Bull. Korean Math. Soc. **53** (2016), No. 6, pp. 1887–1892
http://dx.doi.org/10.4134/BKMS.b160006
pISSN: 1015-8634 / eISSN: 2234-3016

RULED MINIMAL SURFACES IN PRODUCT SPACES

Yuzi Jin, Young Wook Kim, Namkyoung Park, and Heayong Shin

Abstract. It is well known that the helicoids are the only ruled minimal surfaces in $\mathbb{R}^3$. The similar characterization for ruled minimal surfaces can be given in many other 3-dimensional homogeneous spaces. In this note we consider the product space $M \times \mathbb{R}$ for a 2-dimensional manifold M and prove that $M \times \mathbb{R}$ has a nontrivial minimal surface ruled by horizontal geodesics only when M has a Clairaut parametrization. Moreover such minimal surface is the trace of the longitude rotating in M while translating vertically in constant speed in the direction of $\mathbb{R}$.

1. Introduction

In Euclidean 3-space the only ruled minimal surfaces are the planes and the helicoids which are the surfaces obtained by rotating a geodesic in $\mathbb{R}^2$ while translating vertically in constant speed. The similar result can be derived in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ (cf. [1, 2]). For other homogeneous space such as $\mathbb{S}^3$, $\mathbb{H}^3$, Nil^3, Berger sphere and $SL(2, R)$, we also have complete characterization for the ruled minimal surfaces (cf. [3, 5, 6, 7]). In this note we will consider ruled minimal surfaces in the product space $M \times \mathbb{R}$ where M is a 2-dimensional Riemannian manifold.

When M is a surface of revolution we can construct a "helicoid" in $M \times \mathbb{R}$ by rotating the longitude in M while translating in the direction of $\mathbb{R}$ in a constant speed. In Sec. 2, we show that such helicoids are ruled minimal surfaces in $M \times \mathbb{R}$. In fact these are the only nontrivial minimal surfaces ruled by horizontal geodesics. More generally, when M has a parametrization $\varphi(x, y)$ with the metric of the form

$$(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & \beta(x)^2 \end{bmatrix},$$

the surfaces given by $X(s, t) = (\varphi(t, s), as)$ in $M \times \mathbb{R}$ are the only nontrivial minimal surfaces ruled by horizontal geodesics. Here, we call these surfaces as 'helicoids' for convenience. In general, a parametrization is called a Clairaut

Received January 4, 2016.
2010 *Mathematics Subject Classification.* 53A35.
Key words and phrases. ruled surface, minimal surface, helicoid.
Heayong Shin was supported by NRF 2014R1A2A2A01007324.

parametrization when its metric coefficients satisfy $g_{12} = 0$ and both g_{11} and g_{22} are functions of x only. (cf. [4, p. 340]) Our $\varphi(x, y)$ above is a Clairaut parametrization.

In Section 3, we will give the main theorem which states that this is the only possible case of nontrivial horizontally ruled minimal surfaces in product space $M \times \mathbb{R}$. More precisely, if there exist a nontrivial horizontally ruled minimal surface in general product space $M \times \mathbb{R}$, then M must have a Clairaut parametrization and the minimal surface should be one of the helicoids.

2. Helicoids in $M \times \mathbb{R}$ for surface M with Clairaut parametrization

Let M be a 2-dimensional manifold given by Clairaut parametrization $\varphi(x, y)$ with Riemannian metric

$$(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & \beta(x)^2 \end{bmatrix}.$$

From elementary computation, we note that the x-parameter curves of φ are geodesics which we will call longitudes. Any surface of revolution in $\mathbb{R}^3$ has such parametrization with longitude as x-parameter curves. More generally, such manifolds can be characterized as having a non-zero Killing field on M.

In the product space $M \times \mathbb{R}$, the natural generalization of the helicoids is the surfaces given by the parametrization $X(s, t) = (\varphi(t, s), as)$ which we will call as helicoids in $M \times \mathbb{R}$. The next lemma states that the helicoids are in fact a ruled minimal surface in $M \times \mathbb{R}$. Even though the proof of the lemma is a straight forward computation, we give a proof for the sake of completeness.

Lemma 2.1. *Let M be a 2-dimensional manifold given by Clairaut parametrization $\varphi(x, y)$ as above. In the product space $M \times \mathbb{R}$, the parametrization*

$$X(s, t) = (\varphi(t, s), as)$$

gives a ruled minimal surface for any $a \in \mathbb{R}$.

Proof. We take $\Psi(x, y, z) = (\varphi(x, y), z)$ as a coordinate of $M \times \mathbb{R}$. Then, the coefficients for the first fundamental form are

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta(x)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the parametrization of the surface is $X(s, t) = \Psi(t, s, as)$. For the coordinate frame $\{\partial_x, \partial_y, \partial_z\}$ the Riemannian connection becomes

$$\nabla_{\partial_x}\partial_x = 0, \qquad \nabla_{\partial_y}\partial_y = -\beta(x)\beta'(x)\partial_x, \qquad \nabla_{\partial_z}\partial_z = 0$$

$$\nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = \frac{\beta'(x)}{\beta(x)}\partial_y, \quad \nabla_{\partial_y}\partial_z = \nabla_{\partial_z}\partial_y = 0, \quad \nabla_{\partial_z}\partial_x = \nabla_{\partial_x}\partial_z = 0.$$

Now for the surface $X(s, t) = \Psi(t, s, as)$, we have

$$X_s = \partial_y + a\partial_z, \qquad X_t = \partial_x$$

and

$$\begin{aligned} X_{ss} &= \nabla_{X_s} X_s = \nabla_{\partial_y}\partial_y + 2a\nabla_{\partial_y}\partial_z + a^2\nabla_{\partial_z}\partial_z = -\beta(t)\beta'(t)\partial_x, \\ X_{st} &= \nabla_{X_t} X_s = \nabla_{\partial_x}\partial_y + a\nabla_{\partial_x}\partial_z = \frac{\beta'(t)}{\beta(t)}\partial_y = X_{ts}, \\ X_{tt} &= \nabla_{X_t} X_t = \nabla_{\partial_x}\partial_x = 0. \end{aligned}$$

Taking the unit normal vector field $\mathbf{n}$ to the surface as

$$\mathbf{n} = \frac{1}{\sqrt{a^2+\beta^2(t)}}\left(\frac{a}{\beta(t)}\partial_y - \beta(t)\partial_z\right),$$

we have

$$\begin{aligned} E &= \langle X_s, X_s\rangle = a^2 + \beta^2(t), \\ F &= \langle X_s, X_t\rangle = 0, \\ G &= \langle X_t, X_t\rangle = 1 \end{aligned}$$

and

$$\begin{aligned} l &= \langle X_{ss}, \mathbf{n}\rangle = 0, \\ m &= \langle X_{st}, \mathbf{n}\rangle = \frac{a\beta'(t)}{\sqrt{a^2+\beta^2(t)}}, \\ n &= \langle X_{tt}, \mathbf{n}\rangle = 0. \end{aligned}$$

Therefore the mean curvature H of the surface is

$$H = \frac{1}{2}\frac{Gl - 2Fm + En}{EG - F^2} = 0$$

and the surface is minimal in $M \times \mathbb{R}$. Noting that $X_{tt} = 0$ and $\langle X_t, \partial_z\rangle = 0$, the t-parameter curves of X are horizontal geodesics. □

As mentioned before, the 2-dimensional manifold given by Clairaut parametrization can be characterized as a Riemannian 2-manifold with a non-zero Killing field. And from the exactly same computation, the above lemma can be stated as the following.

When M is a 2-manifold with a nonzero Killing field K, let F_s be the flow of K on M. Then, $\tilde{F}_s(p,z) = (F_s(p), z)$ and $T_s(p,z) = (p, z+s)$ are flows of Killing fields in $M \times \mathbb{R}$. Moreover for each $z \in \mathbb{R}$ the orthogonal trajectories of orbits $\{\tilde{F}_s(p,z)\} \subset M \times \{z\}$, $p \in M$ are all geodesics in $M \times \mathbb{R}$ which is horizontal in the sense that it is perpendicular to $\frac{\partial}{\partial z}$ everywhere. Let $\gamma(t)$ be one of such geodesics, then the surface in $M \times \mathbb{R}$ given by the parametrization

$$X(s,t) = T_{as}(\tilde{F}_s(\gamma(t)))$$

is a minimal surface ruled by horizontal geodesics which we will call as horizontally ruled minimal surfaces. In fact, these are the only possible cases of nontrivial horizontally ruled minimal surfaces in a product space $M \times \mathbb{R}$ which we will prove in the next section.

There are examples of the minimal surfaces ruled by non horizontal geodesics in product space. If we consider the Euclidean 3-space $\mathbb{R}^2 \times \mathbb{R}$ with a distinguished vertical direction, then a usual helicoid with oblique (and therefore non vertical) axis serves as one.

3. Horizontally ruled minimal surfaces in $M \times \mathbb{R}$

In this section we consider a ruled minimal surfaces in product space $M \times \mathbb{R}$ for general 2-dimensional Riemannian manifold M. Of course the horizontal section $M \times \{z_0\}$ and the vertical cylinder $\{\gamma(t)\} \times \mathbb{R}$ over a geodesic γ of M are ruled minimal surfaces in $M \times \mathbb{R}$ ruled by horizontal geodesics. These surfaces are referred as the trivial ruled minimal surfaces. Note that these surfaces are totally geodesic in $M \times \mathbb{R}$. For the existence of nontrivial horizontally ruled minimal surfaces in $M \times \mathbb{R}$, the next theorem states that M must have a Clairaut parametrization and the ruled minimal surfaces must be the helicoid considered in Sec. 2 at least locally.

Theorem 3.1. *If there is a ruled minimal surface Σ in $M \times \mathbb{R}$ through $P = (p_0, z_0)$ ruled by horizontal geodesics and $T_P\Sigma$ is neither parallel nor perpendicular to the vertical direction of $\mathbb{R}$, then $p_0 \in M$ has a neighborhood U with Clairaut parametrization and the surface Σ is a part of a helicoid in $U \times \mathbb{R}$ near (p_0, z_0).*

Proof. Since $T_P\Sigma$ is transversal to the vertical direction, there exist a neighborhood of P in Σ on which the projection map $\pi : M \times \mathbb{R} \to M$ is a diffeomorphism. Noting that the ruling geodesics are projected to the geodesics in M, we can take a ruled parametrization of Σ on a neighborhood of P such that

$$\begin{cases} X(s,t) = (\varphi(s,t), h(s)) \subset M \times \mathbb{R}, \\ \varphi(s,t) = \exp_{\alpha(s)}(t\mathbf{v}(s)) \end{cases}$$

for some functions $h(s)$ where $\alpha(s)$ is a unit speed curve with $\alpha(0) = p_0$ in M and $\mathbf{v}(s)$ is a tangent vector field to M along α of unit length with $\langle \mathbf{v}(s), \alpha'(s) \rangle \equiv 0$. Noting that $\varphi(x,y)$ is a geodesic coordinate in some neighborhood U of p_0 in M, we can take a coordinate patch $\Psi(x,y,z) = (\varphi(x,y), z)$ on $U \times \mathbb{R}$. For this coordinate, the coefficients for the first fundamental form are

$$(g_{ij}) = \begin{bmatrix} f^2(x,y) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for some function $f(x,y) > 0$ with $f(x,0) = 1$ and the Riemannian connection becomes

$$\nabla_{\partial_x}\partial_x = \frac{1}{f}\frac{\partial f}{\partial x}\partial_x - f\frac{\partial f}{\partial y}\partial_y \quad \nabla_{\partial_y}\partial_y = 0, \qquad \nabla_{\partial_z}\partial_z = 0,$$
$$\nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = \frac{1}{f}\frac{\partial f}{\partial y}\partial_x, \quad \nabla_{\partial_y}\partial_z = \nabla_{\partial_z}\partial_y = 0, \quad \nabla_{\partial_z}\partial_x = \nabla_{\partial_x}\partial_z = 0.$$

Now for the ruled parametrization $X(s,t) = \Psi(s,t,h(s))$ of the surface Σ, we have

$$X_s = \partial_x + h'(s)\partial_z, \qquad X_t = \partial_y$$

and

$$\begin{aligned} X_{ss} &= \nabla_{X_s} X_s = h''(s)\partial_z + \nabla_{\partial_x}\partial_x + 2h'(s)\nabla_{\partial_x}\partial_z + (h')^2(s)\nabla_{\partial_z}\partial_z \\ &= \frac{1}{f(s,t)}\frac{\partial f(s,t)}{\partial s}\partial_x - f(s,t)\frac{\partial f(s,t)}{\partial t}\partial_y + h''(s)\partial_z, \\ X_{st} &= \nabla_{X_t} X_s = \nabla_{\partial_y}\partial_x + h'(s)\nabla_{\partial_y}\partial_z = \frac{1}{f(s,t)}\frac{\partial f(s,t)}{\partial t}\partial_x \left(= X_{ts}\right), \\ X_{tt} &= \nabla_{X_t} X_t = \nabla_{\partial_y}\partial_y = 0. \end{aligned}$$

Taking the unit normal vector field $\mathbf{n}$ to the surface as

$$\mathbf{n} = \frac{1}{\sqrt{f^2(s,t) + (h')^2(s)}}\left(\frac{h'(s)}{f(s,t)}\partial_x - f(s,t)\partial_z\right),$$

we have

$$\begin{aligned} E &= \langle X_s, X_s\rangle = f^2(s,t) + (h')^2(s), \\ F &= \langle X_s, X_t\rangle = 0, \\ G &= \langle X_t, X_t\rangle = 1 \end{aligned}$$

and

$$\begin{aligned} l &= \langle X_{ss}, \mathbf{n}\rangle = \frac{1}{\sqrt{f^2(s,t) + (h')^2(s)}}\left(h'(s)\frac{\partial f(s,t)}{\partial s} + h''(s)f(s,t)\right), \\ m &= \langle X_{st}, \mathbf{n}\rangle = \frac{h'(s)}{\sqrt{f^2(s,t) + (h')^2(s)}}\frac{\partial f(s,t)}{\partial s}, \\ n &= \langle X_{tt}, \mathbf{n}\rangle = 0. \end{aligned}$$

Therefore the mean curvature H of the surface is

$$\begin{aligned} H =& \frac{1}{2}\frac{Gl - 2Fm + En}{EG - F^2} \\ =& \frac{1}{2}\frac{1}{\sqrt{f^2(s,t) + (h')^2(s)}}\left(h'(s)\frac{\partial f(s,t)}{\partial s} + h''(s)f(s,t)\right). \end{aligned}$$

Since the mean curvature $H = 0$,

$$h'(s)\frac{\partial f(s,t)}{\partial s} + h''(s)f(s,t) = \frac{\partial}{\partial s}\big(h'(s)f(s,t)\big) = 0.$$

This implies $h'(s)f(s,t) = \zeta(t)$ for some function $\zeta(t)$ and since $f(s,0) = 1$, $h'(s) = \zeta(0)$ is a constant. But from the fact that $T_P\Sigma$ is not horizontal, $h'(0) \neq 0$ and $h(s) = c_1 s + c_0$ for some constants c_0 and $c_1 = \zeta(0) \neq 0$. Therefore $f(s,t) = \frac{1}{c_1}\zeta(t)$ is independent of s and the coordinate φ of M is a Clairaut parametrization and clearly $X(s,t)$ gives a helicoid in $U \times \mathbb{R}$. $\square$

For the ruled parametrization $X(s,t) = (\varphi(s,t), h(s))$ in above proof, $h'(s) = c_1$ is a constant and the angle between $T_{X(s,t)}\Sigma$ and ∂_z is given by

$$\arccos\left(\frac{h'(s)}{\sqrt{f^2(s,t) + (h'(s))^2}}\right) = \arccos\left(\frac{c_1}{\sqrt{\xi^2(t) + c_1^2}}\right)$$

which is independent to s. From this we can conclude that for any horizontally ruled minimal surface $\Sigma \subset M \times \mathbb{R}$, the angle between the tangent space of Σ and the vertical direction is constant along any orthogonal trajectories of the ruling geodesics in Σ.

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Yuzi Jin
Department of Mathematics
Jilin Institute of Chemical Technology
Jilin, P. R. China
E-mail address: jinyuzi2011@sina.cn

Young Wook Kim
Department of Mathematics
Korea University
Seoul 136-701, Korea
E-mail address: ywkim@korea.ac.kr

Namkyoung Park
Department of Mathematics
Chung-Ang University
Seoul 156-756, Korea
E-mail address: carstone@nate.com

Heayong Shin
Department of Mathematics
Chung-Ang University
Seoul 156-756, Korea
E-mail address: hshin@cau.ac.kr