Bull. Korean Math. Soc. **53** (2016), No. 6, pp. 1869–1878
http://dx.doi.org/10.4134/BKMS.b151057
pISSN: 1015-8634 / eISSN: 2234-3016

ON THE CLOSED EINSTEIN-WEYL STRUCTURE AND COMPACT K-CONTACT MANIFOLD

Amalendu Ghosh

Abstract. We study closed Einstein-Weyl structure on compact K-contact manifolds and prove that a compact K-contact manifold admitting a closed Einstein-Weyl structure is Einstein and Sasakian.

1. Introduction

In recent years, much attention has been given to the classification of Riemannian manifolds admitting several generalizations of Einstein metric. An interesting generalization of such metric is the so-called Einstein Weyl metric. This appears as a conformal generalization of Einstein metric, defined in the background of Weyl manifold. The notion of Weyl manifold was proposed by H. Weyl [19] as a generalization of the Riemannian connection to find a good geometric model for space-time geometry to unify field theory. Although his attempt failed, but his connection provides an instructive example of non-Riemannian geometry which helps us to explain several physical problems.

A Weyl structure on a manifold M is a class $[g] = \{e^{2f}g : f \in C^{\infty}(M)\}$ of conformally related Riemannian metric g of M which satisfies the conformally invariant equation: $D^{[g]}g = -2\theta \otimes g$, where θ is a unique 1-form associated with the metric g such that $g(X, E) = \theta(X)$, and $D^{[g]}$ is the unique torsion free connection known as Weyl connection. The vector field E dual to the Lee form θ with respect to g is known as the Einstein-Weyl vector field. A Riemannian manifold admitting such structure referred as *Weyl manifold*. Thus a Weyl manifold is a manifold equipped with a conformal structure and a compatible Weyl connection. We denote it by $(M, [g], D^{[g]})$. This Weyl structure is said to be closed if the unique 1-form θ is closed, i.e., $d\theta = 0$. Unlike the Riemannian manifold, the Ricci tensor of a Weyl manifold need not be symmetric. Therefore, to define Einstein like condition in the background of Weyl manifold, one needs to consider the symmetric part of the Ricci tensor of the Weyl connection. If this is proportional to a Riemannian metric of the conformal class

Received December 26, 2015; Revised May 6, 2016.
2010 *Mathematics Subject Classification.* 53C25, 53C15, 53C20.
Key words and phrases. closed Einstein-Weyl structure, K-contact manifold, Sasakian manifold.

©2016 Korean Mathematical Society

$[g]$, then we say that the structure is *Einstein-Weyl*. It is evident that every Einstein manifold can be regarded as an Einstein-Weyl manifold. But there exists manifold having no Einstein metric admits an Einstein-Weyl structure (e.g. see [4], [15]). Indeed, by applying a D-homothetic deformation (Tanno [17]) to the metric of the unit sphere $S^{2n+1}(1)$ one can prove that the resulting metric becomes η-Einstein (i.e., the Ricci tensor S satisfies $S = \alpha g + \beta\eta \otimes \eta$ for some constants α, β and η is the contact form), and it admits an Einstein-Weyl structure $(g, f\eta)$ with $\beta < 0$ for some constant f. For details we refer to [4] and [8].

Recently, Einstein-Weyl structures have received a lot of attention in the frame work of η-Einstein contact metric manifold. Because it has been observed by several authors (see [8], [11], [12], [14] and [15]) that there is a nice connection between Einstein-Weyl structure and η-Einstein contact geometry. In this direction, the first breakthrough was provided by Pedersen-Swann [16] (see also Higa [9]) who constructed such structures on the principal circle bundle over Kaehler Einstein manifolds with positive scalar curvature. Later on, Narita ([14], [15]) and Boyer-Galicki-Matzeu [4] constructed several examples of Einstein-Weyl structures in the framework of η-Einstein Sasakian manifolds. It is known (see [9]) that a Riemannian manifold admits a pair of Einstein-Weyl structures $(g, \pm\theta)$ if and only if it satisfies some additional conditions on the Ricci tensor and the covariant derivative of the 1-form θ. Recall that any η-Einstein K-contact manifold admits an Einstein-Weyl structure $(g, f\eta)$ with $\beta < 0$ for some constant f (see [4]). Then it also admits an Einstein-Weyl structure $(g, -f\eta)$. Particularly, in [4] the authors proved that *"a Sasakian manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ admits a pair of Einstein Weyl structures $(g, \pm\theta)$ (where θ is the 1-form associated with the metric g of the conformal class $[g]$) if and only if it is η-Einstein with Einstein constants (α, β), where $\beta < 0$ and $\theta = \pm\mu\eta$ such that $\mu^2 = -\beta/(2n-1)$"*. Later on, the author [8] extended this result for all classes of complete K-contact manifolds by proving that *"if a complete K-contact manifold admits the pair of Einstein-Weyl structures $(g, \pm\theta)$, then it is compact η-Einstein and Sasakian with Einstein constants (α, β) where $\beta < 0$"*. These results imply that there is a strong connection between the pair of Einstein-Weyl structures $(g, \pm\theta)$ and η-Einstein K-contact geometry.

In this paper, we study closed Einstein-Weyl structure on compact K-contact manifold and improved a result obtained by Matzeu [12]. Actually, Matzeu (see [12]) proved that *"a compact K-contact manifold $M(\varphi, \xi, \eta, g)$ of dimension $2n+1 \geq 3$ admitting a closed Einstein-Weyl structure with θ as its 1-form associated with the metric $g \in [g]$ is Sasakian if and only if it is η-Einstein."* This establishes again a strong connection between η-Einstein Sasakian geometry and closed Einstein-Weyl structure. However, by using the full strength of K-contact geometry we have been able to prove that any compact K-contact manifold admitting a closed Einstein-Weyl structure is Einstein-Sasakian. We address these issues in Section 4.

2. Rudimets of K-contact geometry

Now, we review some basic contact metric geometry. A $(2n+1)$ dimensional Riemannian manifold (M,g) is said to be an almost contact metric manifold if there exist a $(1,1)$ tensor field φ, a unit vector field ξ (called the Reeb vector field) and a 1-form η such that

$$\varphi^2 X = -X + \eta(X)\xi, \quad g(X,\xi) = \eta(X), \quad g(\xi,\xi) = 1$$

for any vector field X. From these it is easy to verify that $\varphi\xi = 0$ and $\eta o \varphi = 0$. For such a manifold, one can always define a 2-form ϕ by $\phi(X,Y) = g(X,\varphi Y)$, called the fundamental 2-form. Further, an almost contact metric structure of M is said to be contact metric if $\phi = d\eta$. If, in addition ξ is Killing, then M is said to be K-contact. The following formulas are valid for a K-contact (Sasakian) manifold (see [1]):

$$\nabla_X \xi = -\varphi X, \tag{2.1}$$

$$Q\xi = 2n\xi, \tag{2.2}$$

$$R(X,\xi)\xi = X - \eta(X)\xi, \tag{2.3}$$

$$R(\xi,Y)X = (\nabla_Y \varphi)X, \tag{2.4}$$

$$(\nabla_Y \varphi)X + (\nabla_{\varphi Y}\varphi)\varphi X = 2g(Y,X)\xi - \eta(X)(Y + \eta(Y)\xi), \tag{2.5}$$

where Q is the Ricci operator associated with the Ricci tensor S, R is the Riemann curvature tensor of g and ∇ is the operator of covariant differentiation of g. We also note that the almost contact metric structure is said to be normal if $[\varphi,\varphi](X,Y) + 2d\eta(X,Y)\xi = 0$, for any vector field X,Y on M, where $[\varphi,\varphi](X,Y) = [\varphi X,\varphi Y] + \varphi^2[X,Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y]$. A normal almost contact metric manifold is called a Sasakian, equivalently, an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(X)Y$$

for any vector field X,Y on M. From the definition it follows that $\eta \wedge (d\eta)^n$ is non vanishing everywhere on M, which is also the volume form on M. In other words, a contact metric manifold $M(\varphi,\xi,\eta,g)$ is Sasakian if and only the metric cone $C(M)(dr^2 + r^2 g, d(r^2\eta))$ is Kaehler. A Sasakian manifold is K-contact, but the converse is not true, except in dimension 3. Another equivalent characterization of a Sasakian manifold is that a contact metric manifold is said to be Sasakian if and only if the curvature tensor R satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.6}$$

For details about contact metric manifolds we refer to [1].

3. Einstein-Weyl geometry

We now assume that $M^{2n+1}(\varphi, \xi, \eta, g)$ admits an Einstein-Weyl structure, i.e., the metric g represents the Weyl structure. Then from $D^{[g]}g = -2\theta \otimes g$ it is easy to deduce that

$$D^{[g]}_X Y = \nabla_X Y + \theta(X)Y + \theta(Y)X - g(X,Y)E, \tag{3.1}$$

where ∇ is the Riemannian connection of g and X, Y are arbitrary vector fields of M. We note that the equation (3.1) and $D^{[g]}g = -2\theta \otimes g$ are invariant under the transformations $\bar{g} = e^{2f}g$ and $\bar{\theta} = \theta - df$, where f is a smooth function on M. In particular, if the 1-form θ is closed, then the Weyl connection becomes locally the Riemannian connection. From (3.1) one can derive the relation between the Riemann curvature tensor R and the Weyl curvature tensor $R^{D^{[g]}}$ (see Higa[9]) as follows:

$$\begin{aligned} R^{D^{[g]}}(X,Y)Z = {} & R(X,Y)Z + \{(\nabla_X\theta)Z - \theta(X)\theta(Z)\}Y \\ & - \{(\nabla_Y\theta)Z - \theta(Y)\theta(Z)\}X - g(Y,Z)\{(\nabla_X E \\ & - \theta(X)E\} + g(X,Z)\{(\nabla_Y E - \theta(Y)E\} + d\theta(X,Y)Z \\ & - |\theta|^2\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{3.2}$$

for every vector field X, Y, Z on M. As a consequence equation (3.2) provides the following relation between the Weyl Ricci tensor $S^{[g]}$ of the connection $D^{[g]}$ and the Ricci tensor S of the Riemann connection ∇ (see [9])

$$\begin{aligned} S^{D^{[g]}}(X,Y) = {} & S(X,Y) - 2n(\nabla_X\theta)Y + (\nabla_Y\theta)X \\ & + (2n-1)\theta(X)\theta(Y) + (\delta\theta - (2n-1)|\theta|^2)g(X,Y), \end{aligned} \tag{3.3}$$

where $\delta\theta$ is the co-differential of θ and $|\theta|$ is the pointwise norm of θ with respect to g. A Weyl manifold $(M, [g], D^{[g]})$ is said to be Einstein-Weyl if there exists a smooth function Λ on M such that:

$$S^{D^{[g]}}(X,Y) + S^{D^{[g]}}(Y,X) = \Lambda g(X,Y). \tag{3.4}$$

Thus, from (3.4) and (3.3) it follows that the Weyl manifold $(M, [g], D^{[g]})$ is Einstein-Weyl if and only if

$$S(X,Y) - \frac{2n-1}{2}((\nabla_X\theta)Y + (\nabla_Y\theta)X) + (2n-1)\theta(X)\theta(Y) = \sigma g(X,Y) \tag{3.5}$$

for every vector field X,Y on M and σ is smooth function on M. We refer this equation as Einstein-Weyl equation. Note that if θ is closed, then θ becomes locally exact, and, hence the metric g is locally conformal to an Einstein metric. For instance, the product $N \times S^1$ is locally conformal to an Einstein manifold, where N is any Einstein manifold of positive scalar curvature. As a consequence, Einstein-Weyl structure is considered as a nice generalization of Einstein metric from the view point of conformal geometry. In [5] Gauduchon proved a fundamental theorem on Einstein-Weyl structures, when M is compact. Particularly, he proved that *"On a compact Weyl manifold, up to homothety, there is always possible to find a unique metric g_0 in the conformal class $[g]$ such that the corresponding* 1*-form θ_0 is co-closed* (*i.e.*, $\delta\theta_0 = 0$)*"*. We shall refer this metric as the *Gauduchon metric.* By virtue of this, Pedersen-Swann [16] (see also Tod [18]) proved that on a compact Einstein-Weyl manifold this co-closed 1-form turns out to be the dual of a Killing field. Consequently, on every compact manifold one can split the Einstein-Weyl equation (3.5) into the simplified Einstein-Weyl equation and the Killing dual field equation:

$$S(X,Y) + (2n-1)\theta(X)\theta(Y) = \sigma g(X,Y),$$
$$(\nabla_X\theta)Y + (\nabla_Y\theta)X = 0.$$

Closed Einstein-Weyl structure on a compact manifold enjoys a nice property. In fact, Gauduchon [6] showed that if the 1-form associated with the Einstein-Weyl structure is closed and exact, then the Weyl curvature tensor $R^{D^{[g]}}$ and the Weyl Ricci tensor $S^{D^{[g]}}$ vanish identically (see also Matzeu [12]), and as a result equations (3.2) and (3.3) reduces to

$$\begin{aligned} R(X,Y)Z = {} & \{(\nabla_Y\theta)Z - \theta(Y)\theta(Z)\}X - \{(\nabla_X\theta)Z - \theta(X)\theta(Z)\}Y \\ & + g(Y,Z)\{(\nabla_X E - \theta(X)E\} - g(X,Z)\{(\nabla_Y E - \theta(Y)E\} \\ & + |\theta|^2\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} (2n-1)(\nabla_X\theta)Y = {} & S(X,Y) + (2n-1)\theta(X)\theta(Y) \\ & + (\delta\theta - (2n-1)|\theta|^2)g(X,Y) \end{aligned} \tag{3.7}$$

for every vector field X, Y, Z on M. Now, we are at a position to derive some equations for our latter use. First, we can write equation (3.7) as

$$(2n-1)\nabla_X E = QX + (2n-1)\theta(X)E + \lambda X, \tag{3.8}$$

where λ is a function given by

$$\lambda = \delta\theta - (2n-1)|\theta|^2. \tag{3.9}$$

Differentiating equation (3.8), using the resulting equation in the well-known formula:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

and the repeated use of (3.8), one can easily deduce

$$\begin{aligned} (2n-1)R(X,Y)E = {} & (\nabla_X Q)Y - (\nabla_Y Q)X + \theta(Y)QX \\ & - \theta(X)QY + \lambda[\theta(Y)X - \theta(X)Y] \\ & + (X\lambda)Y - (Y\lambda)X. \end{aligned} \tag{3.10}$$

Finally, we remark that the odd dimensional spheres and products of spheres $S^1 \times S^{2n+1}$ admit Einstein-Weyl structures while $S^1 \times S^2$ and $S^1 \times S^3$ do not carry any Einstein metric [10]. In particular, $S^1 \times S^{2n+1}$ admits flat Weyl structures, which are therefore closed Einstein-Weyl.

4. Main results

We now consider a compact K-contact manifold admitting closed Einstein-Weyl structure to prove the following:

Theorem 4.1. *Let $M(\varphi, \xi, \eta, g)$ be a compact K-contact manifold of dimension $2n+1 \geq 3$. If g represents a closed Einstein-Weyl structure with θ is a 1-form associated with $g \in [g]$, then M is Einstein and Sasakian.*

Proof. Differentiating covariantly (2.2) along an arbitrary vector field X and then recalling (2.1) we find that

$$(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X. \tag{4.1}$$

Taking scalar product of (3.10) with ξ, using (4.1) and (2.2) provides

$$\begin{aligned}(2n-1)g(R(X,Y)E,\xi) = {} & g((Q\varphi + \varphi Q)X, Y) - 4ng(\varphi X, Y) \\ & + (2n+\lambda)[\theta(Y)\eta(X) - \theta(X)\eta(Y)] \\ & + (X\lambda)\eta(Y) - (Y\lambda)\eta(X).\end{aligned} \tag{4.2}$$

Replacing Y by ξ in (4.2) and making use of (2.2), (2.3) we obtain

$$(\lambda+1)\{E - \eta(E)\xi\} = D\lambda - (\xi\lambda)\xi \tag{4.3}$$

for all X in M and D is the gradient operator. Differentiating this along an arbitrary vector field X and using (2.1) gives

$$\begin{aligned}&(\lambda+1)\{\nabla_X E + \eta(X)\varphi X + g(E, \varphi X)\xi - g(\nabla_X E, \xi)\xi\} \\ &= (X\lambda)\varphi^2 E + \nabla_X D\lambda + (\xi\lambda)\varphi X - X(\xi\lambda)\xi.\end{aligned}$$

Inner product of this equation with an arbitrary vector field Y, symmetrizing the resulting equation and by virtue of the Poincare lemma: $g(\nabla_X D\lambda, Y) = g(\nabla_Y D\lambda, X)$ shows

$$\begin{aligned}&(\lambda+1)\{2\eta(E)g(\varphi X, Y) + g(E, \varphi X)\eta(Y) \\ &\quad - g(E, \varphi Y)\eta(X) - g(\nabla_X E, \xi)\eta(Y) + g(\nabla_Y E, \xi)\eta(X)\} \\ &= (X\lambda)g(\varphi^2 E, Y) - (Y\lambda)g(\varphi^2 E, X) \\ &\quad + 2(\xi\lambda)g(\varphi X, Y) + Y(\xi\lambda)\eta(X) - X(\xi\lambda)\eta(Y),\end{aligned}$$

where we have also used the fact that the 1-form θ is closed. Now, we replace X by φX and Y by φY in the foregoing equation to achieve

$$\begin{aligned}2(\lambda+1)\eta(E)g(\varphi X, Y) = {} & (\varphi X\lambda)g(\varphi E, Y) - (\varphi Y\lambda)g(\varphi E, X) \\ & + 2(\xi\lambda)g(\varphi X, Y).\end{aligned} \tag{4.4}$$

Taking inner product of (4.3) with φX one can deduce

$$(\varphi X\lambda) + (\lambda+1)g(\varphi E, X) = 0.$$

By virtue of this, the first two term of the left hand side of (4.4) vanishes. So, we have $[(\lambda+1)\eta(E)-\xi\lambda]d\eta(X,Y)=0$. Moreover, since $d\eta(X,Y)=g(X,\varphi Y)$ is non-vanishing on any K-contact manifold, we obtain

$$(\lambda+1)\eta(E)=\xi\lambda. \tag{4.5}$$

Next, we contract equation(3.10) to get

$$QE+\frac{1}{4n}Dr+D\lambda+(\delta\theta)E=0. \tag{4.6}$$

As ξ is Killing, $\xi r=0$. Thus, the inner product of (4.6) with ξ yields

$$g(QE,\xi)+\xi\lambda+(\delta\theta)\eta(E)=0.$$

Finally, applying (2.2) and (4.5), the last equation entails that

$$(\delta\theta+\lambda+2n+1)\eta(E)=0. \tag{4.7}$$

At this point, let us assume that $\eta(E)=0$ in some open set $\mathcal{O}$ of M. Then differentiating $\eta(E)=0$ along φE and using (2.1) we immediately obtain

$$g(\nabla_{\varphi E}E,\xi)+g(\varphi E,\varphi E)=0.$$

Taking (2.2) and (3.8) into account we compute

$$(2n-1)g(\nabla_X E,\xi)=(2n+\lambda)\eta(X)+(2n-1)\theta(X)\eta(E).$$

From which it follows that $g(\nabla_{\varphi E}E,\xi)=0$. Therefore, we have $g(\varphi E,\varphi E)=0$ on $\mathcal{O}$. This gives $\varphi E=0$. Operating by φ and since $\eta(E)=0$ we ultimately find $E=0$ on $\mathcal{O}$. Hence, from (3.7) the Ricci tensor vanishes identically on $\mathcal{O}$. But by virtue of (2.2), this is impossible. Consequently, $\eta(E)$ is non-vanishing everywhere on M. So, we have $\delta\theta+\lambda+2n+1=0$. Since ξ is Killing, we have $\pounds_\xi S=0$. Making use of (4.1) it follows that $\nabla_\xi Q=Q\varphi-\varphi Q$. Next, replacing X by ξ in (3.10), using (4.1) and the foregoing equation gives

$$\begin{aligned}(2n-1)R(\xi,Y)E=&(\nabla_\xi Q)Y-(\nabla_Y Q)\xi+\theta(Y)Q\xi-\theta(\xi)QY\\&+\lambda[\theta(Y)\xi-\theta(\xi)Y]+(\xi\lambda)Y-(Y\lambda)\xi\\=&-\varphi QY+2n\varphi Y+(\lambda+2n)\theta(Y)\xi\\&-\theta(\xi)[QY+\lambda Y]+(\xi\lambda)Y-(Y\lambda)\xi.\end{aligned} \tag{4.8}$$

Scalar product of (4.8) with an arbitrary vector field X and making use of (2.6) provides

$$\begin{aligned}(2n+1)g((\nabla_Y\varphi)X,E)=&g(\varphi QY,X)-2ng(\varphi Y,X)\\&-(\lambda+2n)\theta(Y)\eta(X)+\theta(\xi)g(QY+\lambda Y,X)\\&-(\xi\lambda)g(Y,X)-(Y\lambda)\eta(X).\end{aligned} \tag{4.9}$$

Replacing X by φX, Y by φY in (4.9), adding the resulting equation with (4.9) and then recalling (2.5), we obtain

$$\begin{aligned}&(2n+1)\{2g(Y,X)\xi-\eta(X)(Y+\eta(Y)\xi\}\\&=g((Q\varphi+\varphi Q)Y,X)-4ng(\varphi Y,X)-(\lambda+2n)\theta(Y)\eta(X)-\theta(\xi)g(QY+\lambda Y,X)\end{aligned}$$

$$+\,\theta(\xi)g(\varphi Q\varphi+\lambda\varphi^2Y,X)-(\xi\lambda)g(Y,X)-(\xi\lambda)g(\varphi Y,\varphi X)+(Y\lambda)\eta(X).$$

Anti-symmetrizing the previous equation shows that

$$\begin{aligned}&2g((Q\varphi+\varphi Q)Y,X)-4ng(\varphi Y,X)\\ &=(\lambda+1)\{\eta(X)\theta(Y)-\eta(Y)\theta(X)\}-(Y\lambda)\eta(X)+(X\lambda)\eta(Y).\end{aligned}$$

Choosing $X, Y\perp\xi$, the foregoing equation yields

$$g((Q\varphi+\varphi Q)Y,X)=4ng(\varphi Y,X).$$

Clearly for any vector field X, $X-\eta(X)\xi$ is orthogonal to ξ. Therefore, replacing X by $X-\eta(X)\xi$ and Y by $Y-\eta(Y)\xi$ in the preceding equation and using (2.2), we infer

$$(Q\varphi+\varphi Q)Y=4n\varphi Y \tag{4.10}$$

for any vector field Y. Let $\{e_k,\varphi e_k,\xi\}$, $k=1,2,\dots,n$, be a φ-basis of M such that $Qe_k=\mu_k e_k$. From which, we have $\varphi Qe_k=\mu_k\varphi e_k$. Substituting e_k for Y in (4.10) and using the foregoing equations we obtain $Q\varphi e_k=(4n-\mu_k)\varphi e_k$. Computing the scalar curvature $r=g(Q\xi,\xi)+\sum_{k=1}^{n}[g(Qe_k,e_k)+g(Q\varphi e_k,\varphi e_k)]$ with the use of (2.2) we get $r=2n(2n+1)$. Going back to equations (4.3) and (4.5) we deduce that $D\lambda=(\lambda+1)E$. Utilizing this and noting that the scalar curvature is constant, it follows from (4.6) that $QE+(\lambda+\delta\theta+1)E=0$. But we have already deduced that $\delta\theta+\lambda+2n+1=0$. Therefore, we obtain $QE=2nE$. Taking covariant differentiation of this along an arbitrary vector field X and using (3.8) one obtains

$$(2n-1)(\nabla_X Q)E+Q^2X=(2n-\lambda)QX+2n\lambda X. \tag{4.11}$$

We now take the trace of (4.11) over X and use $r=2n(2n+1)$ to achieve $|Q|^2=2nr$. Use of this and $r=2n(2n+1)$, we compute

$$\begin{aligned}|Q-\frac{r}{2n+1}I|^2&=|Q|^2-\frac{2r^2}{2n+1}+\frac{r^2}{2n+1}=2nr-\frac{r^2}{2n+1}\\ &=4n^2(2n+1)-4n^2(2n+1)=0.\end{aligned}$$

Since the length of the symmetric tensor $Q-\frac{r}{2n+1}I$ vanishes, we must have $Q=\frac{r}{2n+1}I=2nI$. Since M is Einstein and compact, applying Boyer-Galicki's Theorem ([3]) "A compact Einstein K-contact manifold is Sasakian", we complete the proof. □

Since any Sasakian manifold is K-contact, the above result is also true for Sasakian manifolds. Thus, in view of the above Theorem we can extend the well known result of Boyer-Galicki [2] (see also [3] and [13]) which says that any simply connected Sasakian Einstein manifold is a spin manifold

Theorem 4.2. *If a simply connected compact K-contact manifold admits a closed Einstein-Weyl structure, then it is necessarily spin.*

Remark 4.1. After the submission of this paper, Gauduchon-Moroianu upload a paper [7] in arXiv (appeared on 6th January 2016) which contains the same result as Theorem 4.1. However, the proof of our result is slightly different from that of [7]. In our proof, we have used some tensorial computation that involves the formulas of closed Einstein-Weyl structures and the full strength of the K-contactness property (i.e., the formulas involving Ricci curvature and curvature tensor). On the other hand, the proof in [7] requires two more results, e.g., Proposition 2.1 and Proposition 2.2. For details we refer to [7].

Acknowledgement. The author is very much thankful to the reviewer for some valuable remarks.

References

[1] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Math. 203 Birkhäuser, Boston-Basel-Berlin, 2002.

[2] C. P. Boyer and K. Galicki, *On Sasakian Einstein geometry*, Internat. J. Math. **11** (2000), no. 7, 873–909.

[3] ———, *Einstein manifolds and contact geometry*, Proc. Amer. Math. Soc. **129** (2001), no. 8, 2419–2430.

[4] C. P. Boyer, K. Galicki, and P. Matzeu, *On η-Einstein Sasakian geometry*, Comm. Math. Phys. **262** (2006), no. 1, 177–208.

[5] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267** (1984), no. 4, 495–518.

[6] ———, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$*, J. Reine Angew. Math. **469** (1995), 1–50.

[7] P. Gauduchon and A. Moroianu, *Weyl-Einstein structures on K-contact manifolds*, arXiv:1601.00892v1 (5th Jan, 2016).

[8] A. Ghosh, *Einstein-Weyl structures on contact metric manifolds*, Ann. Glob. Anal. Geom. **35** (2009), no. 4, 431–441.

[9] T. Higa, *Weyl manifolds and Einstein-Weyl manifolds*, Comment. Math. Univ. St. Paul. **42** (1993), no. 2, 143–160.

[10] N. J. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differentail Geom. **9** (1974), 435–441.

[11] P. Matzeu, *Almost contact Einstein-Weyl structures*, Manuscripta Math. **108** (2002), no. 3, 275–288.

[12] ———, *Closed Einstein-Weyl structures on compact Sasakian and cosymplectic manifolds*, Proc. Edinb. Math. Soc. **54** (2011), no. 1, 149–160.

[13] A. Moroianu, *Parallel and Killing Spinors on Spin^c manifolds*, Comm. Math. Phys. **187** (1997), no. 2, 417–427.

[14] F. Narita, *Riemannian submersions and Riemannian manifolds with Einstein-Weyl structures*, Geom. Dedicata **65** (1997), no. 1, 103–116.

[15] ———, *Einstein-Weyl structures on almost contact metric manifolds*, Tsukuba J. Math. **22** (1998), no. 1, 87–98.

[16] H. Pedersen and A. Swann, *Riemannian submersions, four manifolds and Einstein-Weyl geometry*, Proc. London Math. Soc. **66** (1993), no. 2, 381–399.

[17] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math. **12** (1968), 700–717.

[18] K. P. Tod, *Compact 3-dimensional Einstein-Weyl structures*, J. London Math. Soc. **45** (1992), no. 2, 341–351.

[19] H. Weyl, *Space-Time-Matter*, Dover, New York, 1952 (translation of the fourth edition of Raum, Zeit, Materie, the first edition which was published in 1918 by Springer, Berlin).

Amalendu Ghosh
Department Of Mathematics
Chandernagore College
Chandannagar: 712 136, W.B., India
E-mail address: aghosh_70@yahoo.com