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NOTES ON THE PAPER "A CRITERION FOR BOUNDED FUNCTIONS" [BULL. KOREAN MATH. SOC. 53 (2016), NO. 1, 215–225]

Yi-Ling Cang and Jin-Lin Liu

Abstract. In this note we point out some mistakes in a recent paper by M. Nunokawa et al. [Bull. Korean Math. Soc. 53 (2016), no. 1, 215–225].

Let H denote the class of functions analytic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Very recently, M. Nunokawa, S. Owa and J. Sokól published a paper [2] titled "A criterion for bounded functions" in Bull. Korean Math. Soc. **53** (2016), no. 1, pp. 215–225. In this paper, they prove the following theorem [2, Theorem 2.5].

Theorem A. *Let* $h(z) = \{(1+z)/(1-z)\}^{\alpha}$, $\alpha \in (0, 1]$, *and* $p(z)$ *be analytic in* D *with* $h(0) = p(0) = 1$. *Assume also that* $\phi(p(z))$ *is analytic in* D, *moreover* $Re\{\phi(h(z))\} \geq 0$ *in* D. *If*

$$p(z) + zp'(z)\phi(p(z)) \prec h(z) \quad (z \in D),$$

then $p(z) \prec h(z)$ $(z \in D)$.

In [2] Nunokawa et al. claimed that the above theorem is a new extension of a result by Hallenback and Ruscheweyh [1] because the function

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (h(0) = 1, \alpha \in (0, 1]) \tag{1}$$

is not convex (see [2, p. 222, line 7]). However, such statement is incorrect.

In this note we shall prove that the function $h(z)$ given by (1) is convex in D. Further, we shall also show that the function

$$g(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2} \quad (g(0) = 1, \alpha \in (0, 1]) \tag{2}$$

considered in [2] is close-to-convex in D.

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Proposition 1. *Let $0 < \alpha \le 1$. Then the function $h(z)$ given by* (1) *is analytic and univalently convex in D and*

$$h(D) = \left\{ w : w \in \mathbb{C} \quad \textit{and} \quad -\frac{\alpha\pi}{2} < \arg w < \frac{\alpha\pi}{2} \right\}.$$

Proof. It is easy to see that the transformation

$$t = w^{\frac{1}{\alpha}} \tag{3}$$

maps the convex region

$$G = \left\{ w : w \in \mathbb{C} \quad \text{and} \quad -\frac{\alpha\pi}{2} < \arg w < \frac{\alpha\pi}{2} \right\}$$

conformally onto the right-half t-plane $-\frac{\pi}{2} < \arg t < \frac{\pi}{2}$ so that $w = 1$ corresponding to $t = 1$. Since

$$z = \frac{t-1}{t+1} \tag{4}$$

maps the right-half t-plane $\mathrm{Re}(t) > 0$ onto D, from (1), (3) and (4) we find that

$$w = t^{\alpha} = \left(\frac{1+z}{1-z} \right)^{\alpha} = h(z)$$

maps D conformally onto $G = h(D)$ with $h(0) = 1$. This completes the proof. □

Proposition 2. *Let $0 < \alpha \le 1$. Then the function $g(z)$ defined by* (2) *is close-to-convex in D.*

Proof. Suppose that

$$g(z) = \left(\frac{1+z}{1-z} \right)^{\alpha} + \frac{2\alpha z}{1-z^2} := h(z) + Q(z).$$

Then the function $h(z)$ is convex in D and satisfies

$$|\arg \{h(z)\}| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1).$$

The function $Q(z)$ is starlike in D. Thus

$$\mathrm{Re} \left\{ \frac{zg'(z)}{Q(z)} \right\} = \mathrm{Re} \left\{ h(z) + \frac{zQ'(z)}{Q(z)} \right\} > 0,$$

which shows that the function $g(z)$ is close-to-convex in D. Now the proof is complete. □

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Yi-Ling Cang
Department of Mathematics
Suqian College
Suqian 223800, P. R. China
E-mail address: cangyiling88@126.com

Jin-Lin Liu
Department of Mathematics
Yangzhou University
Yangzhou 225002, P. R. China
E-mail address: jlliu@yzu.edu.cn