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DOMAINS WITH FLAT BOUNDARY PIECES

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Abstract. In this paper, we investigate domains having a flat piece in the boundary and find a sufficient condition for such a domain to be a cone.

1. Introduction

In [3], we investigated convex domains having flat (codimension 1) faces and found a sufficient and necessary condition for being a cone, which is the existence of an accumulation point in the flat face by the automorphism group:

Theorem 1.1 ([3]). *Let $\Omega \subset \mathbb{RP}^n$ be a convex domain with a face F of codimension 1. Then Ω is a cone over F if and only if the automorphism group of Ω has an orbit accumulating at a point of F.*

An affine set $\Omega = \{(x, y, z) \mid -x^2 - y^2 + z^2 > 1,\ z > 0\}$ is not a cone but a convex domain whose affine boundary is a hyperboloid. If we see Ω in the projective space $\mathbb{RP}^3$, then Ω has a flat face in the infinite boundary $\{[x, y, z, 0] \mid z^2 > x^2 + y^2\}$. From the above theorem we can observe that any interior point of their infinite boundary face cannot be an accumulation point by their projective automorphism groups. Especially the automorphism group of a hyperboloid cannot have a cocompact subgroup. As a 2-dimensional example, a hyperbola $\Omega = \{(x, y) \mid y > 1/x,\ x > 0\}$ has a flat face $F = \{[x, y, 0] \mid x > 0,\ y > 0\}$ in the infinite boundary and any point of F cannot be an accumulation point by the action of the automorphism group. Actually the set of all accumulation points of Ω is $\partial F = \{[1, 0, 0], [0, 1, 0]\}$.

In this paper, we consider more general domains, which are possibly not convex but having flat boundary pieces. We can ask if the existence of an accumulation point in the flat boundary piece is still a sufficient condition for such a domain to be a cone or not. We can easily find non-convex domains as counter examples for this question. We'll see that an additional assumption

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about the flat boundary piece is needed to get the same conclusion for general domains:

Theorem 1.2. *Let Ω be a domain, H a hyperplane in $\mathbb{RP}^n$, and F an open subset of H with no complete line. Suppose that F is a flat boundary piece of Ω and its closure $\overline{F}$ is a component of $H \cap \overline{\Omega}$. Then*

$$\Omega = C(F)$$

if and only if there is an $\mathrm{Aut}_{\mathrm{proj}}(\Omega)$-orbit accumulating at a point of F. Here $C(F)$ is the cone over F (see Section 3 for the definition).

2. Benzécri pseudo-length

We define on the set of all line segments of a projective domain a projective invariant called *pseudo-length* which is originally introduced by Benzécri [1]. Every affine domain is considered canonically as a projective domain. Hence the pseudo-length is defined for an affine domain and it is an affine invariant. We say that a line segment $\overline{pq}$ of a projective domain Ω is *bounded by k in Ω* if there exists a line segment $\overline{p'q'}$ of Ω containing $\overline{pq}$ such that the absolute value of the logarithm of (p', p, q, q') is bounded by k. Here (p', p, q, q') denotes the cross ratio of four points, more precisely,

$$(p', p, q, q') = \frac{|\,\overline{p'q}\,|\;|\,\overline{pq'}\,|}{|\,\overline{p'p}\,|\;|\,\overline{qq'}\,|}.$$

Definition 2.1. The pseudo-length $l_\Omega(p, q)$ of a line segment $\overline{pq}$ of a projective domain Ω is defined as follows:

$$l_\Omega(p, q) = \inf\{k \,|\, \overline{pq} \text{ is bounded by } k\}.$$

Let Ω be a projective domain, $p \in \Omega$ and r a real positive number. By the pseudo-ball of center p and of radius r, we mean the set of all points q of Ω such that there exists at least one closed line segment with extremities p and q which is bounded by r. We will denote this pseudo-ball of center p and of radius r by $B_\Omega(p, r)$. Note that $B_\Omega(p, r)$ is an open connected subset of Ω (See [1]).

Fact 2.2 ([1])**.** Let Ω, Ω_1 and Ω_2 be projective domains. Then

(1) For any $g \in \mathrm{Aut}_{\mathrm{proj}}(\Omega)$ and a line segment $\overline{xy}$,

$$l_\Omega(gx, gy) = l_\Omega(x, y).$$

(2) If $\Omega_1 \subset \Omega_2$ and a line segment $\overline{xy}$ is in Ω_1, then

$$l_{\Omega_2}(x, y) \le l_{\Omega_1}(x, y).$$

(3) If the sequence of line segments $\overline{x_i y_i}$ converges to a line segment $\overline{xy}$, then

$$\overline{\lim}\quad l_\Omega(x_i, y_i) \le l_\Omega(x, y).$$

3. Flat boundary pieces

We say $\partial\Omega$ is *locally flat* at p if there is a hyperplane H containing p and an open ball B_p centered at p such that $\Omega \cap B_p$ is an open half ball with $p \in H \cap B_p = B_p \cap \partial\Omega$.

Definition 3.1. Let Ω be a domain in $\mathbb{RP}^n$. We say that Ω has a *flat boundary piece* F if F is a connected subset of a hyperplane H and $\partial\Omega$ is locally flat at each point p in F.

Most simple domains with flat boundary pieces are polyhedrons. And all the affine open cones also have flat boundary pieces in their infinite boundary when we see them as projective domains. Here we define more special kind of cones in the projective space than usual cones in a vector space, actually we define them more geometrically.

Definition 3.2. Let Ω be a domain in $\mathbb{RP}^n$ and B a domain of a hyperplane H of $\mathbb{RP}^n$. We say that Ω is a *cone over* B, denoted by $\Omega = C(B)$, if it is a cone with the infinite boundary $\overline{B}$ in the affine space $\mathbb{A}^n = \mathbb{RP}^n \setminus H$, i.e., there is a point a in the boundary of Ω such that Ω consists of open line segments whose endpoints are a and a point of B. In this case, we sometimes denote it by $\Omega = \{a\} \vee B$ to specify the cone point a.

Two cones over B with a cone point a are projectively equivalent and furthermore all cones over B are projectively equivalent no matter what the cone point is. Therefore $C(B)$ is well-defined in the quotient of the space of all domains by the action of projective transformations. Actually all cones over B are projectively equivalent to each component of $\pi^{-1}(B)$ in $\mathbb{R}^n$ regardless of their cone points, where π is the quotient map from $\mathbb{R}^n \setminus \{0\}$ to $H \simeq \mathbb{RP}^{n-1}$.

Now we can see the difference between usual cones and our cones. In general, it is not true that every affine cone is always a cone in our sense. For example, the affine cone $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0 \text{ or } x > 0\}$ is not a cone in our sense. Actually, we cannot construct $\overline{\Omega}$ adding line segments starting from a subset of $\mathbb{RP}^1$, that is, we cannot find a base B for Ω. (But if we define cones similarly in the sphere $\mathbb{S}^n$ with the action of $PGL^+(n+1, \mathbb{R}) = GL(n+1, \mathbb{R})/\mathbb{R}^+$, then they contain all the cones in an n-dimensional real vector space.)

Remark 3.3. Note that

- (i) if a properly convex domain Ω is a convex sum of a point ξ and a face F with codimension 1, i.e., $\Omega = \{\xi\} \dot{+} F$, then $\Omega = C(F) = \{\xi\} \vee F$,
- (ii) $C(B)$ has no complete line if B has no complete line,
- (iii) $C(B)$ is convex if B is convex.

The following easy lemmas will be needed in the next section.

Lemma 3.4. *Let Ω be a domain in $\mathbb{RP}^n$ with a flat boundary piece P and $\{w_i\} \subset \Omega$. Then the limit of a sequence of ϵ-balls $B_\Omega(w_i, \epsilon) \subset \Omega$ must contain* *an open subset of P if the sequence $\{w_i\}$ converges to a point of the interior of P.*

Proof. Let $w = \lim_{i\to\infty} w_i$ and H the hyperplane containing P. We can choose an affine chart such that H is the set of all points with the last coordinate 0, $C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, 0 < x_k < 1,\ k = 1, \ldots, n\}$ is contained in $\overline{\Omega}$, and the following are satisfied:

(i) there is $N > 0$ such that $w_i \in C$ for all $i \geq N$,
(ii) $S = \overline{C} \cap H = \overline{C} \cap P$,
(iii) $(\overline{C} \setminus S) \subset \Omega$.

Then $\lim_{i\to\infty} B_C(w_i, \epsilon) = B_S(w, \epsilon)$. Since $B_C(w_i, \epsilon) \subset B_\Omega(w_i, \epsilon)$ for all i by Fact 2.2,

$$\lim_{i\to\infty} B_C(w_i, \epsilon) \subset \lim_{i\to\infty} B_\Omega(w_i, \epsilon),$$

which implies that the limit of $B_\Omega(w_i, \epsilon)$ should contain an open set $B_S(w, \epsilon)$ of P. □

4. Singular projective transformations

Since $\mathrm{PM}(n+1, \mathbb{R})$, which is the projectivization of the group of all $(n+1)$ by $(n+1)$ matrices, is a compactification of $\mathrm{PGL}(n+1, \mathbb{R})$, any infinite sequence of non-singular projective transformations contains a convergent subsequence. Note that the limit projective transformation may be singular. For a singular projective transformation g we will denote the projectivization of the kernel and range of g by $\mathrm{Ker}(g)$ and $\mathrm{Ran}(g)$. Then g maps $\mathbb{RP}^n \setminus \mathrm{Ker}(g)$ onto $\mathrm{Ran}(g)$.

When g_i is a sequence of projective transformations in $\mathrm{PGL}(n+1, \mathbb{R})$ which converges to a singular projective transformation g, for any compact subset $C \subset \mathbb{RP}^n$ which does not meet $\mathrm{Ker}(g)$ the sequence $g_i(C)$ converges uniformly to $g(C)$ with the topology induced from the Hausdorff metric on the set of all closed subsets of $\mathbb{RP}^n$ ([1]).

When $\{g_i\}$ preserves a domain and one of its orbit accumulates at a point of a flat boundary piece of the domain, we get the following:

Lemma 4.1. *Let Ω be a domain, $x \in \Omega$, H a hyperplane in $\mathbb{RP}^n$, and F an open subset of H with no complete line. Suppose that F is a flat boundary piece of Ω and its closure $\overline{F}$ is a component of $H \cap \overline{\Omega}$. If $\{g_i\} \subset \mathrm{Aut}_{\mathrm{proj}}(\Omega)$ converges to a singular projective transformation g and $g_i(x)$ converges to a point p of F, then*

(i) $\mathrm{Ran}(g) = H$,
(ii) $\mathrm{Ker}(g)$ *is a point set in $\partial\Omega$,*
(iii) $g(\Omega) \subset F$.

Proof. First we show that $B_\Omega(x, \epsilon)$ cannot intersect $\ker(g)$ for any ϵ. Suppose $y \in \mathrm{Ker}(g) \cap B_\Omega(x, \epsilon)$. Then we can choose two points a and b of Ω satisfying the following:

(i) the open line segment $\overline{ab}$ contains $\overline{xy}$,

(ii) $\overline{ab} \subset \Omega$,

(iii) $\overline{ab} \cap \mathrm{Ker}(g) = \{y\}$.

By Lemma 2.7 of [4], $g_i(\overline{ab})$ must converge to a complete line passing through p, which contradicts that F has no complete line.

Now we show $\mathrm{Ran}(g) = H$. By Lemma 2.6 in [4], $\mathrm{Ran}(g) \cap \overline{\Omega}$ generates $\mathrm{Ran}(g)$ and p must be an interior point of $\mathrm{Ran}(g) \cap \overline{\Omega}$. So $\mathrm{Ran}(g)$ should be a projective subspace of H because any projective subspace L intersecting H transversally cannot contain p as an interior point of $L \cap \overline{\Omega}$.

From the fact $B_\Omega(x, \epsilon) \cap \mathrm{Ker}(g) = \emptyset$ for all ϵ, we see that $\overline{B_\Omega(x, \epsilon)}$ is a subset of Ω not intersecting $\mathrm{Ker}(g)$. So $g_i(\overline{B_\Omega(x, \epsilon)}) = \overline{B_\Omega(g_i(x), \epsilon)}$ converges uniformly to $g(\overline{B_\Omega(x, \epsilon)})$ and thus $g(\overline{B_\Omega(x, \epsilon)})$ contains an open subset of F by Lemma 3.4, which completes the proof of $\mathrm{Ran}(g) = H$.

Since $\mathrm{Ran}(g) = H$, $\mathrm{Ker}(g)$ must be a point set $\{z\}$. If $z \in \Omega$, then Ω has an open line segment $\overline{a'b'}$ containing z, which is a part of the line l_{zx} passing through z and x. (Note that the line segment $\overline{zx}$ might not be in Ω.) The sequence $g_i(\overline{a'b'})$ converges to a complete line and $g(a') = g(b') = g(x) = p$, which contradicts that there is no complete line passing through p. This implies that $\mathrm{Ker}(g) \cap \Omega = \emptyset$. Hence for any point $y \in \Omega$, $g_i(y)$ converges to a point $g(y)$ in H, which implies $g(\Omega) \subset F$ because Ω is connected.

If $\overline{\Omega} \cap \mathrm{Ker}(g) = \emptyset$, then $g_i(\overline{\Omega})$ have to converge uniformly to $g(\overline{\Omega}) \subset H$. But this cannot happen because $g_i(\overline{\Omega}) = \overline{\Omega}$ for all i. This contradiction implies that z must be a boundary point of Ω. $\square$

5. The proof of Theorem 1.2

Since every cone has an accumulation point in the base, we only need a proof for the opposite direction. Now we may assume that there is a point $x \in \Omega$ and a sequence of projective transformations $\{g_i\} \subset \mathrm{Aut}_{\mathrm{proj}}(\Omega)$ such that $\{g_i\}$ converges to a singular projective transformation g and $g_i(x)$ converges to a point $p \in F$. Then by Lemma 4.1, we know that

$$\mathrm{Ran}(g) = H, g(\Omega) \subset F$$

and there is a point $z \in \partial\Omega$ such that

$$\mathrm{Ker}(g) = \{z\} \subset \partial\Omega.$$

So g is a map from $\mathbb{RP}^n \setminus \{z\}$ to H, and g maps each line passing through z to a point of H.

As in the convex case ([3]), we can divide into two cases according to whether z is in H or not.

Case 1 : $z \notin H$

In this case g maps H to itself homeomorphically and g can be considered as an automorphism on H since every line passing through z meets exactly one point of H. We have evidently $g(\overline{F}) \subset \overline{F}$ and $g(F)$ is an open subset of F, because $g(\Omega) \subset F$ and $g(F)$ is an open subset of H.

We will next prove that $g(\overline{F}) = \overline{F}$, $g(F) = F$ and $g(\Omega) = F$. Since $\overline{F}$ does not intersect $\mathrm{Ker}(g)$, $g_i(\overline{F})$ converges uniformly to $g(\overline{F}) \subset \overline{F}$ and thus there is a positive integer N such that $g_i(\overline{F}) \subset \overline{F}$ for all $i \geq N$. But $g_i(\overline{F})$ cannot be a proper subset of $\overline{F}$ because the g_i-invariance of Ω implies that $g_i(F)$ is also another maximal flat boundary piece of Ω. This implies $g_i(F) = F$ for all $i \geq N$ and so our claim is proved.

There are two cones C_1, C_2 over F in $\mathbb{RP}^n$ and another two cones C_1', C_2' over $H \setminus \overline{F}$ in $\mathbb{RP}^n$ whose cone points are all z. The interiors of the four cones are all disjoint and the union of their closures $\overline{C}_1 \cup \overline{C}_2 \cup \overline{C}_1' \cup \overline{C}_2'$ is the whole projective space $\mathbb{RP}^n$. Note that $g^{-1}(F) = C_1 \cup C_2$ and $g^{-1}(H \setminus \overline{F}) = C_1' \cup C_2'$. So either $\Omega \subset C_1$ or $\Omega \subset C_2$ by connectivity of Ω. We may assume that $\Omega \subset C_1$ and denote C_1 by $\{z\} \vee F$.

Suppose $\Omega \neq C_1$, i.e., there is a point w in $\partial\Omega$ which is not contained in $\partial(\{z\} \vee F)$. Then we can choose a point η in F such that w is contained in the open line segment $\overline{z\eta}$. Since $g(w) = g(\eta) \in g(F) = F$, the sequence of boundary points $\{g_i(w)\}$ must accumulate at $g(\eta)$ and hence $g_i(w)$ is contained in F except for finite points. So w must be a locally flat point and $g_i(w) \in F$ for some i, which is impossible because $g_i(F) = F$ and $w \notin F$. This contradiction implies that

$$\Omega = C_1 = \{z\} \vee F.$$

Case 2 : $z \in H$

If b is a boundary point of Ω such that the line l_{bz} connecting b and z intersects Ω, then the limit point $g(b)$ of the sequence $g_i(b)$ is in $g(\Omega) \subset F$. Since F is a flat boundary piece, $g_i(b)$ must be in F for sufficiently large i and so $\partial\Omega$ must be also locally flat at b.

Now we consider the set E which consists of such b's, i.e.,

$$E = \{b \in \partial\Omega \,|\, b \notin F,\ l_{bz} \cap \Omega \neq \emptyset\}.$$

Then for each $b \in E$, there is a maximal flat boundary piece F_b containing b and a natural number n_b such that $g_i(F_b) = F$ for all $i \geq n_b$.

If we assume that $E = \emptyset$, then for each point $w \in \Omega$, whole line l_{wz} connecting w and z must intersect $\partial\Omega$ only at $\overline{F}$. But this is impossible, because z and $\overline{F}$ is contained in a hyperplane H and Ω is a domain.

Now we may assume that there is a point $b_* \in E$. If we suppose that there is a point b' in E such that $b' \notin F_{b_*}$, then

$$F_{b'} \cap F_{b_*} = \emptyset,$$

and $g_i(F_{b'}) = g_i(F_{b_*}) = F$ for all i greater than both $n_{b'}$ and n_{b_*}, which contradicts that g_i is a non-singular projective transformation. So we get $E \subset F_{b_*}$.

Now we will prove Ω is a cone over F_{b_*}. At least one of two cones C_1 and C_2 over F_{b_*} with a cone point z intersects Ω, say C_1 (if both of them intersect Ω, choose any of them). We may denote C_1 by $\{z\} \vee F_{b_*}$. If $t \in \partial(\{z\} \vee F_{b_*}) \cap \Omega$, then there is a point $s \in \partial F_{b_*}$ such that $t \in \overline{zs}$, which is a contradiction because any boundary point of F_{b_*} cannot be a point of E. So $\partial(\{z\} \vee F_{b_*}) \cap \Omega = \emptyset$ and thus Ω must be contained in $\{z\} \vee F_{b_*}$. But if we suppose that there is a point w of $\partial\Omega$ inside $\{z\} \vee F_{b_*}$, then there is a point d in F_{b_*} such that $l_{wz} = l_{dz}$, which means w is a point of $E \setminus F_{b_*}$ and this cannot happen. So we have proved

$$\Omega = \{z\} \vee F_{b_*}.$$

Since $g_i(F_{b_*}) = F$ for $i \geq n_{b_*}$, we see

$$\Omega = g_{n_{b_*}}(\Omega) = g_{n_{b_*}}(\{z\} \vee F_{b_*}) = \{g_{n_{b_*}}(z)\} \vee g_{n_{b_*}}(F_{b_*}) = \{g_{n_{b_*}}(z)\} \vee F,$$

which completes the proof by choosing $g_{n_{b_*}}(z)$ as ξ.

Remark 5.1. The case 2 in the proof of Theorem 1.2 is when the kernel of g is contained in the range of g. This case occurs when Ω is a double cone, that is, the base of Ω is again a cone: In this case, Ω has two different expressions,

$$\Omega = \{z\} \vee F_{b_*} = \{g_{n_{b_*}}(z)\} \vee F.$$

Since $z \in \overline{F}$, F consists of all the open line segments whose endpoints are z and points of $P = \mathrm{int}(\overline{F} \cap \overline{F}_{b_*})$ and thus

$$F = \{z\} \vee P.$$

This means the base F is also a cone and Ω is a double cone,

$$\Omega = \{g_{n_{b_*}}(z)\} \vee F = \{g_{n_{b_*}}(z)\} \vee \{z\} \vee P.$$

Note that Ω is k-times cone if the cardinal number of $\mathrm{Aut}_{proj}(\Omega)$-orbit of z is k. Especially Ω becomes a polygon if $k+1$ equals the dimension of Ω.

Remarks and Examples 5.2. The following examples show that the conditions about flat boundary pieces in Theorem 1.2 are necessary.

(i) $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x > 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1,\ y > 0\}$,

$$g_n = \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}, \quad F = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\ y = 0\}.$$

F is a flat boundary piece of Ω and $\{g_n(1/2,1)\}$ converges to $(1/2,0) \in F$. But Ω is not a cone, which does not contradict Theorem 1.2 because $\overline{F}$ is not a component of $\langle F \rangle \cap \overline{\Omega}$.

(ii) $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x \leq 0,\ y > 0\} \cup \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ 0 < y < 1/x\}$,

$$g_n = \begin{pmatrix} 2^n & 0 \\ 0 & 1/2^n \end{pmatrix}, \quad F = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}.$$

F is a flat boundary piece of Ω and any point in the positive y-axis accumulates to $(0,0) \in F$ under the action of $\{g_n\}$. But Ω is not a cone, which does not contradict Theorem 1.2 because F has a full line.

(iii) $\Omega = \Omega_1 \setminus K$,

$$\Omega_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x > 0,\ y > 0,\ z > 0\},$$

$$K = \bigcup_{n \in \mathbb{Z},\ 1 \leq t,\ s \leq 2} \{(t,s,z) \in \mathbb{R}^3 \mid 3^n \leq z \leq 2 \cdot 3^n\},$$

$$g_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3^n \end{pmatrix},$$

$$F = \{(x, y, 0) \in \mathbb{R}^3 \mid x > 0,\ y > 0\} \setminus \{(x, y, 0) \in \mathbb{R}^3 \mid x \in [1, 2],\ y \in [1, 2]\}.$$

F is a flat boundary piece of Ω and $\{g_n(1/2, 1/2, 1)\}$ converges to $(1/2, 1/2, 0) \in F$. But Ω is not a cone, which does not contradict Theorem 1.2 because $\overline{F}$ is not a component of $\langle F \rangle \cap \overline{\Omega}$. In fact, we see that $g(\Omega)$ equals $\{(x, y, 0) \in \mathbb{R}^3 \mid x > 0,\ y > 0$ which contains F properly.

Now we get a generalized version of Corollary 6 in [3].

Corollary 5.3. *Let Ω be an affine domain in $\mathbb{R}^n$ and H a hyperplane and F an open subset of H with no complete line. Suppose that F is a flat boundary piece of Ω and its closure $\overline{F}$ is a component of $H \cap \overline{\Omega}$. Then*

$$C = \mathbb{R}^+ \times F.$$

if and only if there is an $\mathrm{Aut}_{\mathrm{aff}}(\Omega)$-orbit accumulating to a point of F.

Proof. By the hypothesis, there is a point $x \in \Omega$ and a sequence $\{g_i\}$ in $\mathrm{Aut}_{\mathrm{aff}}(\Omega)$ such that $g_i(x)$ converges to a point in F. Let g be a limit singular projective transformation of the sequence $\{g_i\}$. Then by Theorem 1.2, there is a point $\xi \in \partial\Omega$ such that

$$\Omega = C(F) = \{\xi\} \vee F,$$

when Ω is considered as a subset of $\mathbb{RP}^n$. So it suffices to show that ξ lies in the infinite boundary of Ω, that is, $\xi \in \partial\Omega \cap \mathbb{RP}^{n-1}_\infty$.

As we can see in the proof of Theorem 1.2, $\{\xi\}$ is either the kernel $\mathrm{Ker}(g)$ or the g_i-image of $\mathrm{Ker}(g)$ for some i. Since $\mathrm{Ran}(g) \cap \mathbb{R}^n$ contains F, $\mathrm{Ker}(g)$ is a subset of $\mathbb{RP}^{n-1}_\infty$ by Lemma 3.5 of [2]. This implies that $\xi \in \mathbb{RP}^{n-1}_\infty$ because g_i preserves $\mathbb{R}^n$ for all i. □

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