\vec{h}_k) \Phi(e_{l,m}) &= \Phi(\vec{h}_k) \Phi(e_{i,j}) \Phi(e_{l,m}), \text{ if } k \neq i, j
\end{aligned}$$

Assume that  $j = k, m \neq k, i \neq l$ , then

$$\begin{aligned}
\Phi(e_{i,j}) \Phi(\vec{h}_k) \Phi(e_{l,m}) &= \Phi'(e_{i,k}) \Phi''(\vec{h}_k) \Phi'(e_{l,m}), \\
&= \Phi'(g_{i-1}) \Phi'(g_{k-1}) \Phi'(e_{i-1, k-1}) \Phi'(g_{k-1}) \Phi'(g_{i-1}) \Phi''(\vec{h}_k) \\
&\quad \Phi'(g_{l-1}) \Phi'(g_{m-1}) \Phi'(e_{l-1, m-1}) \Phi'(g_{m-1}) \Phi'(g_{l-1}), \\
&= \Phi'(g_{i-1}) \Phi'(g_{k-1}) \Phi'(e_{i-1, k-1}) \Phi''(\vec{h}_{k-1}) \Phi'(g_{k-1}) \Phi'(g_{i-1}) \\
&\quad \Phi'(g_{l-1}) \Phi'(g_{m-1}) \Phi'(e_{l-1, m-1}) \Phi'(g_{m-1}) \Phi'(g_{l-1}), \\
&= \Phi'(g_{i-1}) \Phi'(g_{k-1}) \Phi'(g_{l-1}) \Phi'(g_{m-1}) \Phi'(e_{i-1, k-1}) \Phi''(\vec{h}_{k-1}) \\
&\quad \Phi'(e_{l-1, m-1}) \Phi'(g_{k-1}) \Phi'(g_{i-1}) \Phi'(g_{m-1}) \Phi'(g_{l-1}),
\end{aligned}$$

$$\begin{aligned}
 \Phi(e_{i,j})\Phi(\vec{h}_k)\Phi(e_{l,m}) &= \Phi'(g_{i-1})\Phi'(g_{k-1})\Phi'(g_{l-1})\Phi'(g_{m-1})\Phi'(e_{i-1,k-1}) \\
 &\quad \Phi'(e_{l-1,m-1})\Phi''(\vec{h}_{i-1})\Phi'(g_{k-1})\Phi'(g_{i-1})\Phi'(g_{m-1})\Phi'(g_{l-1}), \\
 &= \Phi'(g_{i-1})\Phi'(g_{k-1})\Phi'(g_{l-1})\Phi'(g_{m-1})\Phi'(e_{i-1,k-1}) \\
 &\quad \Phi'(e_{l-1,m-1})\Phi'(g_{k-1})\Phi'(g_{i-1})\Phi''(\vec{h}_i)\Phi'(g_{m-1})\Phi'(g_{l-1}), \\
 &= \Phi'(e_{i,k})\Phi'(e_{l,m})\Phi''(\vec{h}_i), \\
 &= \Phi(e_{i,k})\Phi(e_{l,m})\Phi(\vec{h}_i).
 \end{aligned}$$

The following identity can be proved in the similar way,  
 $\Phi(e_{i,j})\Phi(\vec{h}_k)\Phi(e_{l,m}) = \Phi(\vec{h}_m)\Phi(e_{i,j})\Phi(e_{l,m})$  if  $k = l, i \neq k, j \neq m$ .

Now, assume that  $i = k = l$  and  $j = m$ , then

$$\begin{aligned}
 \Phi(e_{i,j})\Phi(\vec{h}_k)\Phi(e_{l,m}) &= \Phi'(e_{i,j})\Phi''(\vec{h}_i)\Phi'(e_{i,j}), \\
 &= \Phi'(g_{i-1})\Phi'(g_{j-1})\Phi'(e_{i-1,j-1})\Phi'(g_{j-1})\Phi'(g_{i-1})\Phi''(\vec{h}_i) \\
 &\quad \Phi'(g_{i-1})\Phi'(g_{j-1})\Phi'(e_{i-1,j-1})\Phi'(g_{j-1})\Phi'(g_{i-1}), \\
 \Phi(e_{i,j})\Phi(\vec{h}_k)\Phi(e_{l,m}) &= \Phi'(g_{i-1})\Phi'(g_{j-1})\Phi'(e_{i-1,j-1})\Phi''(\vec{h}_i)\Phi'(e_{i-1,j-1})\Phi'(g_{j-1}) \\
 &\quad \Phi'(g_{i-1}), \\
 &= \Phi'(g_{i-1})\Phi'(g_{j-1})\Phi'(e_{i-1,j-1})\Phi'(g_{j-1})\Phi'(g_{i-1}), \\
 &= x\Phi'(e_{i,j}) = x\Phi(e_{i,j})
 \end{aligned}$$

Now,

$$\begin{aligned}
 e_{i,j}\vec{h}e_{l,m} &= \prod_p e_{i_p,j_p} \prod_n \vec{h}_{k_n} \prod_q e_{l_q,m_q} \\
 &= \lambda \vec{h}' \prod_p e_{i_p,j_p} \prod_q e_{l_q,m_q} \vec{h}'', \lambda \in k(x), \text{ from (3.1.2)}
 \end{aligned}$$

$$\begin{aligned}
 \Phi(e_{i,j})\Phi(\vec{h})\Phi(e_{l,m}) &= \Phi'(e_{i,j})\Phi''(\vec{h})\Phi'(e_{l,m}), \\
 &= \prod_p \Phi'(e_{i_p,j_p}) \prod_n \Phi''(\vec{h}_{k_n}) \prod_q \Phi'(e_{l_q,m_q}), \\
 &\quad \text{since } \Phi' \text{ and } \Phi'' \text{ are homomorphism,} \\
 &= \lambda \Phi''(\vec{h}') \prod_p \Phi'(e_{i_p,j_p}) \prod_q \Phi'(e_{l_q,m_q})\Phi''(\vec{h}''), \\
 &\quad \text{since } \Phi \text{ preserves the relations of (3.1.2) and } \vec{h}', \vec{h}'' \in H, \\
 &= \Phi(e_{i,j}\vec{h}e_{l,m}),
 \end{aligned}$$

proving our claim.

Let  $\vec{d}_1, \vec{d}_2 \in \vec{D}_{r,s}(x)$  then there exist  $\vec{h}^{(i)}, \vec{h}^{(i')}, i = 1, 2 \in H$  such that  $\vec{d}_1 = \vec{h}^{(1)}d_1\vec{h}^{(1)'}$ ,  $\vec{d}_2 = \vec{h}^{(2)}d_2\vec{h}^{(2)'}$ , where  $d_1, d_2$  are the underlying walled Brauer diagrams.

Since  $d_1 = d_{\sigma_1} \cdot e_{i_1,j_1} e_{i_2,j_2} \dots e_{i_k,j_k}$ , for some  $\sigma_1 \in \Sigma_r \times \Sigma_s$ , where  $k$  is the number of horizontal edges in  $d_1$ ,  $1 \leq i_1 < i_2 < \dots < i_k \leq r$ ,

$r + 1 \leq j_1, j_2, \dots, j_k \leq r + s$  and  $j_i$ 's are all different and  $d_2 = e_{l_1, m_1} e_{l_2, m_2} \dots e_{l_{k'}, m_{k'}} d_{\sigma_2}$ , for some  $\sigma_2 \in \Sigma_r \times \Sigma_s$ , where  $k'$  is the number of horizontal edges in  $d_2$ ,  $1 \leq l_1 < l_2 < \dots < l_{k'} \leq r$ ,  $r + 1 \leq m_1, m_2, \dots, m_{k'} \leq r + s$  and  $m_i$ 's are all different.

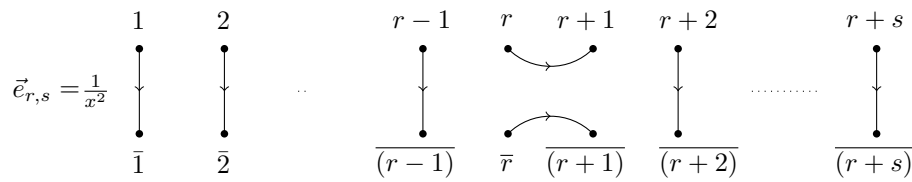
$$\begin{aligned} \Phi(\vec{d}_1, \vec{d}_2) &= \Phi(\vec{h}^{(1)} d_{\sigma_1} e_{i_1, j_1} e_{i_2, j_2} \dots e_{i_k, j_k} \vec{h}^{(1)'} \vec{h}^{(2)} e_{l_1, m_1} e_{l_2, m_2} \dots e_{l_{k'}, m_{k'}} d_{\sigma_2} \vec{h}^{(2)'}) \\ &= \Phi(\vec{h}^{(1)} d_{\sigma_1} \vec{h}^{(3)} e_{i_1, j_1} e_{i_2, j_2} \dots e_{i_k, j_k} e_{l_1, m_1} e_{l_2, m_2} \dots e_{l_{k'}, m_{k'}} \vec{h}^{(3)'} d_{\sigma_2} \vec{h}^{(2)'}) \\ &\quad \text{where, } e_{i_1, j_1} \dots e_{i_k, j_k} \vec{h}^{(1)'} \vec{h}^{(2)} e_{l_1, m_1} \dots e_{l_{k'}, m_{k'}} \\ &\quad = \vec{h}^{(3)} e_{i_1, j_1} \dots e_{i_k, j_k} e_{l_1, m_1} \dots e_{l_{k'}, m_{k'}} \vec{h}^{(3)'}, \\ &= \Phi''(\vec{h}^{(1)} d_{\sigma_1} \vec{h}^{(3)}) \Phi'(e_{i_1, j_1} e_{i_2, j_2} \dots e_{i_k, j_k} e_{l_1, m_1} e_{l_2, m_2} \dots e_{l_{k'}, m_{k'}}) \\ &\quad \Phi''(\vec{h}^{(3)'} d_{\sigma_2} \vec{h}^{(2)'}) \\ &= \Phi''(\vec{h}^{(1)}) \Phi'(d_{\sigma_1}) \Phi''(\vec{h}^{(3)}) \Phi'(e_{i_1, j_1} e_{i_2, j_2} \dots e_{i_k, j_k}) \\ &\quad \Phi'(e_{l_1, m_1} e_{l_2, m_2} \dots e_{l_{k'}, m_{k'}}) \Phi''(\vec{h}^{(3)'}) \Phi'(d_{\sigma_2}) \Phi''(\vec{h}^{(2)'}) \\ &= \Phi''(\vec{h}^{(1)}) \Phi'(d_{\sigma_1}) \Phi'(e_{i_1, j_1} e_{i_2, j_2} \dots e_{i_k, j_k}) \Phi''(\vec{h}^{(1)'}) \Phi''(\vec{h}^{(2)'}) \\ &\quad \Phi'(e_{l_1, m_1} e_{l_2, m_2} \dots e_{l_{k'}, m_{k'}}) \Phi'(d_{\sigma_2}) \Phi''(\vec{h}^{(2)'}) \text{, from(3.1.1) and(3.1.3)} \\ &= \Phi(\vec{d}_1) \Phi(\vec{d}_2), \end{aligned}$$

hence  $\bar{\Phi}$  is an algebra homomorphism. □

#### 4. Cellularity of Walled Signed Brauer Algebra

In this section, We will show that the walled signed Brauer algebras  $\vec{D}_{r,s}(x)$  form cellular analogous of the towers of recollement introduced in [5] and  $\vec{D}_{r,s}(x)$  are quasi-hereditary.

Suppose that  $k$  is arbitrary with  $r, s > 0$  and  $x \neq 0$  and let  $\vec{e}_{r,s} \in \vec{D}_{r,s}(x)$  be  $x^{-2}$  times the diagram with one positive upper horizontal arc connecting  $r$  and  $r + 1$ , one positive lower horizontal arc connecting  $\bar{r}$  and  $\bar{r} + 1$  and all the remaining edges being positive propagating lines from  $i$  to  $\bar{i}$ .

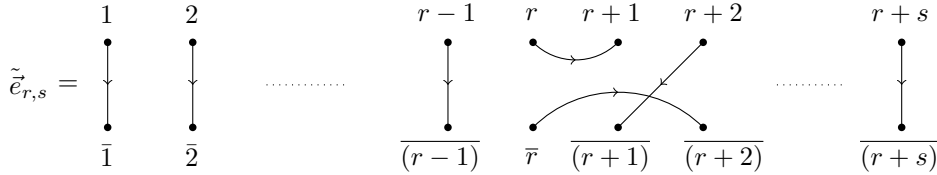


$$(\vec{e}_{r,s})^2 = \vec{e}_{r,s}.$$

$\vec{e}_{r,s}$  is an idempotent in  $\vec{D}_{r,s}(x)$ .

If  $x = 0$  then we cannot define the idempotent  $\vec{e}_{r,s}$  as above. However, if  $r$  or  $s$  is atleast 2 then we can define an alternative idempotent  $\tilde{e}_{r,s}$  as, the diagram with one positive upper horizontal arc connecting  $r$  and  $r + 1$ , one positive lower

horizontal arc connecting  $\bar{r}$  and  $\overline{r+2}$ , vertices  $r+2$  and  $\overline{r+1}$  is connected by a positive propagating line and all the remaining edges being positive propagating lines from  $i$  to  $\bar{i}$ .



$$(\tilde{e}_{r,s})^2 = \tilde{e}_{r,s}.$$

**Proposition 4.1.** *If  $x \neq 0$  then for each  $r, s > 0$ , there is an algebra isomorphism between  $\overrightarrow{D}_{r-1,s-1}(x)$  and  $\vec{e}_{r,s} \overrightarrow{D}_{r,s}(x) \vec{e}_{r,s}$ .*

*If  $x = 0$  and  $r \geq 2$  or  $s \geq 2$ , there is an algebra isomorphism between  $\overrightarrow{D}_{r-1,s-1}(x)$  and  $\tilde{e}_{r,s} \overrightarrow{D}_{r,s}(x) \tilde{e}_{r,s}$ .*

*Proof.* Define a map  $\Phi_{r,s} : \overrightarrow{D}_{r-1,s-1}(x) \rightarrow \vec{e}_{r,s} \overrightarrow{D}_{r,s}(x) \vec{e}_{r,s}$  by  $\Phi_{r,s}(\vec{d}) = \vec{e}_{r,s} \vec{d} \vec{e}_{r,s}$ , for  $\vec{d} \in \overrightarrow{D}_{r-1,s-1}(x)$ ,  $\vec{d} \in \overrightarrow{D}_{r,s}(x)$  is obtained by adding two signed propagating (either positive or negative) lines immediately before and after the wall in  $\vec{d}$  so that  $r$  is connected to  $\bar{r}$  and  $r+1$  to  $\overline{r+1}$ .

It is clear that the map  $\Phi_{r,s}$  is an injective algebra homomorphism and

$$\Phi_{r,s}(\overrightarrow{D}_{r-1,s-1}) = \vec{e}_{r,s} \overrightarrow{D}_{r,s} \vec{e}_{r,s}.$$

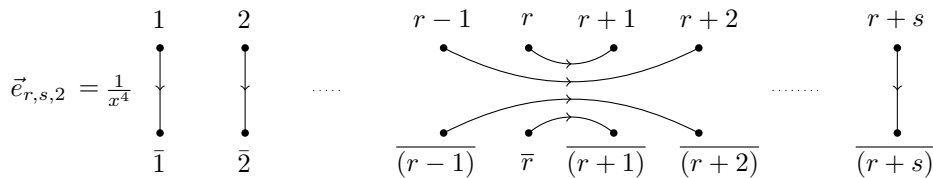
Hence  $\Phi_{r,s}$  is an isomorphism.

The proof of second statement is similar to the first. □

Now we define a sequence of idempotents  $\vec{e}_{r,s,i}$  in  $\overrightarrow{D}_{r,s}(x)$ , set  $\vec{e}_{r,s,0} = 1$  and for  $1 \leq i \leq \min(r, s)$ , set  $\vec{e}_{r,s,i} = \Phi_{r,s}(\vec{e}_{r-1,s-1,i-1})$ . Note that when  $x = 0$  and  $r = s$  the element  $\vec{e}_{r,r,r}$  is not defined.

To these elements we define associate quotients,  $\overrightarrow{D}_{r,s,i} = \overrightarrow{D}_{r,s} / \overrightarrow{D}_{r,s} \vec{e}_{r,s,i} \overrightarrow{D}_{r,s}$ .

When  $x \neq 0$  we can give an alternative description of the  $\vec{e}_{r,s,i}$  (Via our explicit description of  $\Phi_{r,s}$ ) as  $x^{-2i}$  times the diagram with  $i$  positive upper horizontal arcs connecting  $r-t$  to  $r+1+t$ ,  $i$  positive lower horizontal arcs connecting  $\overline{r-t}$  to  $\overline{r+1+t}$  for  $0 \leq t \leq i-1$  and the remaining edges all positive propagating lines connecting  $u$  to  $\bar{u}$  for some  $u$ . A similar explanation can be given in the case  $x = 0$ . Example,



We define the propagating vector of a diagram  $\vec{d} \in \vec{D}_{r,s}$  to be the pair  $\overrightarrow{(a,b)}$ , where  $\vec{d}$  has  $a$  signed propagating lines to the left of the wall and  $b$  to the right and the remaining upper and lower vertices are joined in pairs with signed horizontal arcs.

Note that if we multiply two diagrams with propagating vectors  $\overrightarrow{(a_1,b_1)}$  and  $\overrightarrow{(a_2,b_2)}$  then the result must have propagating vector  $\overrightarrow{(a,b)}$  with  $a \leq \min(a_1,a_2)$  and  $b \leq \min(b_1,b_2)$ .

Let  $J_i = \vec{D}_{r,s} \vec{e}_{r,s,i} \vec{D}_{r,s}$   
 then  $J_0 = \vec{D}_{r,s} \vec{e}_{r,s,0} \vec{D}_{r,s} = \vec{D}_{r,s}$   
 $J_1 = \vec{D}_{r,s} \vec{e}_{r,s,1} \vec{D}_{r,s} \subset \vec{D}_{r,s}$   
 we get the sequence of ideals

$$(4.1) \quad \dots \subset J_i \subset J_{i-1} \subset \dots \subset J_1 \subset J_0 = \vec{D}_{r,s}$$

**Proposition 4.2.** *The ideal  $J_i$  has a basis of all diagrams with propagating vector  $\overrightarrow{(a,b)}$  for some  $a \leq r - i$  and  $b \leq s - i$ .*

*In particular the section  $J_i/J_{i+1}$  in the filtration (4.1) has a basis of all diagram with propagating vector  $\overrightarrow{(r-i, s-i)}$ .*

*Proof.* The proof follows from the definition of  $J_i$ , Theorem(3.2) and the multiplication of walled signed Brauer diagrams defined in Section 3. □

In particular we have that,

$$(4.2) \quad \vec{D}_{r,s}/J_1 \cong k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s),$$

where  $\Sigma_r$  and  $\Sigma_s$  are symmetric groups.

We have some basic results about hyperoctahedral group representation from [7, 14].

For each bi-partition  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  of  $n$  ( $\lambda^{(1)} \vdash k_1, \lambda^{(2)} \vdash k_2$  with  $k_1 + k_2 = n$ ) the specht module  $S^\lambda$  for  $\mathbb{Z}_2 \wr \Sigma_n$  [Definiton (2.11.)] is given by

$S^\lambda = z_\lambda \mathcal{H}$ , where  $\mathcal{H}$  is the Iwahori-Hecke algebra of type  $\mathbf{B}$  and  $\mathcal{H} \cong k(\mathbb{Z}_2 \wr \Sigma_n)$ .

Let  $D^\lambda = S^\lambda / \text{rad } S^\lambda$  then

$\{D^\lambda : \lambda = (\lambda^{(1)}, \lambda^{(2)}) \text{ is } p\text{-regular bi-partition of } n\}$  is a complete set of inequivalent irreducible  $k(\mathbb{Z}_2 \wr \Sigma_n)$ - modules.

As  $k$  is algebraically closed field ( so a splitting field for  $\mathbb{Z}_2 \wr \Sigma_r$  and  $\mathbb{Z}_2 \wr \Sigma_s$ ). Therefore the simple modules for  $k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s)$  are precisely those modules of the form  $D^{\lambda^L} \boxtimes D^{\lambda^R}$  (outer tensor product), where  $D^{\lambda^L}$  is a simple  $k(\mathbb{Z}_2 \wr \Sigma_r)$  - module,  $D^{\lambda^R}$  a simple  $k(\mathbb{Z}_2 \wr \Sigma_s)$  - module [6, Theorem 10.33], and  $\lambda^L$  is a  $p$ -regular bi-partition of  $r$  and  $\lambda^R$  is a  $p$ -regular bi-partition of  $s$ .



Denote,

$\vec{\Lambda}_{reg}^{r,s} = \{(\lambda^L, \lambda^R) : \lambda^L \text{ is } p \text{ regular bi-partition of } r, \lambda^R \text{ is } p \text{ regular bi-partition of } s\}$ .

If  $p = 0$  or  $p > \max(r, s)$  then the group algebra  $k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s)$  is semi simple.

Let  $\vec{\Lambda}^{r,s} = \{(\lambda^L, \lambda^R) \mid \lambda^L \text{ is bi-partition of } r, \lambda^R \text{ is bi-partition of } s\}$ .

Let  $\vec{\Lambda}_{r,s}$  denote the indexing set for the simple  $\vec{D}_{r,s}$ -modules. From proposition (4.1.), we can define an exact localization functor,

$$F_{r,s} : \vec{D}_{r,s}\text{-mod} \longrightarrow \vec{D}_{r-1,s-1}\text{-mod} \text{ by } F_{r,s}(M) = \vec{e}_{r,s} M ; M \in \vec{D}_{r,s}\text{-mod},$$

and a corresponding right exact globalization functor  $G_{r-1,s-1}$  in the opposite direction,

$$G_{r-1,s-1} : \vec{D}_{r-1,s-1}\text{-mod} \longrightarrow \vec{D}_{r,s}\text{-mod} \text{ by } G_{r-1,s-1}(N) = \vec{D}_{r,s} \vec{e}_{r,s} \otimes_{\vec{e}_{r,s} \vec{D}_{r,s} \vec{e}_{r,s}} N ; N \in \vec{D}_{r-1,s-1}\text{-mod}.$$

By Theorem 1 in [5] and (4.2), we have that, for  $r, s > 0$

$$\vec{\Lambda}_{r,s} = \vec{\Lambda}_{r-1,s-1} \sqcup \vec{\Lambda}_{reg}^{r,s}.$$

**Proposition 4.3.** *If  $x \neq 0$  or  $r \neq s$  then*

$$\vec{\Lambda}_{r,s} = \bigsqcup_{i=0}^{\min(r,s)} \vec{\Lambda}_{reg}^{r-i,s-i}.$$

*Proof.* Since

$$\begin{aligned} \vec{\Lambda}_{r,s} &= \vec{\Lambda}_{r-1,s-1} \sqcup \vec{\Lambda}_{reg}^{r,s} \\ &= \vec{\Lambda}_{r-2,s-2} \sqcup \vec{\Lambda}_{reg}^{r-1,s-1} \sqcup \vec{\Lambda}_{reg}^{r,s} \text{ and so on, we have,} \\ (4.3) \quad \vec{\Lambda}_{r,s} &= \bigsqcup_{i=0}^{\min(r,s)} \vec{\Lambda}_{reg}^{r-i,s-i} \text{ as } \vec{D}_{r,0} \cong \vec{D}_{0,r} \cong k(\mathbb{Z}_2 \wr \Sigma_r) \end{aligned}$$

□

We shall describe the walled signed Brauer algebra in terms of partial one-row diagrams.

Given a walled signed Brauer diagram  $\vec{d} \in \vec{D}_{r,s}$  with  $t$  signed upper horizontal and  $t$  signed lower horizontal arcs.

Denote by  $\vec{d}^+$  the configuration formed by the signed upper horizontal arcs in  $\vec{d}$ , and by  $\vec{d}^-$  the configuration formed by the signed lower horizontal arcs in  $\vec{d}$ . Renumber the upper vertices of the propagating lines in  $\vec{d}$  from left to right as  $1, 2, \dots, r - t, r - t + 1, r - t + 2, \dots, r + s - 2t$  and their lower vertices from left to right as  $\bar{1}, \bar{2}, \dots, \bar{r - t}, \bar{r - t + 1}, \dots, \bar{r + s - 2t}$ . Then the propagating lines define an element  $\sigma_d = (\sigma, f) \in \mathbb{Z}_2 \wr \Sigma_{r-t} \times \mathbb{Z}_2 \wr \Sigma_{s-t}$  such that  $\sigma(i) = j$  if the  $i^{th}$  upper vertex on a propagating line is connected to the lower vertex  $j$ , where  $\sigma \in \Sigma_{r-t} \times \Sigma_{s-t}$  and  $f : \{1, 2, \dots, r - t, r - t + 1, \dots, r + s - 2t\} \longrightarrow \mathbb{Z}_2$  with

$$f(i) = \begin{cases} 0, & \text{if the corresponding propagating line is positive;} \\ 1, & \text{if the corresponding propagating line is negative.} \end{cases}$$

Therefore  $\vec{d}$  is uniquely written as

$$\vec{d} = X_{\vec{d}^+, \vec{d}^-, \sigma_d}.$$

We denote the set of elements  $\vec{d}^+$  arising thus by  $\nu_{r,s,t}$  (and by abuse of notation use the same set to refer to the elements  $\vec{d}^-$  that arise), and call this the set of partial one-row  $(r, s, t)$  diagrams.

**Lemma 4.4.** *Let  $V_l$  be the vector space over  $k(x)$  with basis  $\nu_{r,s,l}$ . then for  $l > 0$  the algebra  $J_l/J_{l+1}$  is isomorphic to an inflation  $V_l \otimes V_l \otimes k(\mathbb{Z}_2 \wr \Sigma_{r-l} \times \mathbb{Z}_2 \wr \Sigma_{s-l})$  of  $k(\mathbb{Z}_2 \wr \Sigma_{r-l} \times \mathbb{Z}_2 \wr \Sigma_{s-l})$  along a free  $k$ -module  $V_l$  of rank  $|\nu_{r,s,l}|$  with respect to some bilinear form (we shall define in the proof).*

*Proof.* Let  $\psi : V_l \otimes V_l \otimes k(\mathbb{Z}_2 \wr \Sigma_{r-l} \times \mathbb{Z}_2 \wr \Sigma_{s-l}) \rightarrow J_l/J_{l+1}$  be a map is defined by

$$\psi(\vec{d}^+ \otimes \vec{d}^- \otimes \sigma_d) = X_{\vec{d}^+, \vec{d}^-, \sigma_d}$$

We have to define  $\phi_l$ ;

$$\text{Let } \vec{d}_1^+ \otimes \vec{d}_1^- \otimes \sigma_{d_1}, \vec{d}_2^+ \otimes \vec{d}_2^- \otimes \sigma_{d_2} \in V_l \otimes V_l \otimes k(\mathbb{Z}_2 \wr \Sigma_{r-l} \times \mathbb{Z}_2 \wr \Sigma_{s-l}).$$

$$\text{Then, we have } \psi(\vec{d}_1^+ \otimes \vec{d}_1^- \otimes \sigma_{d_1}) = X_{\vec{d}_1^+, \vec{d}_1^-, \sigma_{d_1}} \in \vec{D}_{r,s}(x) \text{ and}$$

$$\psi(\vec{d}_2^+ \otimes \vec{d}_2^- \otimes \sigma_{d_2}) = X_{\vec{d}_2^+, \vec{d}_2^-, \sigma_{d_2}} \in \vec{D}_{r,s}(x)$$

By definition of the multiplication in  $\vec{D}_{r,s}(x)$ , we have,  $X_{\vec{d}_1^+, \vec{d}_1^-, \sigma_{d_1}} \cdot X_{\vec{d}_2^+, \vec{d}_2^-, \sigma_{d_2}} = x^t \cdot \vec{d}$ ,

where  $t$  is the number of closed loops in the product  $X_{\vec{d}_1^+, \vec{d}_1^-, \sigma_{d_1}} \cdot X_{\vec{d}_2^+, \vec{d}_2^-, \sigma_{d_2}}$  and  $\vec{d} \in \vec{D}_{r,s}(x)$  having  $2l$  or more signed horizontal edges.

$$\text{Since } (\vec{d}_1^+ \otimes \vec{d}_1^- \otimes \sigma_{d_1}) \cdot (\vec{d}_2^+ \otimes \vec{d}_2^- \otimes \sigma_{d_2}) = \vec{d}_1^+ \otimes \vec{d}_2^- \otimes \sigma_{d_1} \phi_l(d_1^-, d_2^+) \sigma_{d_2}.$$

If this product  $X_{\vec{d}_1^+, \vec{d}_1^-, \sigma_{d_1}} \cdot X_{\vec{d}_2^+, \vec{d}_2^-, \sigma_{d_2}}$  does not have propagating vector  $(r - l, s - l)$  then

set  $\phi_l(\vec{d}_1^-, \vec{d}_2^+) = 0$ , other wise,  $\phi_l(\vec{d}_1^-, \vec{d}_2^+) = x^t \sigma_d$ ; where  $\sigma_d \in k(\mathbb{Z}_2 \wr \Sigma_{r-l} \times \mathbb{Z}_2 \wr \Sigma_{s-l})$  such that  $X_{\vec{d}_1^+, \vec{d}_1^-, \sigma_{d_1}} \cdot X_{\vec{d}_2^+, \vec{d}_2^-, \sigma_{d_2}} = x^t \cdot \vec{d} = x^t \cdot X_{\vec{d}_1^+, \vec{d}_2^-, \sigma_{d_1} \sigma_d \sigma_{d_2}}$ . Consider,

$$\begin{aligned} \psi((\vec{d}_1^+ \otimes \vec{d}_1^- \otimes \sigma_{d_1}) \cdot (\vec{d}_2^+ \otimes \vec{d}_2^- \otimes \sigma_{d_2})) &= \psi(\vec{d}_1^+ \otimes \vec{d}_2^- \otimes \sigma_{d_1} \phi_l(d_1^-, d_2^+) \sigma_{d_2}) \\ &= \psi(\vec{d}_1^+ \otimes \vec{d}_2^- \otimes \sigma_{d_1} (x^t \sigma_d) \sigma_{d_2}) \\ &= x^t X_{\vec{d}_1^+, \vec{d}_2^-, \sigma_{d_1} \sigma_d \sigma_{d_2}} \\ &= X_{\vec{d}_1^+, \vec{d}_1^-, \sigma_{d_1}} \cdot X_{\vec{d}_2^+, \vec{d}_2^-, \sigma_{d_2}} \end{aligned}$$

$$\psi((\vec{d}_1^+ \otimes \vec{d}_1^- \otimes \sigma_{d_1}) \cdot (\vec{d}_2^+ \otimes \vec{d}_2^- \otimes \sigma_{d_2})) = \psi(\vec{d}_1^+ \otimes \vec{d}_1^- \otimes \sigma_{d_1}) \cdot \psi(\vec{d}_2^+ \otimes \vec{d}_2^- \otimes \sigma_{d_2})$$

$\psi$  is the algebra homomorphism.

$$\text{Suppose that } \psi(\vec{d}_1^+ \otimes \vec{d}_1^- \otimes \sigma_{d_1}) = \psi(\vec{d}_2^+ \otimes \vec{d}_2^- \otimes \sigma_{d_2})$$

$$\text{then } X_{\vec{d}_1^+, \vec{d}_1^-, \sigma_{d_1}} = X_{\vec{d}_2^+, \vec{d}_2^-, \sigma_{d_2}}$$

$$\vec{d}_1^+ = \vec{d}_2^+, \vec{d}_1^- = \vec{d}_2^-, \sigma_{d_1} = \sigma_{d_2} \text{ and } \vec{d}_1^+ \otimes \vec{d}_1^- \otimes \sigma_{d_1} = \vec{d}_2^+ \otimes \vec{d}_2^- \otimes \sigma_{d_2}$$

$\psi$  is one-to-one.

Now let  $\vec{d} \in J_l/J_{l+1}$  then  $\vec{d} = X_{\vec{d}^+, \vec{d}^-, \sigma_d}$ .

Consider,

$$\psi(\vec{d}^+ \otimes \vec{d}^- \otimes \sigma_d) = X_{\vec{d}^+, \vec{d}^-, \sigma_d} = \vec{d}$$

$\psi$  is onto, since  $\vec{d}^+ \otimes \vec{d}^- \otimes \sigma_d \in V_l \otimes V_l \otimes k(\mathbb{Z}_2 \wr \Sigma_{r-l} \times \mathbb{Z}_2 \wr \Sigma_{s-l})$

Hence  $\psi$  is an isomorphism. □

**Lemma 4.5.** *Let  $\vec{d}_1 \in J_m/J_{m+1}$  and  $\vec{d}_2 \in J_n/J_{n+1}$  be two diagrams in  $\vec{D}_{r,s}$  whose pre image is  $\vec{d}_1^+ \otimes \vec{d}_1^- \otimes \sigma_1$  and  $\vec{d}_2^+ \otimes \vec{d}_2^- \otimes \sigma_2$  respectively, under the bilinear forms for their respective layers. We assume that  $n \geq m$ . Then the product  $\vec{d}_1 \cdot \vec{d}_2$  is either an element of  $J_n/J_{n+1}$ , or is an element of  $J_{n+1}$ . In the former case it corresponds under  $\psi$  to a scalar multiple of an element  $\vec{d}^+ \otimes \vec{d}_2^- \otimes \mu\sigma_2$ ; where  $\vec{d}^+ \in V_n$  and  $\mu \in k(\mathbb{Z}_2 \wr \Sigma_{r-n} \times \mathbb{Z}_2 \wr \Sigma_{s-n})$ .*

*There is a similar statement for  $n \leq m$ .*

*Proof.* The proofs of both statements are very similar to that of Lemma 4.4. □

**Lemma 4.6.** *The involution on  $\vec{D}_{r,s}$  corresponds to the standard involution on  $V_l \otimes V_l \otimes k(\mathbb{Z}_2 \wr \Sigma_{r-l} \times \mathbb{Z}_2 \wr \Sigma_{s-l})$  which sends  $\vec{d}^+ \otimes \vec{d}^- \otimes \sigma$  to  $\vec{d}^- \otimes \vec{d}^+ \otimes \sigma^{-1}$ .*

*Proof.* The proof follows easily from the definition of the involution as the reflection in the horizontal axis. □

**Proposition 4.7.** *The walled signed Brauer algebra  $\vec{D}_{r,s}$  is an iterated inflation of group algebra of the form  $(\mathbb{Z}_2 \wr \Sigma_{r-l} \times \mathbb{Z}_2 \wr \Sigma_{s-l})$  for  $0 \leq l \leq \min(r, s)$  along  $V_l$ .*

*Proof.* By the above lemmas, the fact that  $k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s)$  is cellular, and the proposition 2.17., we have that,

$\vec{D}_{r,s}(x)$  is an iterated inflation of the group algebra of  $(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s)$  and hence as a  $k$ -module  $\vec{D}_{r,s}(x)$  is equal to  $k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s) \oplus (V_1 \otimes V_1 \otimes k(\mathbb{Z}_2 \wr \Sigma_{r-1} \times \mathbb{Z}_2 \wr \Sigma_{s-1})) \oplus (V_2 \otimes V_2 \otimes k(\mathbb{Z}_2 \wr \Sigma_{r-2} \times \mathbb{Z}_2 \wr \Sigma_{s-2})) \oplus \dots$

and the iterated inflation starts with  $k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s)$  inflates it along  $V_1 \otimes V_1 \otimes k(\mathbb{Z}_2 \wr \Sigma_{r-1} \times \mathbb{Z}_2 \wr \Sigma_{s-1})$  and so on, ending with an inflation of  $k = k(\mathbb{Z}_2 \wr \Sigma_1)$  or  $k = k(\mathbb{Z}_2 \wr \Sigma_0)$  as bottom layer (depending on whether  $(r + s)$  is odd or even). □

**Theorem 4.8.**

- (i) *The walled signed Brauer algebra  $\vec{D}_{r,s}$  is cellular with a cell module  $\Delta_{r,s}(\lambda^L, \lambda^R)$  for each  $(\lambda^L, \lambda^R) \in \vec{\Lambda}^{r-l, s-l}$  with  $0 \leq l \leq \min(r, s)$ .*
- (ii) *If  $x \neq 0$  or  $r \neq s$  then the simple modules are indexed by all pair  $(l, \lambda^L, \lambda^R)$ , where  $0 \leq l \leq \min(r, s)$  and  $(\lambda^L, \lambda^R) \in \vec{\Lambda}_{reg}^{r-l, s-l}$ .*
- (iii) *If  $x = 0$  and  $r = s$  we get the same indexing set for simple modules as in (ii); but with the single simple corresponding to  $l = \min(r, s)$ .*

*Proof.* (i) From the basis definition in [8], we have the following result, A cell basis for  $k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s)$  can be obtained as a product of cell basis for

$k(\mathbb{Z}_2 \wr \Sigma_r)$  and  $k(\mathbb{Z}_2 \wr \Sigma_s)$ .

Hence  $k(\mathbb{Z}_2 \wr \Sigma_r \times \mathbb{Z}_2 \wr \Sigma_s)$  is cellular with cell modules of the form  $M \boxtimes N$ , where  $M, N$  are cell modules for  $k(\mathbb{Z}_2 \wr \Sigma_r)$  and  $k(\mathbb{Z}_2 \wr \Sigma_s)$  respectively.

By Proposition 4.7. and Theorem 2.18., we have,

$\vec{D}_{r,s}$  is cellular with cell module  $\Delta_{r,s}(\lambda^L, \lambda^R)$  for each  $(\lambda^L, \lambda^R) \in \vec{\Lambda}^{r-l, s-l}$ .

(ii) by Proposition 4.3, for  $x \neq 0$  or  $r \neq s$ , the simple modules of  $\vec{D}_{r,s}$  are indexed by all pair  $(l, \lambda^L, \lambda^R)$ , where  $0 \leq l \leq \min(r, s)$  and  $(\lambda^L, \lambda^R) \in \vec{\Lambda}_{reg}^{r-l, s-l}$ .

(iii) In the case of  $x = 0$ , the above assertion is also valid except that the case  $l = 0$  (which occurs only for  $r$  even) does not contribute a simple module.  $\square$

**Corollary 4.9** *If either  $p = 0$  or  $p > \max(r, s)$  and either  $x \neq 0$  or  $x = 0$  and  $r \neq s$  then the algebra  $\vec{D}_{r,s}(x)$  is quasi-hereditary with heredity chain induced by the idempotent  $\vec{e}_{r,s,i}$ . In all other cases  $\vec{D}_{r,s}(x)$  is not quasi-hereditary.*

*Proof.* The proof follows immediately from the fact that a cellular algebra is quasi-hereditary precisely when there are the same number of simples as cell modules.

$\vec{D}_{r,s}$  is quasi-hereditary with heredity chain

$\dots \subset J_i \subset J_{i-1} \subset \dots \subset J_1 \subset J_0 = \vec{D}_{r,s}$ , where  $J_i = \vec{D}_{r,s} \vec{e}_{r,s,i} \vec{D}_{r,s}$ .  $\square$

**Conclusion.** We established the cellularity of walled signed Brauer algebras and we also give a necessary and sufficient condition for the walled signed Brauer algebras to be quasi-hereditary. As an application of this paper, we will describe the irreducible representations of a certain Lie superalgebra, by using [8] and Schur-Weyl duality, in our subsequent paper.

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