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# THREE DIFFERENT WAYS TO OBTAIN THE VALUES OF HYPER $m$-ARY PARTITION FUNCTIONS

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Abstract. We consider a natural generalization of $h_2(n)$, denoted $h_m(n)$, which is the number of partitions of $n$ into parts which are power of $m \geq 2$ wherein each power of $m$ is allowed to be used as a part at most $m$ times. In this note, we approach in three different ways using the recurrences, the matrix and the tree to calculate the value of $h_m(n)$.

## 1. Introduction

In the late 1960s, the unrestricted binary partition function was first studied by R. Churchhouse [7]. It is then followed by active studies on numerous functions which enumerate partitions into powers of a fixed number $m$, by Ø. Rødseth [15], G. E. Andrews [1], H. Gupta [12], and G. Dirdal [9, 10]. Thereafter, although a paper entitled "Some binary partition functions" was published by B. Reznick [14] in 1990, there had been quite a long time gap in studies until 2000 when interestingly, the Calkin-Wilf tree was introduced by N. Calkin and H. S. Wilf [6] who found the relevance between the hyperbinary partition and the rational recount using the tree. With this as a momentum, in context of trees, active studies are continued such as a restudy on the Stern-Brocot tree, its relevance to the Calkin-Wilf tree, a $q$-analogue of the Calkin-Wilf, and so on [2, 3, 4, 5]. Also, in perspective of partition functions, numerous authors continue to study on $m$-ary partition functions, $m$-ary overpartition, hyper $m$-ary partition, and hyper $m$-ary overpartition [8, 11, 13, 16, 17]. In this note, we take a look into arithmetic properties for hyper $m$-ary partition functions of K. M. Courtright and J. A. Sellers [8]. They generalized hyperbinary partition, $h_2(n)$, into $h_m(n)$ as follows. Let $h_m(n)$ be the number of partitions of $n$ into parts which are powers of $m \geq 2$ wherein each power of $m$ is allowed to be used as a part at most $m$ times. For fixed $m \geq 2$, the generating function for $h_m(n)$

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is defined

$$H_m(q) := \sum_{n\geq 0} h_m(n)q^n = \prod_{i\geq 0}\left(1+q^{m^i}+q^{2m^i}+\cdots+q^{m\cdot m^i}\right),$$

when $h_m(0)=1$. Using the following recurrences obtained from the definition above,

$$h_m(Xm) = h_m(X) + h_m(X-1) \tag{1}$$

$$h_m(Xm+r) = h_m(X) \quad \text{for } 1\leq r\leq m-1, \tag{2}$$

they proved the following theorem.

**Theorem 1.1** (Courtright and Sellers [8])**.** *Let $m\geq 3$ and $j\geq 1$ be fixed integers and let $k$ be an integer between $2$ and $m-1$. Then, for all $n\geq 0$*

$$h_m\left(m^j n + m^{j-1}k\right) = j\, h_m(n).$$

In this paper, from the theorem above, we introduce an extended theorem including the case where $k=1$ and give some more general terms of hyper $m$-ary partition. Theorems to prove in Section 2 are:

**Theorem 2.3.** *Let $X$ and $p$ be positive integers. Then*

$$h_m(Xm^p) = p\, h_m(X-1) + h_m(X)$$

and

**Theorem 2.5.** *Let $X$ and $q\geq 2$ be positive integers. Then*

$$h_m\left(Xm^q + \sum_{i=1}^{q-1} m^i\right) = h_m(X-1) + q\, h_m(X).$$

Also, in Section 3 we give proofs with a different perspective and calculate the value of $h_m(n)$ using matrices. In Section 4, we introduce a generalization of the Calkin-Wilf tree, the hyper $m$-ary partition tree, which is composed of enumerating two consecutive terms $\frac{h_m(n-1)}{h_m(n)}$ in fraction.

## 2. Some general terms obtained by using recurrence relations

In this section, we examine how to calculate value of $h_m(n)$ when $n$ is a positive integer and $m\geq 3$ with a special form. It is easy to find Lemma 2.1 below from (2).

**Lemma 2.1.** *Let $X\geq 1$ and $p\geq 0$ be integers. Then for an integer $1\leq r_i\leq m-1$ $(0\leq i\leq p-1)$, we have*

$$h_m\left(Xm^p + \sum_{i=0}^{p-1} r_i m^i\right) = h_m(X). \tag{3}$$

*Proof.* By (2),

$$\begin{aligned} h_m\left(Xm^p + \sum_{i=0}^{p-1} r_i m^i\right) &= h_m\left(Xm^p + r_0 + \sum_{i=1}^{p-1} r_i m^i\right) \\ &= h_m\left(Xm^{p-1} + \sum_{i=1}^{p-1} r_i m^{i-1}\right) \\ &= h_m\left(Xm^{p-1} + r_1 + \sum_{i=2}^{p-1} r_i m^{i-1}\right) \\ &= h_m\left(Xm^{p-2} + \sum_{i=2}^{p-1} r_i m^{i-2}\right) \\ &= \cdots = h_m\left(Xm + r_{p-1}\right) = h_m\left(X\right). \end{aligned}$$ □

This lemma says that if $n$ is written as an $m$-ary number which has no 0 in any digits, then we can evaluate $h_m(n)$ easily.

**Lemma 2.2.** *Let $X$ and $p$ be positive integers. Then*

$$h_m\left(Xm^p - 1\right) = h_m\left(X - 1\right). \tag{4}$$

*Proof.* From Lemma 2.1,

$$\begin{aligned} h_m\left(Xm^p - 1\right) &= h_m\left((X-1)m^p + \sum_{i=0}^{p-1}(m-1)m^i\right) \\ &= h_m\left(X - 1\right). \end{aligned}$$ □

Using Lemma 2.2 above, we can derive the following theorem.

**Theorem 2.3.** *Let $X$ and $p$ be positive integers. Then*

$$h_m\left(Xm^p\right) = p\, h_m\left(X - 1\right) + h_m\left(X\right). \tag{5}$$

*Proof.* For $p = 1$, (5) is same as (1). For $p \geq 2$, by (1) and (4),

$$\begin{aligned} h_m\left(Xm^p\right) &= h_m\left(Xm^{p-1}\right) + h_m\left(Xm^{p-1} - 1\right) \\ &= h_m\left(Xm^{p-1}\right) + h_m\left(X - 1\right) \\ &= h_m\left(Xm^{p-2}\right) + 2\, h_m\left(X - 1\right) \\ &= \cdots = h_m\left(X\right) + p\, h_m\left(X - 1\right). \end{aligned}$$ □

This shows that, if $n$ written as $m$-ary number has 0's in its every digit only except for the highest, then $h_m(n)$ is equal to the number of 0's plus one.

**Corollary 2.4.** *Let $k$ and $p_i$ be positive integers and let $r_i$ be an integer between 2 and $m - 1$ for $1 \leq i \leq k$. Then*

$$h_m\left(\sum_{i=1}^{k} r_i m^{i-1+\sum_{j=1}^{i} p_j}\right) = \prod_{i=1}^{k}(p_i + 1). \tag{6}$$

*Proof.* By applying Theorem 2.3 and (2) successively,

$$\begin{aligned}
&h_m\left(\sum_{i=1}^{k} r_i m^{i-1+p_i+\cdots+p_2+p_1}\right)\\
&= h_m\left(\left(\sum_{i=2}^{k} r_i m^{i-1+p_i+\cdots+p_2+p_1}\right)+r_1 m^{p_1}\right)\\
&= h_m\left(\left(\sum_{i=2}^{k} r_i m^{i-1+p_i+\cdots+p_2}\right)+r_1\right)\\
&\quad + p_1\, h_m\left(\left(\sum_{i=2}^{k} r_i m^{i-1+p_i+\cdots+p_2}\right)+r_1-1\right)\\
&= (p_1+1)\; h_m\left(\sum_{i=2}^{k} r_i m^{i-2+p_i+\cdots+p_2}\right)\\
&= \cdots = (p_1+1)\,(p_2+1)\cdots(p_{k-1}+1)\; h_m\left(r_k m^{p_k}\right)\\
&= \prod_{i=1}^{k}(p_i+1)\,.
\end{aligned}$$

□

By combining Lemma 2.1 and Corollary 2.4, we can evaluate $h_m(n)$, where $n$ has no 1's when it is written as $m$-ary number.

**Theorem 2.5.** *Let $X$ and $q \geq 2$ be positive integers. Then*

$$h_m\left(Xm^q+\sum_{i=1}^{q-1} m^i\right) = h_m\left(X-1\right)+q\,h_m\left(X\right). \tag{7}$$

*Proof.* First, by (1) and (2),

$$\begin{aligned}
h_m\left(Xm^2+m\right) &= h_m\left(Xm+1\right)+h_m\left(Xm\right)\\
&= 2\,h_m\left(X\right)+h_m\left(X-1\right).
\end{aligned}$$

Then

$$\begin{aligned}
&h_m\left(Xm^q+\sum_{i=1}^{q-1} m^i\right)\\
&= h_m\left(Xm^{q-1}+\sum_{i=0}^{q-2} m^i\right)+h_m\left(Xm^{q-1}+\sum_{i=1}^{q-2} m^i\right)\\
&= h_m\left(X\right)+h_m\left(Xm^{q-1}+\sum_{i=1}^{q-2} m^i\right)\\
&= 2\,h_m\left(X\right)+h_m\left(Xm^{q-2}+\sum_{i=1}^{q-3} m^i\right)
\end{aligned}$$

$$
\begin{aligned}
&= \cdots = (q-2)h_m(X) + h_m\left(Xm^2+m\right) \\
&= q\,h_m(X) + h_m(X-1),
\end{aligned}
$$

where we used (1) and (3) alternatively. □

**Corollary 2.6.** *Let $X$, $p$ and $q$ be positive integers and let $r$ be an integer between* 1 *and* $m-1$. *Then*

$$
h_m\left(\left\{rm^{q-1}+\sum_{i=0}^{q-2}m^i\right\}m^p\right) = pq+1. \tag{8}
$$

*Proof.* By (5), (3), and (7),

$$
\begin{aligned}
&h_m\left(\left\{rm^{q-1}+\sum_{i=0}^{q-2}m^i\right\}m^p\right) \\
&= h_m\left(rm^{q-1}+\sum_{i=0}^{q-2}m^i\right) + p\,h_m\left(rm^{q-1}+\sum_{i=1}^{q-2}m^i\right) \\
&= 1+pq.
\end{aligned}
$$

□

If any number $n$ written as $m$-ary number having 1's and 0's on every digits, then by using Theorem 2.3 and Theorem 2.5, we may calculate $h_m(n)$ easily.

## 3. A method using matrices

In this section, we show how to calculate $h_m(X)$ using matrices.

**Lemma 3.1.** *Let $m \geq 3$ and $X$ be positive integers and let $r$ be an integer between* 2 *and* $m-1$. *Then*

$$
\begin{pmatrix} h_m(Xm-1) \\ h_m(Xm) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} h_m(X-1) \\ h_m(X) \end{pmatrix}, \tag{9}
$$

$$
\begin{pmatrix} h_m(Xm) \\ h_m(Xm+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h_m(X-1) \\ h_m(X) \end{pmatrix}, \tag{10}
$$

$$
\begin{pmatrix} h_m(Xm+r-1) \\ h_m(Xm+r) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h_m(X-1) \\ h_m(X) \end{pmatrix}. \tag{11}
$$

*Proof.* Using (2) with $r = m-1$, we deduce that

$$
h_m(Xm-1) = h_m\left((X-1)m+(m-1)\right) = h_m(X-1)
$$

which is (9), (10) and (11) are easy consequences of (1) and (2). □

When $m$ is an integer greater than or equal to 3, and when $r$ is an integer between 2 and $m-1$, we define the matrix as follows.

$$
M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ M_r = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.
$$

*Remark* 3.2. Let $m \geq 3$ and $X$ be positive integers and let $r$ be some integer between 0 and $m-1$. Then we can rewrite Lemma 3.1 as follows.

$$\begin{pmatrix} h_m(Xm+r-1) \\ h_m(Xm+r) \end{pmatrix} = M_r \begin{pmatrix} h_m(X-1) \\ h_m(X) \end{pmatrix}. \tag{12}$$

**Theorem 3.3.** *Let $m \geq 3$ and $k$ be positive integers and let $0 \leq r_i \leq m-1$ $(0 \leq i \leq k-1)$ and $1 \leq r_k \leq m-1$ be integers. Then*

$$h_m\left(\sum_{i=0}^{k} r_i m^i\right) = \begin{pmatrix} 0 & 1 \end{pmatrix} M_{r_0} M_{r_1} \cdots M_{r_{k-1}} \begin{pmatrix} h_m(r_k-1) \\ h_m(r_k) \end{pmatrix}. \tag{13}$$

*Proof.* By (12),

$$\begin{aligned}
\begin{pmatrix} h_m\left(\sum_{i=0}^{k} r_i m^i - 1\right) \\ h_m\left(\sum_{i=0}^{k} r_i m^i\right) \end{pmatrix} &= \begin{pmatrix} h_m\left(\sum_{i=1}^{k} r_i m^i + r_0 - 1\right) \\ h_m\left(\sum_{i=1}^{k} r_i m^i + r_0\right) \end{pmatrix} \\
&= M_{r_0} \begin{pmatrix} h_m\left(\sum_{i=1}^{k} r_i m^{i-1} - 1\right) \\ h_m\left(\sum_{i=1}^{k} r_i m^{i-1}\right) \end{pmatrix} \\
&= M_{r_0} \begin{pmatrix} h_m\left(\sum_{i=2}^{k} r_i m^{i-1} + r_1 - 1\right) \\ h_m\left(\sum_{i=2}^{k} r_i m^{i-1} + r_1\right) \end{pmatrix} \\
&= M_{r_0} M_{r_1} \begin{pmatrix} h_m\left(\sum_{i=2}^{k} r_i m^{i-2} - 1\right) \\ h_m\left(\sum_{i=2}^{k} r_i m^{i-2}\right) \end{pmatrix} \\
&= \cdots = M_{r_0} M_{r_1} \cdots M_{r_{k-1}} \begin{pmatrix} h_m(r_k-1) \\ h_m(r_k) \end{pmatrix}.
\end{aligned}$$

After multiplying $\begin{pmatrix} 0 & 1 \end{pmatrix}$ on both sides, the proof is now complete. □

Theorem 3.3 explains that if a number, expressed as $m$-ary with $k+1$ digits, has a value of $r_i$ in $(i+1)$th digit from right to left, then the value of hyper $m$-ary can be calculated by multiplying pre-defined matrices $M_{r_i}$ from left to right. The following lemma is useful to simplify calculations.

**Lemma 3.4.** *Let $m \geq 3$, $p$ and $q$ be nonnegative integers and let $r$ be an integer between 2 and $m-1$. Then*

$$M_0{}^p M_1{}^q = \begin{pmatrix} 1 & q \\ p & pq+1 \end{pmatrix}, \tag{14}$$

$$M_0{}^p M_r{}^q = \begin{pmatrix} 0 & 1 \\ 0 & p+1 \end{pmatrix}. \tag{15}$$

*Proof.*

$$M_0{}^p M_1{}^q = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q \\ p & pq+1 \end{pmatrix},$$

$$M_0{}^p M_r{}^q = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & p+1 \end{pmatrix}.$$
□

Now we give a new and short proof of theorems by using our method.

*Proof.* (Another proof of Theorem 1.1) Since $2 \le k \le m-1$,

$$\begin{aligned} h_m\left(m^j n + m^{j-1}k\right) &= \begin{pmatrix} 0 & 1 \end{pmatrix} M_0{}^{j-1} M_k \begin{pmatrix} h_m(n-1) \\ h_m(n) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & j \end{pmatrix} \begin{pmatrix} h_m(n-1) \\ h_m(n) \end{pmatrix} \\ &= j\, h_m(n), \end{aligned}$$

where we used Lemma 3.4 in the penultimate step. □

*Proof.* (Another proof of Theorem 2.3)

$$\begin{aligned} h_m\left(X m^p\right) &= \begin{pmatrix} 0 & 1 \end{pmatrix} M_0{}^p \begin{pmatrix} h_m(X-1) \\ h_m(X) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} h_m(X-1) \\ h_m(X) \end{pmatrix} \\ &= p\, h_m(X-1) + h_m(X). \end{aligned}$$
□

*Proof.* (Another proof of Theorem 2.5)

$$\begin{aligned} h_m\left(X m^q + \sum_{i=1}^{q-1} m^i\right) &= \begin{pmatrix} 0 & 1 \end{pmatrix} M_0 M_1{}^{q-1} \begin{pmatrix} h_m(X-1) \\ h_m(X) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & q-1 \\ 1 & q \end{pmatrix} \begin{pmatrix} h_m(X-1) \\ h_m(X) \end{pmatrix} \\ &= h_m(X-1) + q\, h_m(X). \end{aligned}$$
□

*Proof.* (Another proof of Corollary 2.6)

$$\begin{aligned} h_m\left(\left\{ r m^{q-1} + \sum_{i=0}^{q-2} m^i \right\} m^p\right) &= \begin{pmatrix} 0 & 1 \end{pmatrix} M_0{}^p M_1{}^{q-1} \begin{pmatrix} h_m(r-1) \\ h_m(r) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & q-1 \\ p & p(q-1)+1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= pq + 1. \end{aligned}$$
□

By following the procedure described next, we may reduce calculation steps to reach the final result.

Step 0 : Write $n$ as $m$-ary number written by $n = (r_k r_{k-1} \cdots r_1 r_0)_{(m)}$.

Step 1 : If consecutive non-zero numbers appear from the lowest digit, then remove the numbers.

Step 2 : If you see any digit between 2 and $m-1$, remove consecutive non-zero numbers right in front of the digit.

Step 3 : If you read from the lowest term and see any digit between 2 and $m-1$, then cut up to that number and repeat this Step 3 till you reach the highest digit.

Step 4 : Change the number in the highest digit to 1 for ease of calculation.

Note that at the end of Step 4, all the digits are either 0 or 1.

As an example, hyper 4-ary partition value of $2013032101012_{(4)}$ is calculated as

$$\begin{aligned}
& h_4\left(2013032101012_{(4)}\right) \\
= \ & h_4\left(20130321010_{(4)}\right) \qquad \text{by Step 1} \\
= \ & h_4\left(203021010_{(4)}\right) \qquad \text{by Step 2} \\
= \ & h_4\left(20_{(4)}\right) \times h_4\left(30_{(4)}\right) \times h_4\left(21010_{(4)}\right) \qquad \text{by Step 3} \\
= \ & h_4\left(10_{(4)}\right) \times h_4(10_{(4)}) \times h_4(11010_{(4)}) \qquad \text{by Step 4} \\
= \ & 2 \times 2 \times 8 = 32. \qquad \text{by Corollary 2.6 and Theorem 3.3}
\end{aligned}$$

## 4. Another proof using a tree

In this section, we construct a tree structure of fraction $\frac{h_m(n-1)}{h_m(n)}$ and then use this tree to reprove Theorems 2.3 and 2.5 for hyper $m$-ary partition function from a different perspective.

Let us explain the rules how to construct hyper $m$-ary tree.

- $\frac{1}{1}$ is at the top of the tree, and
- A parent $\frac{a}{b}$ has $m$ children : from left to right, the first child is $\frac{a}{a+b}$, the second $\frac{a+b}{b}$, and the rest $\frac{b}{b}$.
- Make $m-2$ copies of the tree constructed by the previous rules, so that a total of $m-1$ same trees appear.

For example, Figure 1 below is the hyper trinary tree constructed by the rules explained above, and we can see how each vertex $\frac{h_m(n-1)}{h_m(n)}$ is labeled. The equalities in Figure 1 will be explained soon.

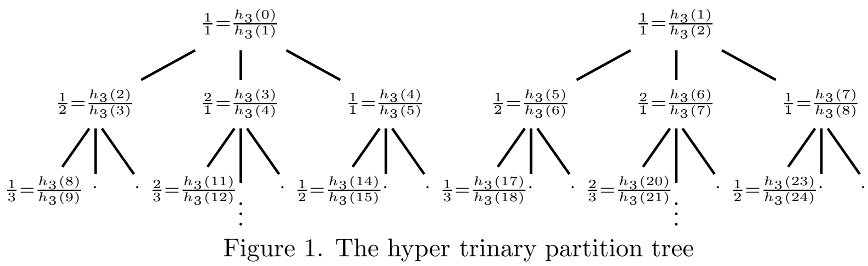



Figure 1. The hyper trinary partition tree

Values of $\frac{h_3(0)}{h_3(1)}$ and $\frac{h_3(1)}{h_3(2)}$, vertex of the root of the two trees, are same as $\frac{1}{1}$ because $h_3(0) = h_3(1) = h_3(2) = 1$. Hence the hyper trinary partition tree is composed of two identical trees.

We call $\frac{h_3(0)}{h_3(1)}$ and $\frac{h_3(1)}{h_3(2)}$, roots of two trees, level 1, and six vertexes, two sets of three children under each root, level 2. With this leveling, there are $2 \times 3^{k-1}$ vertexes in total at level $k$. Vertexes at level $k$ from the left are as follows:

$$\frac{h_3(3^{k-1}-1)}{h_3(3^{k-1})},\ \frac{h_3(3^{k-1})}{h_3(3^{k-1}+1)},\ \dots\ ,\frac{h_3(3^k-2)}{h_3(3^k-1)}.$$

If we express the numbers $3^{k-1},\ 3^{k-1}+1,\ \dots\ ,3^k-1$ as trinary number, then they are $k$ digits number starting with either 1 or 2. Also, regarding $h_3(n)$, the denominators of vertexes at level $k$, we can express $n$ in trinary as

$$n = \sum_{i=0}^{k-1} r_i\, 3^i$$
$$(r_i \in \{0,\ 1,\ 2\}\,,\ 0 \le i \le k-2 \text{ and } r_{k-1} \in \{1,\ 2\}).$$

Here, values of $r_i$ show the location of vertex $\frac{h_3(n-1)}{h_3(n)}$ in the tree. That is, at level 1, it is the first tree on left if the highest digit, $r_{k-1}$, equals 1, and the second tree on right if $r_{k-1} = 2$. Also, when moving down from level $i$ to $i+1$ $(i = 1,\ 2,\ \dots\ ,\ k-1)$, we choose $(r_{k-i-1}+1)$th vertex from the left.
We can similarly construct hyper $m$-ary tree as follows.

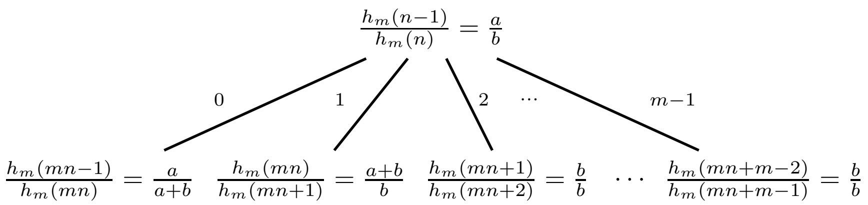
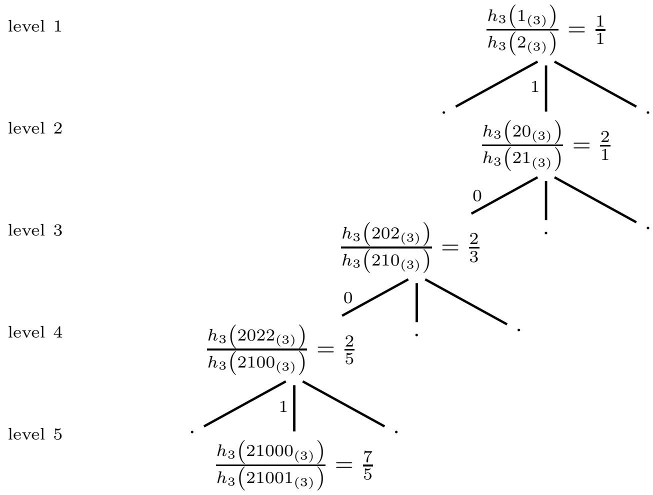



Figure 2. The hyper $m$-ary partition tree

The tree structure above satisfies recurrences (1), (2) in Section 1, thus we can find the value of $h_m(n)$ using this hyper $m$-ary partition tree. For example, a vertex $\frac{h_3(189)}{h_3(190)}$ with its denominator $h_3(190) = h_3\left(21001_{(3)}\right)$ is located at where you start from the second tree, the second vertex from the left when going down to level 2, then choose the first one from left to go level 3, the first from left to level 4 again, and select the second from the left to the level 5.

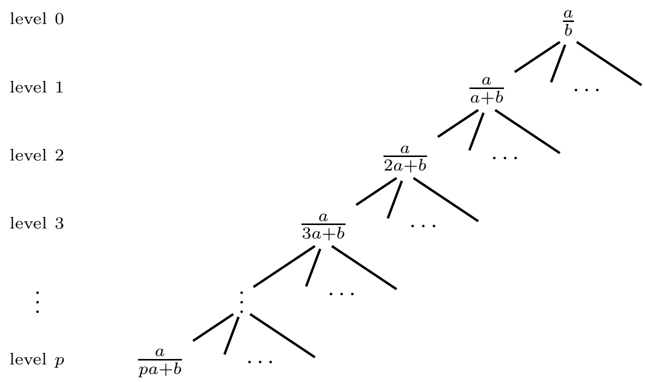



From the tree above, we find $h_3(190) = 5$. On the other hand, as the hyper trinary partition tree is composed of two identical trees, when $n$ is expressed in trinary number we can consider only the case of the highest digit being 1. It means, the identical result is obtained by changing trinary number $21001_{(3)}$ to $11001_{(3)}$ when evaluating $h_3(21001_{(3)})$.

We provide new proofs of Theorem 2.3 and Theorem 2.5 using hyper $m$-ary tree we constructed.

*Proof.* (Another proof of Theorem 2.3 using $m$-ary tree)
A term $\frac{h_m(X-1)}{h_m(X)}$ is located at a vertex in hyper $m$-ary partition tree. Let us assume this vertex as level 0, and keep creating $m$-ary tree from it.

Assuming $h_m(X-1)$ as $a$ and $h_m(X)$ as $b$ for convenience' sake, keeping construction of the leftmost branch from level 0, $\frac{a}{b}$, to level $p$ results in the following.

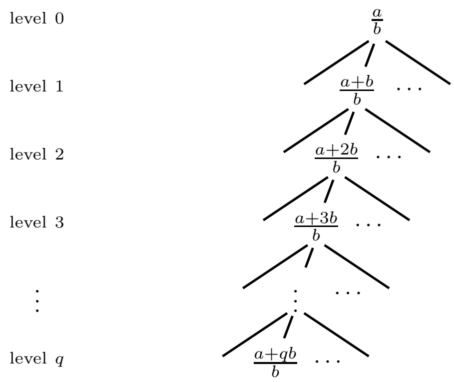


From the tree above, the leftmost term of level $p$ corresponds to $\frac{h_m(Xm^p-1)}{h_m(Xm^p)}$, thus it becomes

$$h_m\left(Xm^p\right) = p\, h_m(X-1) + h_m(X). \qquad \square$$

*Proof.* (Another proof of Theorem 2.5 using $m$-ary tree)

Similarly, assuming $h_m(X-1)$ as $a$ and $h_m(X)$ as $b$ for convenience' sake, let us create $m$-ary tree, keeping construction of the second leftmost branch from level 0, $\frac{a}{b}$, to level $q$ leads to the following.




From the tree above, the second leftmost term of level $q$ corresponds to $\frac{h_m\left(Xm^q+\sum_{i=1}^{q-1} m^i\right)}{h_m\left(Xm^q+\sum_{i=0}^{q-1} m^i\right)}$, thus it becomes

$$h_m\left(Xm^q + \sum_{i=1}^{q-1} m^i\right) = q\, h_m\left(X\right) + h_m\left(X-1\right). \qquad \square$$

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