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THETA SUMS OF HIGHER INDEX

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Abstract. In this paper, we obtain some behaviours of theta sums of higher index for the Schrödinger-Weil representation of the Jacobi group associated with a positive definite symmetric real matrix of degree m.

1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \left\{ \Omega \in \mathbb{C}^{(n,n)} \,\middle|\, \Omega = {}^t\Omega, \ \ \operatorname{Im}\Omega > 0 \right\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \left\{ g \in \mathbb{R}^{(2n,2n)} \,\middle|\, {}^t g J_n g = J_n \right\}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose of a matrix M, $\operatorname{Im}\Omega$ denotes the imaginary part of Ω and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here I_n denotes the identity matrix of degree n. We see that $Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \left\{ (\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)}, \ \kappa \in \mathbb{R}^{(m,m)}, \ \ \kappa + \mu\, {}^t\lambda \text{ symmetric} \right\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') = (\lambda+\lambda', \mu+\mu'; \kappa+\kappa'+\lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

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We let

$$G^J = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \quad \text{(semi-direct product)}$$

be the Jacobi group endowed with the following multiplication law

$$\Big(g,(\lambda,\mu;\kappa)\Big)\cdot\Big(g',(\lambda',\mu';\kappa')\Big) = \Big(gg',(\widetilde{\lambda}+\lambda',\widetilde{\mu}+\mu';\kappa+\kappa'+\widetilde{\lambda}\,{}^t\mu'-\widetilde{\mu}\,{}^t\lambda')\Big)$$

with $g,g'\in Sp(n,\mathbb{R}), (\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\widetilde{\lambda},\widetilde{\mu})=(\lambda,\mu)g'$. Then we have the *natural transitive action* of G^J on the Siegel-Jacobi space $\mathbb{H}_{n,m} := \mathbb{H}_n\times\mathbb{C}^{(m,n)}$ defined by

$$\Big(g,(\lambda,\mu;\kappa)\Big)\cdot(\Omega,Z) = \Big((A\Omega+B)(C\Omega+D)^{-1},(Z+\lambda\,\Omega+\mu)(C\,\Omega+D)^{-1}\Big),$$

where $g=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z)\in\mathbb{H}_{n,m}$. Thus $\mathbb{H}_{n,m}$ is a homogeneous Kähler space which is not symmetric. In fact, $\mathbb{H}_{n,m}$ is biholomorphic to the homogeneous space G^J/K^J, where $K^J\cong U(n)\times S(m,\mathbb{R})$. Here $U(n)$ denotes the unitary group of degree n and $S(m,\mathbb{R})$ denote the abelian additive group consisting of all $m\times m$ symmetric real matrices. We refer to [1, 2, 4], [12]-[29] for more details on materials related to the Siegel-Jacobi space, e.g., Jacobi forms, invariant metrics, invariant differential operators and Maass-Jacobi forms.

The Weil representation for a symplectic group was first introduced by A. Weil in [10] to reformulate Siegel's analytic theory of quadratic forms (cf. [9]) in terms of the group theoretical theory. It is well known that the Weil representation plays a central role in the study of the transformation behaviors of theta series. In [27], Yang constructed the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of the Jacobi group G^J associated with a positive definite symmetric real matrix $\mathcal{M}$ of degree n explicitly.

This paper is organized as follows. In Section 2, we review the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of the Jacobi group G^J associated with a symmetric positive definite matrix $\mathcal{M}$ and recall the basic actions of $\omega_{\mathcal{M}}$ on the representation space $L^2\big(\mathbb{R}^{(m,n)}\big)$ which were expressed explicitly in [27]. In Section 3, we define the theta sum $\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)$ of higher index and obtain some properties of the theta sum. The theta sum $\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)$ is a generalization of the theta sum defined by J. Marklof [6].

Notations: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers respectively. $\mathbb{C}^\times$ denotes the multiplicative group of nonzero complex numbers and $\mathbb{Z}^\times$ denotes the set of all nonzero integers. T denotes the multiplicative group of complex numbers of modulus one. The symbol ":=" means that the expression on the right is the definition of that on the left. For a square matrix $A\in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. We put $i=\sqrt{-1}$.

2. The Schrödinger-Weil representation

In this section we review the Schrödinger-Weil representation of the Jacobi group G^J (cf. [27], Section 3).

Throughout this section we assume that $\mathcal{M}$ is a positive definite symmetric real $m \times m$ matrix. We let

$$L = \left\{ (0, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \,\middle|\, \mu \in \mathbb{R}^{(m,n)},\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \right\}$$

be a commutative normal subgroup of $H_{\mathbb{R}}^{(n,m)}$ and $\chi_{\mathcal{M}} : L \longrightarrow \mathbb{C}^\times$ be the unitary character of L defined by

$$\chi_{\mathcal{M}}((0, \mu; \kappa)) := e^{\pi\, i\, \sigma(\mathcal{M}\kappa)}, \quad (0, \mu; \kappa) \in L.$$

The representation $\mathscr{W}_{\mathcal{M}} = \mathrm{Ind}_L^{H_{\mathbb{R}}^{(n,m)}} \chi_{\mathcal{M}}$ induced by $\chi_{\mathcal{M}}$ from L is realized on the Hilbert space $H(\chi_{\mathcal{M}}) \cong L^2\left(\mathbb{R}^{(m,n)}, d\xi\right)$. $\mathscr{W}_{\mathcal{M}}$ is irreducible (cf. [11], Theorem 3) and is called the Schrödinger representation $\mathscr{W}_{\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ with the central character $\chi_{\mathcal{M}}$. We refer to [11, 14, 17, 18, 19, 24] for more representations of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ and their related topics. Then $\mathscr{W}_{\mathcal{M}}$ is expressed explicitly as

$$[\mathscr{W}_{\mathcal{M}}(h_0)f]\,(\lambda) = e^{\pi\, i\, \sigma\{\mathcal{M}(\kappa_0 + \mu_0\, {}^t\lambda_0 + 2\lambda\, {}^t\mu_0)\}}\, f(\lambda + \lambda_0), \tag{2.1}$$

where $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$ and $\lambda \in \mathbb{R}^{(m,n)}$. See Formula (2.4) in [27] for more detail on $\mathscr{W}_{\mathcal{M}}$. We note that the symplectic group $Sp(n, \mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,m)}$ by conjugation inside G^J. For a fixed element $g \in Sp(n, \mathbb{R})$, the irreducible unitary representation $\mathscr{W}_{\mathcal{M}}^g$ of $H_{\mathbb{R}}^{(n,m)}$ defined by

$$\mathscr{W}_{\mathcal{M}}^g(h) = \mathscr{W}_{\mathcal{M}}(ghg^{-1}), \quad h \in H_{\mathbb{R}}^{(n,m)} \tag{2.2}$$

has the property that

$$\mathscr{W}_{\mathcal{M}}^g((0, 0; \kappa)) = \mathscr{W}_{\mathcal{M}}((0, 0; \kappa)) = e^{\pi i\, \sigma(\mathcal{M}\kappa)}\, \mathrm{Id}_{H(\chi_{\mathcal{M}})}, \quad \kappa \in S(m, \mathbb{R}).$$

Here $\mathrm{Id}_{H(\chi_{\mathcal{M}})}$ denotes the identity operator on the Hilbert space $H(\chi_{\mathcal{M}})$. According to Stone-von Neumann theorem, there exists a unitary operator $R_{\mathcal{M}}(g)$ on $H(\chi_{\mathcal{M}})$ with $R_{\mathcal{M}}(I_{2n}) = \mathrm{Id}_{H(\chi_{\mathcal{M}})}$ such that

$$R_{\mathcal{M}}(g)\mathscr{W}_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}^g(h)R_{\mathcal{M}}(g) \qquad \text{for all } h \in H_{\mathbb{R}}^{(n,m)}. \tag{2.3}$$

We observe that $R_{\mathcal{M}}(g)$ is determined uniquely up to a scalar of modulus one.

From now on, for brevity, we put $G = Sp(n, \mathbb{R})$. According to Schur's lemma, we have a map $c_{\mathcal{M}} : G \times G \longrightarrow T$ satisfying the relation

$$R_{\mathcal{M}}(g_1 g_2) = c_{\mathcal{M}}(g_1, g_2) R_{\mathcal{M}}(g_1) R_{\mathcal{M}}(g_2) \quad \text{for all} \ \ g_1, g_2 \in G. \tag{2.4}$$

We recall that T denotes the multiplicative group of complex numbers of modulus one. Therefore $R_{\mathcal{M}}$ is a projective representation of G on $H(\chi_{\mathcal{M}})$ and $c_{\mathcal{M}}$ defines the cocycle class in $H^2(G, T)$. The cocycle $c_{\mathcal{M}}$ yields the central extension $G_{\mathcal{M}}$ of G by T. The group $G_{\mathcal{M}}$ is a set $G \times T$ equipped with the following multiplication

$$(g_1, t_1) \cdot (g_2, t_2) = \big(g_1 g_2, t_1 t_2\, c_{\mathcal{M}}(g_1, g_2)^{-1}\big), \quad g_1, g_2 \in G,\ t_1, t_2 \in T. \tag{2.5}$$

We see immediately that the map $\widetilde{R}_{\mathcal{M}} : G_{\mathcal{M}} \longrightarrow GL(H(\chi_{\mathcal{M}}))$ defined by

$$\widetilde{R}_{\mathcal{M}}(g, t) = t\, R_{\mathcal{M}}(g) \quad \text{for all } (g, t) \in G_{\mathcal{M}} \tag{2.6}$$

is a *true* representation of $G_{\mathcal{M}}$. As in Section 1.7 in [5], we can define the map $s_{\mathcal{M}} : G \longrightarrow T$ satisfying the relation

$$c_{\mathcal{M}}(g_1, g_2)^2 = s_{\mathcal{M}}(g_1)^{-1} s_{\mathcal{M}}(g_2)^{-1} s_{\mathcal{M}}(g_1 g_2) \quad \text{for all } g_1, g_2 \in G.$$

Thus we see that

$$G_{2,\mathcal{M}} = \big\{(g, t) \in G_{\mathcal{M}} \mid t^2 = s_{\mathcal{M}}(g)^{-1}\big\} \tag{2.7}$$

is the metaplectic group associated with $\mathcal{M}$ that is a two-fold covering group of G. The restriction $R_{2,\mathcal{M}}$ of $\widetilde{R}_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ is the *Weil representation* of G associated with $\mathcal{M}$.

If we identify $h = (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ (resp. $g \in Sp(n, \mathbb{R})$) with $(I_{2n}, (\lambda, \mu; \kappa)) \in G^J$ (resp. $(g, (0, 0; 0)) \in G^J$), every element $\tilde{g}$ of G^J can be written as $\tilde{g} = hg$ with $h \in H_{\mathbb{R}}^{(n,m)}$ and $g \in Sp(n, \mathbb{R})$. In fact,

$$(g, (\lambda, \mu; \kappa)) = (I_{2n}, ((\lambda, \mu) g^{-1}; \kappa))\, (g, (0, 0; 0)) = ((\lambda, \mu) g^{-1}; \kappa) \cdot g.$$

Therefore we define the *projective* representation $\pi_{\mathcal{M}}$ of the Jacobi group G^J with cocycle $c_{\mathcal{M}}(g_1, g_2)$ by

$$\pi_{\mathcal{M}}(hg) = \mathscr{W}_{\mathcal{M}}(h)\, R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n,m)},\ g \in G. \tag{2.8}$$

We let

$$G_{\mathcal{M}}^J = G_{\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n,m)}$$

be the semidirect product of $G_{\mathcal{M}}$ and $H_{\mathbb{R}}^{(n,m)}$ with the multiplication law

$$\begin{aligned} &\big((g_1, t_1), (\lambda_1, \mu_1; \kappa_1)\big) \cdot \big((g_2, t_2), (\lambda_2, \mu_2\,; \kappa_2)\big) \\ &= \big((g_1, t_1)(g_2, t_2), (\tilde{\lambda} + \lambda_2, \tilde{\mu} + \mu_2\,; \kappa_1 + \kappa_2 + \tilde{\lambda}\,{}^t\mu_2 - \tilde{\mu}\,{}^t\lambda_2)\big), \end{aligned}$$

where $(g_1, t_1), (g_2, t_2) \in G_{\mathcal{M}},\ (\lambda_1, \mu_1; \kappa_1), (\lambda_2, \mu_2\,; \kappa_2) \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu) g_2$. If we identify $h = (\lambda, \mu\,; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ (resp. $(g, t) \in G_{\mathcal{M}}$) with $((I_{2n}, 1), (\lambda, \mu\,; \kappa)) \in G_{\mathcal{M}}^J$ (resp. $((g, t), (0, 0; 0)) \in G_{\mathcal{M}}^J$), we see easily that every element $\big((g, t), (\lambda, \mu\,; \kappa)\big)$ of $G_{\mathcal{M}}^J$ can be expressed as

$$\big((g, t), (\lambda, \mu\,; \kappa)\big) = \big((I_{2n}, 1), ((\lambda, \mu) g^{-1}; \kappa)\big)\big((g, t), (0, 0; 0)\big) = ((\lambda, \mu) g^{-1}; \kappa)(g, t).$$

Now we can define the *true* representation $\widetilde{\omega}_{\mathcal{M}}$ of $G_{\mathcal{M}}^J$ by

$$\widetilde{\omega}_{\mathcal{M}}(h \cdot (g, t)) = t\, \pi_{\mathcal{M}}(hg) = t\, \mathscr{W}_{\mathcal{M}}(h)\, R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n,m)},\ (g, t) \in G_{\mathcal{M}}. \tag{2.9}$$

We recall that the following matrices

$$t(b)=\begin{pmatrix} I_n & b\\ 0 & I_n\end{pmatrix} \text{ with any } b={}^t b\in\mathbb{R}^{(n,n)},$$
$$g(\alpha)=\begin{pmatrix} {}^t\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix} \text{ with any } \alpha\in GL(n,\mathbb{R}),$$
$$\sigma_n=\begin{pmatrix} 0 & -I_n\\ I_n & 0\end{pmatrix}$$

generate the symplectic group $G=Sp(n,\mathbb{R})$ (cf. [3, p. 326], [7, p. 210]). Therefore the following elements $h_t(\lambda,\mu\,;\kappa),\ t(b\,;t),\ g(\alpha\,;t)$ and $\sigma_{n\,;t}$ of $G_{\mathcal{M}}\ltimes H_{\mathbb{R}}^{(n,m)}$ defined by

$$h_t(\lambda,\mu\,;\kappa)=\big((I_{2n},t),(\lambda,\mu;\kappa)\big) \text{ with } t\in T,\ \lambda,\mu\in\mathbb{R}^{(m,n)} \text{ and } \kappa\in\mathbb{R}^{(m,m)},$$
$$t(b\,;t)=\big((t(b),t),(0,0;0)\big) \text{ with any } b={}^t b\in\mathbb{R}^{(n,n)},\ t\in T,$$
$$g(\alpha\,;t)=\big((g(\alpha),t),(0,0;0)\big) \text{ with any } \alpha\in GL(n,\mathbb{R}) \text{ and } t\in T,$$
$$\sigma_{n\,;t}=((\sigma_n,t),(0,0;0)) \text{ with } t\in T$$

generate the group $G_{\mathcal{M}}\ltimes H_{\mathbb{R}}^{(n,m)}$. We can show that the representation $\widetilde{\omega}_{\mathcal{M}}$ is realized on the representation $H(\chi_{\mathcal{M}})=L^2\big(\mathbb{R}^{(m,n)}\big)$ as follows: for each $f\in L^2\big(\mathbb{R}^{(m,n)}\big)$ and $x\in\mathbb{R}^{(m,n)}$, the actions of $\widetilde{\omega}_{\mathcal{M}}$ on the generators are given by

$$\big[\widetilde{\omega}_{\mathcal{M}}\big(h_t(\lambda,\mu\,;\kappa)\big)f\big](x)=t\,e^{\pi i\,\sigma\{\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\}}\,f(x+\lambda), \tag{2.10}$$
$$\big[\widetilde{\omega}_{\mathcal{M}}\big(t(b\,;t)\big)f\big](x)=t\,e^{\pi i\,\sigma(\mathcal{M}\,x\,b\,{}^t x)}f(x), \tag{2.11}$$
$$\big[\widetilde{\omega}_{\mathcal{M}}\big(g(\alpha\,;t)\big)f\big](x)=t\,|\det\alpha|^{\frac{m}{2}}\,f(x\,{}^t\alpha), \tag{2.12}$$
$$\big[\widetilde{\omega}_{\mathcal{M}}\big(\sigma_{n\,;t}\big)f\big](x)=t\,\big(\det\mathcal{M}\big)^{\frac{n}{2}}\int_{\mathbb{R}^{(m,n)}} f(y)\,e^{-2\,\pi i\,\sigma(\mathcal{M}\,y\,{}^t x)}\,dy. \tag{2.13}$$

Let

$$G^J_{2,\mathcal{M}}=G_{2,\mathcal{M}}\ltimes H_{\mathbb{R}}^{(n,m)}$$

be the semidirect product of $G_{2,\mathcal{M}}$ and $H_{\mathbb{R}}^{(n,m)}$. Then $G^J_{2,\mathcal{M}}$ is a subgroup of $G^J_{\mathcal{M}}$ which is a two-fold covering group of the Jacobi group G^J. The restriction $\omega_{\mathcal{M}}$ of $\widetilde{\omega}_{\mathcal{M}}$ to $G^J_{2,\mathcal{M}}$ is called the *Schrödinger-Weil representation* of G^J associated with $\mathcal{M}$.

Remark 2.1. In the case $n=m=1,\ \omega_{\mathcal{M}}$ is dealt in [1] and [6].

Remark 2.2. The Schrödinger-Weil representation is applied usefully to the theory of Maass-Jacobi forms [8].

3. Theta sums of higher index

Let $\mathcal{M}$ be a positive definite symmetric real matrix of degree m. We recall the Schrödinger representation $\mathscr{W}_{\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ associate with $\mathcal{M}$ given by Formula (2.1) in Section 2. We note that for an element $(\lambda,\mu;\kappa)$ of $H_{\mathbb{R}}^{(n,m)}$, we have the decomposition

$$(\lambda,\mu;\kappa)=(\lambda,0;0)\circ(0,\mu;0)\circ(0,0;\kappa-\lambda\,{}^t\mu).$$

We consider the embedding $\Phi_n : SL(2,\mathbb{R})\longrightarrow Sp(n,\mathbb{R})$ defined by

$$\Phi_n\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right):=\begin{pmatrix} aI_n & bI_n\\ cI_n & dI_n\end{pmatrix},\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R}). \tag{3.1}$$

For $x,y\in\mathbb{R}^{(m,n)}$, we put

$$(x,y)_{\mathcal{M}}:=\sigma({}^t x\mathcal{M}y)\qquad\text{and}\qquad \|x\|_{\mathcal{M}}:=\sqrt{(x,x)_{\mathcal{M}}}.$$

According to Formulas (2.11)-(2.13), for any $M=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{R})\hookrightarrow Sp(n,\mathbb{R})$ and $f\in L^2\left(\mathbb{R}^{(m,n)}\right)$, we have the following explicit representation

(3.2)

$$[R_{\mathcal{M}}(M)f](x)=\begin{cases} |a|^{\frac{mn}{2}}e^{ab\|x\|^2_{\mathcal{M}}\pi i}f(ax) & \text{if } c=0,\\ (\det\mathcal{M})^{\frac{n}{2}}\,|c|^{-\frac{mn}{2}}\int_{\mathbb{R}^{(m,n)}} e^{\frac{\alpha(M,x,y,\mathcal{M})}{c}\pi i}f(y)dy & \text{if } c\neq 0,\end{cases}$$

where

$$\alpha(M,x,y,\mathcal{M})=a\,\|x\|^2_{\mathcal{M}}+d\,\|y\|^2_{\mathcal{M}}-2(x,y)_{\mathcal{M}}.$$

Indeed, if $a=0$ and $c\neq 0$, using the decomposition

$$M=\begin{pmatrix} 0 & -c^{-1}\\ c & d\end{pmatrix}=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} c & d\\ 0 & c^{-1}\end{pmatrix}$$

and if $a\neq 0$ and $c\neq 0$, using the decomposition

$$M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a & c^{-1}\\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} ac & ad\\ 0 & (ac)^{-1}\end{pmatrix},$$

we obtain Formula (3.2).

If

$$M_1=\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix},\quad M_2=\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}\quad\text{and}\quad M_3=\begin{pmatrix} a_3 & b_3\\ c_3 & d_3\end{pmatrix}\in SL(2,\mathbb{R})$$

with $M_3=M_1M_2$, the corresponding cocycle is given by

$$c_{\mathcal{M}}(M_1,M_2)=e^{-i\,\pi\,mn\,\mathrm{sign}(c_1c_2c_3)/4}, \tag{3.3}$$

where

$$\mathrm{sign}(x)=\begin{cases} -1 & (x<0)\\ \ \ 0 & (x=0)\\ \ \ 1 & (x>0).\end{cases}$$

In the special case when

$$M_1 = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix},$$

we find

$$c_{\mathcal{M}}(M_1, M_2) = e^{-i\,\pi\, mn\,(\sigma_{\phi_1}+\sigma_{\phi_2}-\sigma_{\phi_1+\phi_2})/4},$$

where

$$\sigma_\phi = \begin{cases} 2\nu & \text{if } \phi = \nu\pi \\ 2\nu+1 & \text{if } \nu\pi < \phi < (\nu+1)\pi. \end{cases}$$

It is well known that every $M \in SL(2,\mathbb{R})$ admits the unique Iwasawa decomposition

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \tag{3.4}$$

where $\tau = u + iv \in \mathbb{H}_1$ and $\phi \in [0, 2\pi)$. This parametrization $M = (\tau, \phi)$ in $SL(2,\mathbb{R})$ leads to the natural action of $SL(2,\mathbb{R})$ on $\mathbb{H}_1 \times [0, 2\pi)$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, \phi) := \left(\frac{a\tau + b}{c\tau + d}, \phi + \arg(c\tau + d) \bmod 2\pi \right). \tag{3.5}$$

Lemma 3.1. *For two elements g_1 and g_2 in $SL(2,\mathbb{R})$, we let*

$$g_1 = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1^{1/2} & 0 \\ 0 & v_1^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix}$$

and

$$g_2 = \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_2^{1/2} & 0 \\ 0 & v_2^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix}$$

be the Iwasawa decompositions of g_1 and g_2 respectively, where $u_1, u_2 \in \mathbb{R}$, $v_1 > 0$, $v_2 > 0$ and $0 \le \phi_1, \phi_2 < 2\pi$. Let

$$g_3 = g_1 g_2 = \begin{pmatrix} 1 & u_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_3^{1/2} & 0 \\ 0 & v_3^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi_3 & -\sin\phi_3 \\ \sin\phi_3 & \cos\phi_3 \end{pmatrix}$$

be the Iwasawa decomposition of $g_3 = g_1 g_2$. Then we have

$$u_3 = \frac{A}{(u_2 \sin\phi_1 + \cos\phi_1)^2 + (v_2 \sin\phi_1)^2},$$
$$v_3 = \frac{v_1 v_2}{(u_2 \sin\phi_1 + \cos\phi_1)^2 + (v_2 \sin\phi_1)^2}$$

and

$$\phi_3 = \tan^{-1}\left[\frac{(v_2\cos\phi_2 + u_2 \sin\phi_2)\tan\phi_1 + \sin\phi_2}{(-v_2 \sin\phi_2 + u_2\cos\phi_2)\tan\phi_1 + \cos\phi_2} \right],$$

where

$$A = u_1(u_2\sin\phi_1 + \cos\phi_1)^2 + (u_1 v_2 - v_1 u_2)\sin^2\phi_1 + v_1 u_2 \cos^2\phi_1 + v_1(u_2^2 + v_2^2 - 1)\sin\phi_1\cos\phi_1.$$

Proof. If $g \in SL(2,\mathbb{R})$ has the unique Iwasawa decomposition (3.4), then we get the following

$$\begin{aligned} a &= v^{1/2}\cos\phi + uv^{-1/2}\sin\phi, \\ b &= -v^{1/2}\sin\phi + uv^{-1/2}\cos\phi, \\ c &= v^{-1/2}\sin\phi, \quad d = v^{-1/2}\cos\phi, \\ u &= (ac+bd)\left(c^2+d^2\right)^{-1}, \quad v = \left(c^2+d^2\right)^{-1}, \quad \tan\phi = \frac{c}{d}. \end{aligned}$$

We set

$$g_3 = g_1 g_2 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}.$$

Since

$$u_3 = (a_3c_3 + b_3d_3)\left(c_3^2+d_3^2\right)^{-1}, \quad v = \left(c_3^2+d_3^2\right)^{-1}, \quad \tan\phi_3 = \frac{c_3}{d_3},$$

by an easy computation, we obtain the desired results. □

Now we use the new coordinates $(\tau = u+iv, \phi)$ with $\tau \in \mathbb{H}_1$ and $\phi \in [0, 2\pi)$ in $SL(2,\mathbb{R})$. According to Formulas (2.11)-(2.13), the projective representation $R_{\mathcal{M}}$ of $SL(2,\mathbb{R}) \hookrightarrow Sp(n,\mathbb{R})$ reads in these coordinates $(\tau = u+iv, \phi)$ as follows:

$$[R_{\mathcal{M}}(\tau,\phi)f](x) = v^{\frac{mn}{4}}\, e^{u\|x\|_{\mathcal{M}}^2 \pi\, i}\, [R_{\mathcal{M}}(i,\phi)f]\left(v^{1/2}x\right), \tag{3.6}$$

where $f \in L^2\left(\mathbb{R}^{(m,n)}\right),\ x \in \mathbb{R}^{(m,n)}$ and

$$\begin{aligned} &[R_{\mathcal{M}}(i,\phi)f](x) \\ &= \begin{cases} f(x) & \text{if } \phi \equiv 0 \bmod 2\pi, \\ f(-x) & \text{if } \phi \equiv \pi \bmod 2\pi, \\ (\det\mathcal{M})^{\frac{n}{2}}\,|\sin\phi|^{-\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{B(x,y,\phi,\mathcal{M})\pi i} f(y)dy & \text{if } \phi \not\equiv 0 \bmod \pi. \end{cases} \end{aligned} \tag{3.7}$$

Here

$$B(x,y,\phi,\mathcal{M}) = \frac{\left(\|x\|_{\mathcal{M}}^2 + \|y\|_{\mathcal{M}}^2\right)\cos\phi - 2(x,y)_{\mathcal{M}}}{\sin\phi}.$$

Now we set

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We note that

$$\left[R_{\mathcal{M}}\left(i,\frac{\pi}{2}\right)f\right](x) = [R_{\mathcal{M}}(S)f](x) = (\det\mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} f(y)\, e^{-2\,(x,\,y)_{\mathcal{M}}\,\pi\, i}\, dy \tag{3.8}$$

for $f \in L^2\left(\mathbb{R}^{(m,n)}\right)$.

Remark 3.1. For Schwartz functions $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we have

$$\lim_{\phi \longrightarrow 0\pm} |\sin\phi|^{-\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{B(x,y,\phi,\mathcal{M})\,\pi\, i} f(y)dy = e^{\pm i\,\pi\, mn/4} f(x) \neq f(x).$$

Therefore the projective representation $R_{\mathcal{M}}$ is not continuous at $\phi = \nu\pi\ (\nu \in \mathbb{Z})$ in general. If we set

$$\tilde{R}_{\mathcal{M}}(\tau,\phi) = e^{-i\,\pi\, mn\sigma_\phi/4} R_{\mathcal{M}}(\tau,\phi),$$

$\tilde{R}_{\mathcal{M}}$ corresponds to a unitary representation of the double cover of $SL(2,\mathbb{R})$ (cf. (2.6) and [5]). This means in particular that

$$\tilde{R}_{\mathcal{M}}(i,\phi)\tilde{R}_{\mathcal{M}}(i,\phi') = \tilde{R}_{\mathcal{M}}(i,\phi+\phi'),$$

where $\phi \in [0,4\pi)$ parametrises the double cover of $SO(2) \subset SL(2,\mathbb{R})$.

We observe that for any element $(g,(\lambda,\mu;\kappa)) \in G^J$ with $g \in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$, we have the following decomposition

$$(g,(\lambda,\mu;\kappa)) = (I_{2n},((\lambda,\mu)g^{-1};\kappa))\,(g,(0,0;0)) = ((\lambda,\mu)g^{-1};\kappa)\cdot g.$$

Thus $Sp(n,\mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,m)}$ naturally by

$$g\cdot(\lambda,\mu;\kappa) = \left((\lambda,\mu)g^{-1};\kappa\right), \qquad g \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}.$$

Definition 3.1. For any Schwartz function $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we define the function $\Theta_f^{[\mathcal{M}]}$ on the Jacobi group $SL(2,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \hookrightarrow G^J$ by

$$\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa) := \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\left((\lambda,\mu;\kappa)(\tau,\phi)\right) f\right](\omega), \tag{3.9}$$

where $(\tau,\phi) \in SL(2,\mathbb{R})$ and $(\lambda,\mu\,;\kappa) \in H_{\mathbb{R}}^{(n,m)}$. The function $\Theta_f^{[\mathcal{M}]}$ is called the theta sum of index $\mathcal{M}$ associated to a Schwartz function f. The projective representation $\pi_{\mathcal{M}}$ of the Jacobi group G^J was already defined by Formula (2.8). More precisely, for $\tau = u+iv \in \mathbb{H}_1$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$, we have

$$\begin{aligned}\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa) &= v^{\frac{mn}{4}}\ e^{2\,\pi\, i\,\sigma(\mathcal{M}(\kappa+\mu^t\lambda))} \\ &\times \sum_{\omega\in\mathbb{Z}^{(m,n)}} e^{\pi\, i\left\{u\|\omega+\lambda\|_{\mathcal{M}}^2 + 2(\omega,\mu)_{\mathcal{M}}\right\}}\ [R_{\mathcal{M}}(i,\phi)f]\left(v^{1/2}(\omega+\lambda)\right).\end{aligned}$$

Lemma 3.2. *We set* $f_\phi := \tilde{R}_{\mathcal{M}}(i,\phi)f$ *for* $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$. *Then for any* $R>1$, *there exists a constant* C_R *such that for all* $x \in \mathbb{R}^{(m,n)}$ *and* $\phi \in \mathbb{R}$,

$$|f_\phi(x)| \leq C_R\,\left(1+\|x\|_{\mathcal{M}}\right)^{-R}.$$

Proof. Following the arguments in the proof of Lemma 4.3 in [6], pp. 428–429, we get the desired result. □

Theorem 3.1 (Jacobi 1)**.** *Let $\mathcal{M}$ be a positive definite symmetric integral matrix of degree m such that $\mathcal{M}\mathbb{Z}^{(m,n)} = \mathbb{Z}^{(m,n)}$. Then for any Schwartz function $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we have*

$$\Theta_f^{[\mathcal{M}]}\left(-\frac{1}{\tau},\, \phi + arg\,\tau\,; -\mu, \lambda, \kappa\right) = \left(\det \mathcal{M}\right)^{-\frac{n}{2}}\, c_{\mathcal{M}}(S,(\tau,\phi))\, \Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa),$$

where

$$c_{\mathcal{M}}(S,(\tau,\phi)) := e^{i\,\pi mn\, sign(\sin\phi\, \sin(\phi+\arg\tau))}.$$

Proof. First we recall that for any Schwartz function $\varphi \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, the Fourier transform $\mathscr{F}\varphi$ of φ is given by

$$\left(\mathscr{F}\varphi\right)(x) = \int_{\mathbb{R}^{(m,n)}} \varphi(y)\, e^{-2\pi i\, \sigma(y\, {}^t x)} dy.$$

Now we put

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z}) \hookrightarrow Sp(n,\mathbb{R})$$

and for any $F \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we put

$$F_{\mathcal{M}}(x) := F(\mathcal{M}^{-1}x), \quad x \in \mathbb{R}^{(m,n)}.$$

According to Formula (2.13), for any $F \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$,

$$\begin{aligned} \left[R_{\mathcal{M}}(S)F\right](x) &= \left(\det \mathcal{M}\right)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} F(y)\, e^{-2\pi i\, \sigma(\mathcal{M} y\, {}^t x)} dy \\ &= \left(\det \mathcal{M}\right)^{-\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} F(\mathcal{M}^{-1}y)\, e^{-2\pi i\, \sigma(y\, {}^t x)} dy \\ &= \left(\det \mathcal{M}\right)^{-\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} F_{\mathcal{M}}(y)\, e^{-2\pi i\, \sigma(y\, {}^t x)} dy \\ &= \left(\det \mathcal{M}\right)^{-\frac{n}{2}} \left[\mathscr{F} F_{\mathcal{M}}\right](x). \end{aligned}$$

Thus we have

$$\mathscr{F} F_{\mathcal{M}} = \left(\det \mathcal{M}\right)^{\frac{n}{2}} R_{\mathcal{M}}(S) F \qquad \text{for } F \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right). \tag{3.10}$$

By Lemma 3.1, we get easily

$$S \cdot (\tau,\phi) = \left(-\frac{1}{\tau}, \phi + \arg\tau\right). \tag{3.11}$$

If we take $F = \pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi))f$ for $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, a fixed element $(\lambda,\mu\,;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and an fixed element $(\tau,\phi) \in SL(2,\mathbb{R})$, then it is easily seen that $F \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$.

According to Formulas (3.11), if we take $F = \pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi))f$ for $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$,

$$\begin{aligned} \left[R_{\mathcal{M}}(S)F\right](x) &= \left[R_{\mathcal{M}}(S)\pi_{\mathcal{M}}\big((\lambda,\mu\,;\kappa)(\tau,\phi)\big)f\right](x), \quad x \in \mathbb{R}^{(m,n)} \\ &= \left[R_{\mathcal{M}}(S)\mathscr{W}_{\mathcal{M}}(\lambda,\mu\,;\kappa)R_{\mathcal{M}}(\tau,\phi)f\right](x) \end{aligned}$$

$$= \left[\mathscr{W}_{\mathcal{M}}\big((\lambda,\mu)S^{-1};\kappa\big)R_{\mathcal{M}}(S)R_{\mathcal{M}}(\tau,\phi)f\right](x)$$
$$= c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\ \left[\mathscr{W}_{\mathcal{M}}(-\mu,\lambda\,;\kappa)R_{\mathcal{M}}\big(S\cdot(\tau,\phi)\big)f\right](x)$$
$$= c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\ \left[\mathscr{W}_{\mathcal{M}}(-\mu,\lambda\,;\kappa)R_{\mathcal{M}}\left(-\frac{1}{\tau},\phi+\arg\tau\right)f\right](x)$$
$$= c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\ \left[\pi_{\mathcal{M}}\left((-\mu,\lambda\,;\kappa)\left(-\frac{1}{\tau},\phi+\arg\tau\right)\right)f\right](x).$$

Thus we obtain

(3.12)
$$\big[R_{\mathcal{M}}(S)F\big](x) = c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\ \left[\pi_{\mathcal{M}}\left((-\mu,\lambda\,;\kappa)\left(-\frac{1}{\tau},\phi+\arg\tau\right)\right)f\right](x).$$

According to Poisson summation formula, we have

(3.13)
$$\sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\mathscr{F}F_{\mathcal{M}}\right](\omega) = \sum_{\omega\in\mathbb{Z}^{(m,n)}}F_{\mathcal{M}}(\omega).$$

It follows from (3.10) and (3.12) that

$$\sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\mathscr{F}F_{\mathcal{M}}\right](\omega)$$
$$= \big(\det\mathcal{M}\big)^{\frac{n}{2}}\sum_{\omega\in\mathbb{Z}^{(m,n)}}\big[R_{\mathcal{M}}(S)F\big](\omega)$$
$$= \big(\det\mathcal{M}\big)^{\frac{n}{2}}\,c_{\mathcal{M}}(S,(\tau,\phi))^{-1}$$
$$\times\sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\pi_{\mathcal{M}}\left((-\mu,\lambda\,;\kappa)\left(-\frac{1}{\tau},\phi+\arg\tau\right)\right)f\right](x)$$
$$= \big(\det\mathcal{M}\big)^{\frac{n}{2}}\,c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\,\Theta_f^{[\mathcal{M}]}\left(-\frac{1}{\tau},\ \phi+\arg\tau\,;-\mu,\lambda,\kappa\right).$$

On the other hand,

$$\sum_{\omega\in\mathbb{Z}^{(m,n)}}F_{\mathcal{M}}(\omega)$$
$$= \sum_{\omega\in\mathbb{Z}^{(m,n)}}F\big(\mathcal{M}^{-1}\omega\big)$$
$$= \sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi))f\right]\big(\mathcal{M}^{-1}\omega\big)$$
$$= \sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi))f\right](\omega)\qquad\Big(\because\ \mathcal{M}^{-1}\mathbb{Z}^{(m,n)}=\mathbb{Z}^{(m,n)}\Big)$$
$$= \Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa).$$

Hence from (3.13) we obtain the desired formula

$$\Theta_f^{[\mathcal{M}]}\left(-\frac{1}{\tau},\,\phi+\arg\tau\,;-\mu,\lambda,\kappa\right)=\big(\det\mathcal{M}\big)^{-\frac{n}{2}}\,c_{\mathcal{M}}(S,(\tau,\phi))\,\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa).$$

If

$$S=\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix},\quad (\tau,\phi)=\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}\quad\text{and}\quad S\cdot(\tau,\phi)=\begin{pmatrix} a_3 & b_3\\ c_3 & d_3\end{pmatrix}\in SL(2,\mathbb{R}),$$

according to Lemma 3.1, we get easily

$$c_1c_2c_3=\big(u^2+v^2\big)^{1/2}\sin\phi\,\sin(\phi+\arg\tau),$$

where

$$(\tau,\phi)=\begin{pmatrix} 1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix} v^{1/2} & 0\\ 0 & v^{-1/2}\end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$$

is the Iwasawa decomposition of $(\tau,\phi)\in SL(2,\mathbb{R})$. Thus we obtain

$$c_{\mathcal{M}}(S,(\tau,\phi))=e^{i\,\pi mn\,\mathrm{sign}(c_1c_2c_3)}=e^{i\,\pi mn\,\mathrm{sign}(\sin\phi\,\sin(\phi+\arg\tau))}.$$

This completes the proof. □

Theorem 3.2 (Jacobi 2)**.** *Let $\mathcal{M}=(\mathcal{M}_{kl})$ be a positive definite symmetric integral $m\times m$ matrix and let $s=(s_{kj})\in\mathbb{Z}^{(m,n)}$ be integral. Then we have*

$$\Theta_f^{[\mathcal{M}]}(\tau+2,\phi\,;\lambda,s-2\,\lambda+\mu,\kappa-s\,{}^t\lambda)=\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)$$

for all $(\tau,\phi)\in SL(2,\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$.

Proof. For brevity, we put $T_*=\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$. According to Lemma 3.1, for any $(\tau,\phi)\in SL(2,\mathbb{R})$, the multiplication of T_* and (τ,ϕ) is given by

$$T_*(\tau,\phi)=(\tau+2,\phi). \tag{3.14}$$

For $s\in\mathbb{R}^{(m,n)}$, $(\lambda,\mu\,;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\tau,\phi)\in SL(2,\mathbb{R})$, according to (3.14),

$$\begin{aligned}
&\pi_{\mathcal{M}}((0,s;0)T_*)\,\pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi))\\
&=\mathscr{W}_{\mathcal{M}}(0,s;0)R_{\mathcal{M}}(T_*)\mathscr{W}_{\mathcal{M}}(\lambda,\mu\,;\kappa)R_{\mathcal{M}}(\tau,\phi)\\
&=\mathscr{W}_{\mathcal{M}}(0,s;0)\mathscr{W}_{\mathcal{M}}\big((\lambda,\mu)T_*^{-1}\,;\kappa\big)R_{\mathcal{M}}(T_*)R_{\mathcal{M}}(\tau,\phi)\\
&=c_{\mathcal{M}}(T_*,(\tau,\phi))^{-1}\mathscr{W}_{\mathcal{M}}(\lambda,s-2\,\lambda+\mu\,;\kappa-s\,{}^t\lambda)R_{\mathcal{M}}\big(T_*(\tau,\phi)\big)\\
&=\mathscr{W}_{\mathcal{M}}(\lambda,s-2\,\lambda+\mu\,;\kappa-s\,{}^t\lambda)R_{\mathcal{M}}(\tau+2,\phi)\\
&=\pi_{\mathcal{M}}\big((\lambda,s-2\,\lambda+\mu\,;\kappa-s\,{}^t\lambda)(\tau+2,\phi)\big).
\end{aligned}$$

Here we used the fact that $c_{\mathcal{M}}(T_*,(\tau,\phi))=1$ because T_* is upper triangular.

On the other hand, according to the assumptions on $\mathcal{M}$ and s, for $f\in\mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$ and $\omega\in\mathbb{Z}^{(m,n)}$, using Formulas (2.1), (2.11) or (3.6), we have

$$\begin{aligned}
&\big[\pi_{\mathcal{M}}\big((0,s;0)T_*\big)\,\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\big]\,(\omega)\\
&=\ \big[\mathscr{W}_{\mathcal{M}}(0,s;0)R_{\mathcal{M}}(T_*)\,\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\big]\,(\omega)\\
&=e^{2\pi i\,\sigma(\mathcal{M}\omega\,{}^ts)}\cdot e^{2\,\|\omega\|_{\mathcal{M}}^2\pi\,i}\ \big[R_{\mathcal{M}}(i,0)\,\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\big]\,(\omega)
\end{aligned}$$

$$= \left[\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\right](\omega).$$

Here we used the facts that

$$e^{2\pi i\,\sigma(\mathcal{M}\omega\,{}^ts)} = 1, \quad e^{2\,\|\omega\|^2_{\mathcal{M}}\pi\, i} = 1 \quad \text{and} \quad R_{\mathcal{M}}(i,0)f = f \text{ (cf. (3.7))}.$$

Therefore for $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$,

$$\begin{aligned}
&\Theta_f^{[\mathcal{M}]}(\tau+2,\phi\,;\lambda, s-2\,\lambda+\mu, \kappa - s\,{}^t\lambda)\\
=& \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\big((\lambda, s-2\,\lambda+\mu, \kappa - s\,{}^t\lambda)(\tau+2,\phi)\big)f\right](\omega)\\
=& \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\big((0,s;0)T_*\big)\,\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\right](\omega)\\
=& \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\right](\omega)\\
=& \,\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa).
\end{aligned}$$

This completes the proof. □

Theorem 3.3 (Jacobi 3)**.** *Let $\mathcal{M} = (\mathcal{M}_{kl})$ be a positive definite symmetric integral $m\times m$ matrix and let $(\lambda_0,\mu_0;\kappa_0) \in H_{\mathbb{Z}}^{(m,n)}$ be an integral element of $H_{\mathbb{R}}^{(n,m)}$. Then we have*

$$\begin{aligned}
&\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda+\lambda_0, \mu+\mu_0, \kappa+\kappa_0+\lambda_0\,{}^t\mu - \mu_0\,{}^t\lambda)\\
=& \, e^{\pi\, i\,\sigma(\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0))}\,\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)
\end{aligned}$$

for all $(\tau,\phi) \in SL(2,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$.

Proof. For any $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we have

$$\begin{aligned}
&\sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\mathscr{W}_{\mathcal{M}}(\lambda_0,\mu_0;\kappa_0)\pi_{\mathcal{M}}\big((\lambda,\mu\,;\kappa)(\tau,\phi)\big)f\right](\omega)\\
=& \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\mathscr{W}_{\mathcal{M}}(\lambda_0,\mu_0;\kappa_0)\mathscr{W}_{\mathcal{M}}(\lambda,\mu\,;\kappa)R_{\mathcal{M}}(\tau,\phi)f\right](\omega)\\
=& \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\mathscr{W}_{\mathcal{M}}(\lambda_0+\lambda,\mu_0+\mu;\kappa_0+\kappa+\lambda_0\,{}^t\mu-\mu_0\,{}^t\lambda))R_{\mathcal{M}}(\tau,\phi)f\right](\omega)\\
=& \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\big((\lambda_0+\lambda,\mu_0+\mu;\kappa_0+\kappa+\lambda_0\,{}^t\mu-\mu_0\,{}^t\lambda)(\tau,\phi)\big)f\right](\omega)\\
=& \,\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda+\lambda_0,\mu+\mu_0,\kappa+\kappa_0+\lambda_0\,{}^t\mu-\mu_0\,{}^t\lambda).
\end{aligned}$$

On the other hand, for any $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we have

$$\sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\mathscr{W}_{\mathcal{M}}(\lambda_0,\mu_0;\kappa_0)\pi_{\mathcal{M}}\big((\lambda,\mu\,;\kappa)(\tau,\phi)\big)f\right](\omega)$$

$$= \sum_{\omega\in\mathbb{Z}^{(m,n)}} e^{\pi i\sigma\{\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0+2\,\omega\,{}^t\mu_0)\}}\,[\pi_{\mathcal{M}}(\tau,\phi\,;\lambda,\mu,\kappa)f]\,(\omega+\lambda_0)$$
$$= e^{\pi i\sigma\{\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0\}} \sum_{\omega\in\mathbb{Z}^{(m,n)}} [\pi_{\mathcal{M}}(\tau,\phi\,;\lambda,\mu,\kappa)f]\,(\omega+\lambda_0)\quad (\because\ \mu_0 \text{ is integral})$$
$$= e^{\pi i\sigma\{\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0\}} \sum_{\omega\in\mathbb{Z}^{(m,n)}} [\pi_{\mathcal{M}}(\tau,\phi\,;\lambda,\mu,\kappa)f]\,(\omega)\quad (\because\ \lambda_0 \text{ is integral})$$
$$= e^{\pi i\sigma\{\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0\}}\,\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa).$$

Finally we obtain the desired result. □

We put $V(m,n)=\mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)}$. Let

$$G^{(m,n)} := SL(2,\mathbb{R})\ltimes V(m,n)$$

be the group with the following multiplication law

$$(g_1,(\lambda_1,\mu_1))\cdot(g_2,(\lambda_2,\mu_2)) = (g_1g_2,(\lambda_1,\mu_1)g_2+(\lambda_2,\mu_2)), \tag{3.15}$$

where $g_1,g_2\in SL(2,\mathbb{R})$ and $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}^{(m,n)}$.

We define

$$\Gamma^{(m,n)} := SL(2,\mathbb{Z})\ltimes H_{\mathbb{Z}}^{(n,m)}.$$

Then $\Gamma^{(m,n)}$ acts on $G^{(m,n)}$ naturally through the multiplication law (3.15).

Lemma 3.3. *$\Gamma^{(m,n)}$ is generated by the elements*

$$(S,(0,0)),\quad (T_\flat,(0,s))\quad \textit{and}\quad (I_2,(\lambda_0,\mu_0)),$$

where

$$S=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad T_\flat=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\quad \textit{and } s,\lambda_0,\mu_0\in\mathbb{Z}^{(m,n)}.$$

Proof. Since $SL(2,\mathbb{Z})$ is generated by S and $T_\flat$, we get the desired result. □

We define

$$\Theta_f^{[\mathcal{M}]}(\tau,\phi;\lambda,\mu) = v^{\frac{mn}{4}} \sum_{\omega\in\mathbb{Z}^{(m,n)}} e^{\pi\,i\,\left\{u\|\omega+\lambda\|^2_{\mathcal{M}}\,+\,2(\omega,\,\mu)_{\mathcal{M}}\right\}}\,\left[R_{\mathcal{M}}(i,\phi)f\right]\left(v^{1/2}(\omega+\lambda)\right).$$

Theorem 3.4. *Let $\Gamma_{[2]}^{(m,n)}$ be the subgroup of $\Gamma^{(m,n)}$ generated by the elements*

$$(S,(0,0)),\quad (T_*,(0,s))\quad \textit{and}\quad (I_2,(\lambda_0,\mu_0)),$$

where

$$T_*=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\quad \textit{and } s,\lambda_0,\mu_0\in\mathbb{Z}^{(m,n)}.$$

Let $\mathcal{M}=(\mathcal{M}_{kl})$ be a positive definite symmetric unimodular integral $m\times m$ matrix such that $\mathcal{M}\mathbb{Z}^{(m,n)}=\mathbb{Z}^{(m,n)}$. Then for $f,g\in\mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, the function

$$\Theta_f^{[\mathcal{M}]}(\tau,\phi;\lambda,\mu)\,\overline{\Theta_g^{[\mathcal{M}]}(\tau,\phi;\lambda,\mu)}$$

is invariant under the action of $\Gamma_{[2]}^{(m,n)}$ *on* $G^{(m,n)}$.

Proof. The proof follows directly from Theorem 3.1 (Jacobi 1), Theorem 3.2 (Jacobi 2) and Theorem 3.3 (Jacobi 3) because the left actions of the generators of $\Gamma_{[2]}^{(m,n)}$ are given by

$$((\tau,\phi),(\lambda,\mu)) \longmapsto \left(\left(-\frac{1}{\tau},\phi+\arg\tau\right),(-\mu,\lambda)\right),$$
$$((\tau,\phi),(\lambda,\mu)) \longmapsto ((\tau+2,\phi),(\lambda,s-2\,\lambda+\mu))$$

and

$$((\tau,\phi),(\lambda,\mu)) \longmapsto ((\tau,\phi),(\lambda+\lambda_0,\mu+\mu_0)). \qquad \square$$

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