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MONOTONICITY PROPERTIES OF THE BESSEL-STRUVE KERNEL

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Abstract. In this paper our aim is to study the classical Bessel-Struve kernel. Monotonicity and log-convexity properties for the Bessel-Struve kernel, and the ratio of the Bessel-Struve kernel and the Kummer confluent hypergeometric function are investigated. Moreover, lower and upper bounds are given for the Bessel-Struve kernel in terms of the exponential function and some Turán type inequalities are deduced.

1. Introduction and statements of the main results

Bessel and Struve functions arise in many problems of applied mathematics and mathematical physics. The properties of these functions were studied by many researchers in the past years from many different point of views. In this paper we consider the so-called Bessel-Struve kernel function S_ν, which is defined by the series

$$S_\nu(x)=\sum_{n\geq 0}\frac{\Gamma(\nu+1)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}n!\Gamma\left(\frac{n}{2}+\nu+1\right)}x^n,$$

where $\nu>-1$. The significance of this function is that it is a particular case when $\lambda=1$ of the unique solution $S_\nu(\lambda x)$ of the initial value problem

$$\mathcal{L}_\nu u(x)=\lambda^2u(x),\quad u(0)=1,u'(0)=\frac{\lambda\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})},$$

where for $\nu>-\frac{1}{2}$ the expression $\mathcal{L}_\nu$ stands for the Bessel-Struve operator defined by

$$\mathcal{L}_\nu u(x)=\frac{d^2u}{dx^2}(x)+\frac{2\nu+1}{x}\left(\frac{du}{dx}(x)-\frac{du}{dx}(0)\right)$$

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with an infinitely differentiable function u on $\mathbb{R}$. Recently the Bessel-Struve kernel and the so-called Bessel-Struve intertwining operator have been the subject of some research from the point of view of the operator theory, see [4, 5, 6] and the references therein. In these papers the Bessel-Struve intertwining operator (which is actually a topological isomorphism from the space of infinitely differentiable functions on $\mathbb{R}$ into itself, see [6]) has been studied extensively, and some of the properties of the Bessel-Struve kernel were useful in these investigations. For example, in the proof of [4, Theorem 3.15] it was used the inequality $S_\nu(x) < e^x$, where $\nu > -\frac{1}{2}$ and $x > 0$, in order to show that some series related to the Bessel-Struve intertwining operator are convergent and it is possible to characterize the mean-periodic functions on the space of entire functions and to characterize the continuous linear mappings from the above space into itself which commute with the Bessel-Struve operator. Motivated by these results, in this paper our aim is to study the classical Bessel-Struve kernel. By using some classical tools our aim is to investigate the monotonicity and log-convexity properties for the Bessel-Struve kernel, and of the ratio of the Bessel-Struve kernel and the Kummer confluent hypergeometric function $\Phi(a,c;\cdot)$, defined by the infinite series

$$\Phi(a,c;x) = \sum_{n\geq 0} \frac{\Gamma(a+n)\Gamma(c)}{\Gamma(c+n)\Gamma(a)} \frac{x^n}{n!}.$$

Moreover, by using the auxiliary function

$$S_{\nu,a}(x) = \sum_{n\geq 0} \frac{\Gamma(\nu+1)\Gamma\left(\frac{n+1}{2}\right)(a)_n}{\sqrt{\pi}n!\Gamma\left(\frac{n}{2}+\nu+1\right)} \frac{x^n}{n!}$$

our aim is to deduce a Turán type inequality for the Bessel-Struve kernel. The results presented in this paper may be used to deduce many lower and upper bounds for the Bessel-Struve kernel, which may be useful in the study of the Bessel-Struve intertwining operator. For example part (b) of the next theorem yields $S_\nu(x) < S_{-\frac{1}{2}}(x) = e^x$ for $\nu > -\frac{1}{2}$ and $x > 0$, which was used in the proof of [4, Theorem 3.15]. Some more tight bounds are shown in Theorem 2 below.

Our first main result reads as follows.

Theorem 1. *Let $\nu, \mu > -1$. The following assertions are true*:

(a) *If $\mu \geq \nu$, then $x \mapsto S_\mu(x)/S_\nu(x)$ is decreasing on $(0,\infty)$.*
(b) *The function $\nu \mapsto S_\nu(x)$ is decreasing and log-convex on $(-1,\infty)$ for each fixed $x > 0$.*
(c) *The function $x \mapsto xS'_\nu(x)/S_\nu(x)$ is increasing on $(0,\infty)$ for each fixed $\nu > -1$.*
(d) *The function $x \mapsto S_\nu(x)$ is log-convex on $(0,\infty)$ for each fixed $\nu > -\frac{1}{2}$.*
(e) *The function $\nu \mapsto S'_\nu(x)/S_\nu(x)$ is decreasing on $(-1,\infty)$ for each fixed $x > 0$.*

(f) *The function* $\nu \mapsto \varphi_{\nu-1}(x)/\varphi_\nu(x)$ *is decreasing on* $(0,\infty)$ *for* $x>0$, *where* $\varphi_\nu(x)=x^{2\nu}S_\nu(x)$.
(g) *The function* $\nu \mapsto (\varphi_\nu(x))^{1/\nu}$ *is decreasing on* $(0,\infty)$ *for each fixed* $x>0$.
(h) *The function* $x\mapsto S_\nu(x)/\Phi(a,c;x)$ *is decreasing on* $(0,\infty)$ *for* $a\geq c>0$ *and* $\nu\geq -\frac{1}{2}$.
(i) *The function* $x\mapsto S_\nu(x)/\Phi(a,a+1;x)$ *is increasing on* $(0,\infty)$ *for* $\nu\in\left(0,\frac{1}{2}\right]$ *and* $a\in[0,2\nu]$.
(j) *The function* $x\mapsto S_\nu(x)/\Phi(a,a+1;x)$ *is decreasing on* $(0,\infty)$ *for* $\nu>\frac{1}{2}$ *and* $a\geq 2\nu$.
(k) *The function* $a\mapsto S_{\nu,a}(x)$ *is strictly log-concave on* $(0,\infty)$ *for* $x>0$ *and* $\nu>-1$.

It is important to mention here that from the above main result many inequalities can be deduced. For example, parts (b) and (f) imply a reversed Turán type inequality, while part (k) implies a Turán type inequality. Moreover, following the proof of part (b) it can be shown that the function $\nu\mapsto S_{\nu,a}(x)$ is decreasing and log-convex on $(-1,\infty)$ for each fixed $a,x>0$. This result yields also in particular a Turán type inequality. All of these inequalities can be written in the following chain of inequalities

$$S_{\nu,a-1}(x)S_{\nu,a+1}(x)\leq S^2_{\nu,a}(x)\leq S_{\nu-1,a}(x)S_{\nu+1,a}(x),$$

where $x>0$, $a>1$, $\nu>-1$ on the left-hand side, and $x>0$, $a,\nu>0$ on the right-hand side.

Finally, by using the Chebyshev integral inequality we can get some other inequalities and bounds for the Bessel-Struve kernel which can be useful in the study of the Bessel-Struve intertwining operator.

Theorem 2. *The Bessel-Struve kernel satisfies the following inequalities*:
(a) *If* $\nu\geq\frac{1}{2}$ *and* $x>0$, *then* $xS_\nu(x)\leq e^x-1$ *and it is reversed when* $-1<\nu<\frac{1}{2}$.
(b) *If* $\nu\geq\frac{3}{2}$ *and* $x>0$, *then* $S_{\nu-1}(x)S_{\nu+1}(x)\leq S_{\frac{1}{2}}(x)S_{2\nu-\frac{1}{2}}(x)$ *and it is reversed when* $\nu\in\left(\frac{1}{2},\frac{3}{2}\right)$.
(c) *If* $\nu>-\frac{1}{2}$ *and* $x>0$, *then*

$$S_\nu(x)<e^{\frac{x^2}{4(\nu+1)}}+\frac{x\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}e^{\frac{x^2}{4(\nu+2)}}.$$

(d) *If* $\nu\geq-\frac{1}{2}$ *and* $x>0$, *then*

$$e^{\frac{x\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}}<S_\nu(x)<1-\frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}(1-e^x). \tag{1.1}$$

2. Proofs of the main results

The following result of Biernacki and Krzyż [3] will be used in the sequel.

Lemma 1 ([3]). *Consider the power series* $f(x) = \sum_{n\geq 0} a_n x^n$ *and* $g(x) = \sum_{n\geq 0} b_n x^n$, *where* $a_n \in \mathbb{R}$ *and* $b_n > 0$ *for all* n. *Further suppose that both series converge on* $|x| < r$. *If the sequence* $\{a_n/b_n\}_{n\geq 0}$ *is increasing* (*or decreasing*), *then the function* $x \mapsto f(x)/g(x)$ *is also increasing* (*or decreasing*) *on* $(0, r)$.

We note that the above lemma still holds when both f and g are even, or both are odd functions.

Proof of Theorem 1. (a) By using the definition of the Bessel-Struve kernel we have

$$\frac{S_\mu(x)}{S_\nu(x)} = \frac{\sum_{n\geq 0} c_n(\mu)x^n}{\sum_{n\geq 0} c_n(\nu)x^n}, \quad \text{where} \quad c_n(\alpha) = \frac{\Gamma(\alpha+1)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}n!\Gamma\left(\frac{n}{2}+\alpha+1\right)}.$$

With notation $w_n = c_n(\mu)/c_n(\nu)$ we have

$$\frac{w_{n+1}}{w_n} = \frac{\Gamma\left(\frac{n}{2}+\nu+\frac{3}{2}\right)\Gamma\left(\frac{n}{2}+\mu+1\right)}{\Gamma\left(\frac{n}{2}+\mu+\frac{3}{2}\right)\Gamma\left(\frac{n}{2}+\nu+1\right)}.$$

On the other hand, we know that the Euler gamma function is log-convex on $(0,\infty)$, or equivalently the digamma function $x \mapsto \psi(x) = \Gamma'(x)/\Gamma(x)$ is increasing on $(0,\infty)$. This implies that the function $\phi : (-1,\infty) \to (0,\infty)$, defined by

$$\phi(\mu) = \frac{\Gamma\left(\frac{n}{2}+\mu+1\right)}{\Gamma\left(\frac{n}{2}+\mu+\frac{3}{2}\right)},$$

is decreasing since for $\mu > -1$ and $n \in \{0, 1, \dots\}$ we have

$$\phi'(\mu) = \phi(\mu)\left(\psi\left(\frac{n}{2}+\mu+1\right) - \psi\left(\frac{n}{2}+\mu+\frac{3}{2}\right)\right) < 0.$$

Consequently, for $\mu \geq \nu$ and $n \in \{0, 1, \dots\}$ the next inequality is valid

$$\frac{\Gamma\left(\frac{n}{2}+\mu+1\right)}{\Gamma\left(\frac{n}{2}+\mu+\frac{3}{2}\right)} \leq \frac{\Gamma\left(\frac{n}{2}+\nu+1\right)}{\Gamma\left(\frac{n}{2}+\nu+\frac{3}{2}\right)},$$

which is equivalent to $w_{n+1} \leq w_n$. In other words, the sequence $\{w_n\}_{n\geq 0}$ is decreasing and appealing to Lemma 1 it follows that $x \mapsto S_\mu(x)/S_\nu(x)$ is decreasing on $(0,\infty)$.

(b) It is known that the infinite sum of decreasing functions is also decreasing and the infinite sum of log-convex functions is also log-convex. Thus, to show that $\nu \mapsto S_\nu(x)$ is decreasing and log-convex on $(-1,\infty)$ it is enough to show that $\nu \mapsto c_n(\nu)$ is decreasing and log-convex on $(-1,\infty)$ for each fixed $n \in \{0, 1, \dots\}$. Since the digamma function is increasing and concave on $(0,\infty)$ we obtain for all $\nu > -1$ and $n \in \{0, 1, \dots\}$ that

$$\begin{aligned}\frac{\partial \log c_n(\nu)}{\partial \nu} &= \psi(\nu+1) - \psi\left(\frac{n}{2}+\nu+1\right) \leq 0 \leq \psi'(\nu+1) - \psi'\left(\frac{n}{2}+\nu+1\right)\\ &= \frac{\partial^2 \log c_n(\nu)}{\partial \nu^2}.\end{aligned}$$

Thus, for all $n \in \{0, 1, \dots\}$ the coefficients $\nu \to c_n(\nu)$ are decreasing and log-convex on $(-1, \infty)$ and consequently $\nu \mapsto S_\nu(x)$ is decreasing and log-convex on $(-1, \infty)$ for each fixed $x > 0$.

(c) Let $d_n(\nu) = nc_n(\nu)$. Then by using the infinite series of the Bessel-Struve kernel the quotient $xS'_\nu(x)/S_\nu(x)$ can be written as

$$\frac{xS'_\nu(x)}{S_\nu(x)} = \frac{\sum_{n\geq 0} d_n(\nu)x^n}{\sum_{n\geq 0} c_n(\nu)x^n}.$$

It is clear that the sequence $\{d_n(\nu)/c_n(\nu)\}_{n\geq 0} = \{n\}_{n\geq 0}$ is increasing, and hence by using Lemma 1 it follows that the function $x \mapsto xS'_\nu(x)/S_\nu(x)$ is increasing on $(0, \infty)$.

(d) By using the integral representation (see for example [6])

$$S_\nu(x) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{xt} dt, \tag{2.1}$$

where $\nu > -\frac{1}{2}$, and the well-known Hölder-Rogers inequality for integrals for $x, y > 0$, $\nu > -\frac{1}{2}$ and $\lambda \in [0, 1]$ we get

$$\begin{aligned}
&S_\nu(\lambda x + (1-\lambda)y) \\
&= \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{(\lambda x+(1-\lambda)y)t} dt \\
&= \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 \left((1-t^2)^{\nu-\frac{1}{2}} e^{xt}\right)^\lambda \left((1-t^2)^{\nu-\frac{1}{2}} e^{yt}\right)^{1-\lambda} dt \\
&\leq \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \left(\int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{xt}\right)^\lambda \left(\int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{yt} dt\right)^{1-\lambda} \\
&= (S_\nu(x))^\lambda (S_\nu(y))^{1-\lambda},
\end{aligned}$$

that is, S_ν is log-convex on $(0, \infty)$ for each $\nu > -\frac{1}{2}$.

(e) This is a direct consequence of part (a). We know that if $\mu \geq \nu$, then $x \mapsto S_\mu(x)/S_\nu(x)$ is decreasing on $(0, \infty)$. But this is equivalent to $S_\nu(x)S'_\mu(x) - S'_\nu(x)S_\mu(x) \leq 0$ which can be rewritten as $S'_\mu(x)/S_\mu(x) \leq S'_\nu(x)/S_\nu(x)$.

(f) First observe that the Bessel-Struve kernel satisfies the following recurrence relation

$$xS'_\nu(x) = 2\nu S_{\nu-1}(x) - 2\nu S_\nu(x),$$

which can be verified by using the series representation. In view of this and part (e) we have that

$$\nu \mapsto \frac{\varphi_{\nu-1}(x)}{\varphi_\nu(x)} = \frac{S_{\nu-1}(x)}{x^2 S_\nu(x)} = \frac{1}{x^2} + \frac{1}{2\nu} \cdot \frac{S'_\nu(x)}{xS_\nu(x)}$$

is decreasing on $(0, \infty)$ as a product of two positive and decreasing functions.

(g) Consider the function $g : (0, \infty) \to \mathbb{R}$ given by

$$g(x) = \mu \log(\varphi_\nu(x)) - \nu \log(\varphi_\mu(x)), \quad \mu \geq \nu > 0.$$

Now, using the definition of φ_ν, the function g can be rewritten as $g(x) = \mu\log(S_\nu(x)) - \nu\log(S_\mu(x))$. Since $S_\nu(x) \to 1$ as $x \to 0$, it follows that $g(x) \to 0$ as $x \to 0$. Thus, to prove the assertion it is enough to show that g is increasing on $(0,\infty)$. But, by using part (f) and the fact that $\varphi_\nu'(x) = 2\nu x\varphi_{\nu-1}(x)$ we have for $\mu \geq \nu > 0$ and $x > 0$

$$g'(x) = \mu\frac{\varphi_\nu'(x)}{\varphi_\nu(x)} - \nu\frac{\varphi_\mu'(x)}{\varphi_\mu(x)} = 2x\nu\mu\left(\frac{\varphi_{\nu-1}(x)}{\varphi_\nu(x)} - \frac{\varphi_{\mu-1}(x)}{\varphi_\mu(x)}\right) \geq 0.$$

Alternatively, this part can be proved by showing that $\nu \mapsto \log\varphi_\nu^{\frac{1}{\nu}}(x) = \log x^2 + \frac{1}{\nu}\log S_\nu(x)$ is decreasing on $(0,\infty)$ for each $x > 0$ as a product of two positive and decreasing functions. Here we used part (b) and the fact that S_ν in fact maps $(0,\infty)$ into $(1,\infty)$ and thus $\log S_\nu(x)$ is positive for each $\nu > -1$ and $x > 0$.

(h) Recall that the Bessel-Struve kernel and the Kummer confluent hypergeometric functions have the representation

$$S_\nu(x) = \sum_{n\geq 0} c_n(\nu)x^n \quad \text{and} \quad \Phi(a,c;x) = \sum_{n\geq 0} d_n(a,c)x^n,$$

where

$$c_n(\nu) = \frac{\Gamma(\nu+1)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}n!\Gamma\left(\frac{n}{2}+\nu+1\right)} \quad \text{and} \quad d_n(a,c) = \frac{(a)_n}{(c)_n n!} = \frac{\Gamma(c)}{\Gamma(a)}\frac{\Gamma(a+n)}{\Gamma(c+n)n!}$$

with $\nu > -1$, $a, c > 0$. We would like to apply Lemma 1, and for this we consider the sequence $\{w_n\}_{n\geq 0}$, defined by

$$w_n = \frac{c_n(\alpha)}{d_n(a,c)} = \frac{\Gamma(\nu+1)\Gamma(a)}{\sqrt{\pi}\Gamma(c)}\rho(n), \quad \text{where} \quad \rho(x) = \frac{\Gamma\left(\frac{x+1}{2}\right)\Gamma(x+c)}{\Gamma\left(\frac{x}{2}+\nu+1\right)\Gamma(x+a)}.$$

Since the digamma function is increasing on $(0,\infty)$ in view of

$$\frac{\rho'(x)}{\rho(x)} = \frac{1}{2}\psi\left(\frac{x+1}{2}\right) + \psi(x+c) - \frac{1}{2}\psi\left(\frac{x}{2}+\alpha+1\right) - \psi(x+a)$$

it follows that for $\nu \geq -\frac{1}{2}$ and $a \geq c > 0$ the function ρ is decreasing on $(0,\infty)$ and thus the sequence $\{w_n\}_{n\geq 0}$ will be also decreasing. Applying Lemma 1 this implies that indeed the function $x \mapsto S_\nu(x)/\Phi(a,c;x)$ is decreasing on $(0,\infty)$ for $a \geq c > 0$ and $\nu \geq -\frac{1}{2}$.

(i) and (j) The increasing property of ψ yields

$$\frac{1}{2}\psi\left(\frac{x+1}{2}\right) \geq \frac{1}{2}\psi\left(\frac{x}{2}+\nu\right)$$

for $\nu \leq \frac{1}{2}$ and $x > 0$. The well-known difference equation for the digamma function

$$\psi(x+1) - \psi(x) = \frac{1}{x}$$

implies

$$\frac{1}{2}\psi\left(\frac{x}{2}+\nu\right)-\frac{1}{2}\psi\left(\frac{x}{2}+\nu+1\right)=-\frac{1}{x+2\nu}\quad\text{and}$$
$$\psi(x+a+1)-\psi(x+a)=\frac{1}{x+a}.$$

Now, changing c by $a+1$ for $0\leq\nu\leq\frac{1}{2}$ and $a\in[0,2\nu]$ we get

$$\begin{aligned}\frac{\rho'(x)}{\rho(x)}&=\frac{1}{2}\psi\left(\frac{x+1}{2}\right)-\frac{1}{2}\psi\left(\frac{x}{2}+\nu\right)+\frac{1}{2}\psi\left(\frac{x}{2}+\nu\right)-\frac{1}{2}\psi\left(\frac{x}{2}+\nu+1\right)\\&\quad+\psi(x+a+1)-\psi(x+a)\\&\geq\frac{1}{x+a}-\frac{1}{x+2\nu}=\frac{2\nu-a}{(x+a)(x+2\nu)}\geq 0,\end{aligned}\tag{2.2}$$

which implies that ρ is increasing on $(0,\infty)$. Thus, the sequence $\{w_n\}_{n\geq 0}$ is also increasing, and applying Lemma 1 it follows that indeed the function $x\mapsto S_\nu(x)/\Phi(a,a+1;x)$ is increasing on $(0,\infty)$ for $\nu\in\left(0,\frac{1}{2}\right]$ and $a\in[0,2\nu]$. On the other hand if $\nu>\frac{1}{2}$, the inequality (2.2) is reversed

$$\frac{\rho'(x)}{\rho(x)}\leq\frac{1}{x+a}-\frac{1}{x+2\nu}=\frac{2\nu-a}{(x+a)(x+2\nu)}\leq 0,$$

when $a\geq 2\nu$. Thus, the function ρ is decreasing on $(0,\infty)$ and consequently $\{w_n\}_{n\geq 0}$ is also decreasing. Applying again Lemma 1 the proof of part (j) is complete.

(k) Owing to Karp and Sitnik [7] we know that if we let

$$f(a,x)=\sum_{n\geq 0}f_n\frac{(a)_n}{n!}x^n,$$

where $f_n>0$ (and is independent of a) and we suppose that $b>a>0$, $\delta>0$, then the function

$$\phi_{a,b,\delta}(x)=f(a+\delta,x)f(b,x)-f(b+\delta,x)f(a,x)=\sum_{m\geq 2}\phi_m x^m$$

has positive power series coefficient $\phi_m>0$ so that $a\mapsto f(a,x)$ is strictly log-concave for $x>0$ if the sequence $\{f_n/f_{n-1}\}$ is decreasing. In what follows we shall use this result for the function $S_{\nu,a}$. For this let

$$f_n=\frac{\Gamma(\nu+1)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}n!\Gamma\left(\frac{n}{2}+\nu+1\right)}.$$

Thus, it is enough to show that the sequence $b_n=\{f_n/f_{n-1}\}$ is decreasing. A calculation gives

$$b_n=\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}+\nu\right)}{n\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}+\nu+1\right)}$$

and thus we need to show that the function $\xi : (0,\infty) \to \mathbb{R}$, defined by

$$\xi(x) = \frac{\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{x+1}{2}+\nu\right)}{x\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\nu+1\right)},$$

is decreasing for $\nu > -1$. Logarithmic differentiation gives

$$\frac{\xi'(x)}{\xi(x)} = \frac{1}{2}\psi\left(\frac{x+1}{2}\right) + \frac{1}{2}\psi\left(\frac{x+1}{2}+\nu\right) - \frac{1}{2}\psi\left(\frac{x}{2}\right) - \frac{1}{2}\psi\left(\frac{x}{2}+\nu+1\right) - \frac{1}{x}.$$

By using again the known fact that the digamma function is increasing on $(0,\infty)$, and also the fact that it has the series form

$$\psi(y) = -\gamma - \frac{1}{y} + \sum_{k\geq 1} \frac{y}{k(y+k)} \tag{2.3}$$

for $\nu > -1$ and $x > 0$ it follows that

$$\begin{aligned}
\frac{\xi'(x)}{\xi(x)} &< \frac{1}{2}\psi\left(\frac{x+1}{2}\right) - \frac{1}{2}\psi\left(\frac{x}{2}\right) - \frac{1}{x} \\
&= -\frac{\gamma}{2} - \frac{1}{x+1} + \frac{1}{2}\sum_{k\geq 1}\frac{x+1}{k(x+1+2k)} + \frac{\gamma}{2} + \frac{1}{x} + \frac{1}{2}\sum_{k\geq 1}\frac{x}{k(x+2k)} - \frac{1}{x} \\
&= -\frac{1}{x+1} + \frac{1}{2}\sum_{k\geq 1}\frac{1}{k} - \sum_{k\geq 1}\frac{1}{x+1+2k} - \frac{1}{2}\sum_{k\geq 1}\frac{1}{k} + \sum_{k\geq 1}\frac{1}{x+2k} \\
&= \sum_{k\geq 0}\frac{1}{x+2k+2} - \sum_{k\geq 0}\frac{1}{x+2k+1} \\
&= -\sum_{k\geq 0}\frac{1}{(x+2k+2)(x+2k+1)} < 0.
\end{aligned}$$

Thus ξ is indeed decreasing, and hence by using the above result of Karp and Sitnik for the function

$$a \mapsto S_{\nu,a}(x) = \sum_{n\geq 0} f_n \frac{(a)_n}{n!} x^n,$$

the conclusion follows. $\square$

Let us recall the well-known Chebyshev integral inequality [8, p. 40], which will be used in the proof of the second main results: If $f, g : [a,b] \to \mathbb{R}$ are synchronous (both increase or decrease) integrable functions, and $p : [a,b] \to \mathbb{R}$ is a positive integrable function, then

$$\int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt. \tag{2.4}$$

The inequality (2.4) is reversed if f and g are asynchronous.

Proof of Theorem 2. (a) Let us consider the functions $p, f, g : [0,1] \to \mathbb{R}$ defined by

$$p(t) = 1, \quad f(t) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}(1-t^2)^{\nu-\frac{1}{2}}, \quad g(t) = e^{xt},\ x > 0.$$

Clearly g is increasing and f is decreasing for $\nu \geq \frac{1}{2}$ and increasing for $|\nu| < \frac{1}{2}$. Since

$$\int_0^1 p(t)f(t)dt = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}dt = S_\nu(0) = 1,$$

$$\int_0^1 p(t)g(t)dt = \int_0^1 e^{xt}dt = \frac{e^x-1}{x} = S_{\frac{1}{2}}(x),$$

the Chebyshev integral inequality (2.4) implies

$$S_\nu(x) = \int_0^1 p(t)dt \int_0^1 p(t)f(t)g(t)dt < \frac{e^x-1}{x}.$$

The inequality is reversed for $|\nu| < \frac{1}{2}$ as f and g both are increasing. We can see that with the use of the Chebyshev integral inequality we were not able to cover the case when $\nu \in \left(-1, -\frac{1}{2}\right)$. However, by using part (b) of Theorem 1 we have that $S_\nu(x) \leq S_{\frac{1}{2}}(x)$ for $\nu \geq \frac{1}{2}$ and $x > 0$, while for $-1 < \nu < \frac{1}{2}$ the above inequality is reversed.

(b) This part can be obtained by another careful use of the Chebyshev integral inequality (2.4). In this case we consider the functions $p, f, g : [0,1] \to \mathbb{R}$ defined by

$$p(t) = e^{xt}, \quad f(t) = \frac{2\Gamma(\nu+2)}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}(1-t^2)^{\nu+\frac{1}{2}}, \quad g(t) = \frac{2\Gamma(\nu)}{\sqrt{\pi}\Gamma\left(\nu-\frac{1}{2}\right)}(1-t^2)^{\nu-\frac{3}{2}}.$$

Then we have

$$\int_0^1 p(t)f(t)dt = \frac{2\Gamma(\nu+2)}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\int_0^1 e^{xt}(1-t^2)^{\nu+\frac{1}{2}}dt = S_{\nu+1}(x),$$

$$\int_0^1 p(t)g(t)dt = \frac{2\Gamma(\nu)}{\sqrt{\pi}\Gamma\left(\nu-\frac{1}{2}\right)}\int_0^1 e^{xt}(1-t^2)^{\nu-\frac{3}{2}}dt = S_{\nu-1}(x),$$

$$\int_0^1 p(t)dt = \int_0^1 e^{xt}dt = S_{\frac{1}{2}}(x),$$

$$\begin{aligned}\int_0^1 p(t)f(t)g(t)dt &= \frac{2\Gamma(\nu+2)}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\frac{2\Gamma(\nu)}{\sqrt{\pi}\Gamma\left(\nu-\frac{1}{2}\right)}\int_0^1 e^{xt}(1-t^2)^{2\nu-1}dt \\ &= S_{2\nu-\frac{1}{2}}(x).\end{aligned}$$

Since for $\nu \geq \frac{3}{2}$ both f and g are decreasing, the required inequality follows from (2.4).

(c) First we show that the Bessel-Struve kernel can be represented by using the modified Bessel and Struve functions of the first kind. Namely, for $\nu > -1$ we have

$$S_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} \left(I_\nu(x) + \mathbf{L}_\nu(x) \right), \tag{2.5}$$

where I_ν and $\mathbf{L}_\nu$ stand for the modified Bessel and Struve functions of the first kind. To see this observe that

$$\begin{aligned} x^\nu S_\nu(x) = & \sum_{m\geq 0} \frac{\Gamma(\nu+1)\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}(2m)!\Gamma\left(m+\nu+1\right)} x^{2m+\nu} \\ & + \sum_{m\geq 0} \frac{\Gamma(\nu+1)\Gamma(m+1)}{\sqrt{\pi}(2m+1)!\Gamma\left(m+\nu+\frac{3}{2}\right)} x^{2m+1+\nu}. \end{aligned}$$

The Legendre duplication formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

shows that

$$\frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}(2m)!} = \frac{1}{2^{2m}m!} \quad \text{and} \quad \frac{\Gamma(m+1)}{\sqrt{\pi}(2m+1)!} = \frac{1}{2^{2m+1}\Gamma\left(m+\frac{3}{2}\right)},$$

which implies that

$$\begin{aligned} \frac{x^\nu S_\nu(x)}{2^\nu\Gamma(\nu+1)} &= \sum_{m\geq 0} \frac{\left(\frac{1}{2}x\right)^{2m+\nu}}{m!\Gamma\left(m+\nu+1\right)} + \sum_{m\geq 0} \frac{\left(\frac{1}{2}x\right)^{2m+\nu+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\nu+\frac{3}{2}\right)} \\ &= I_\nu(x) + \mathbf{L}_\nu(x). \end{aligned}$$

Now, we shall use (2.5) together with the following inequalities [1, 2]:

$$I_\nu(x) < \frac{x^\nu}{2^\nu\Gamma(\nu+1)} e^{\frac{x^2}{4(\nu+1)}}, \quad \nu > -1,$$

$$\mathbf{L}_\nu(x) < \frac{2\Gamma(\nu+2)}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)} I_{\nu+1}(x), \quad \nu > -\frac{1}{2}.$$

For $\nu > -\frac{1}{2}$ and $x > 0$ we have

$$\begin{aligned} S_\nu(x) &< 2^\nu\Gamma(\nu+1)x^{-\nu}\left(I_\nu(x) + \frac{2\Gamma(\nu+2)}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)} I_{\nu+1}(x) \right) \\ &< e^{\frac{x^2}{4(\nu+1)}} + \frac{x\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} e^{\frac{x^2}{4(\nu+2)}}. \end{aligned}$$

(d) To prove the right-hand side of the inequality (1.1) we consider the function

$$\zeta_\nu(x) = S_\nu(x) - \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)} e^x.$$

By using the series of S_ν we get

$$\begin{aligned}\zeta'_\nu(x) &= S'_\nu(x) - \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}e^x \\ &= \frac{\Gamma(\nu+1)}{\sqrt{\pi}}\sum_{n\geq 0}\left(\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\nu+\frac{3}{2})} - \frac{1}{\Gamma(\nu+\frac{3}{2})}\right)\frac{x^n}{n!}.\end{aligned} \tag{2.6}$$

Since the digamma function is increasing, the function $t \mapsto \Gamma(\frac{t}{2}+1)/\Gamma(\frac{t}{2}+\nu+\frac{3}{2})$ is decreasing on $(0,\infty)$ for each fixed $\nu \geq -\frac{1}{2}$. Thus for $x>0$ each term of the series in (2.6) are nonpositive and hence $\zeta'_\nu(x) \leq 0$, which implies that ζ_ν is decreasing on $(0,\infty)$ for each fixed $\nu \geq -\frac{1}{2}$. Thus, we have that $\zeta_\nu(x) < \zeta_\nu(0)$, which gives the right-hand side of (1.1).

Now, to prove the left-hand side of (1.1), it is enough to show that the function $\lambda_\nu : (0,\infty) \to \mathbb{R}$, defined by

$$\lambda_\nu(x) = e^{-\frac{x\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}}S_\nu(x),$$

is increasing. A logarithmic differentiation of λ_ν yields

$$\frac{\lambda'_\nu(x)}{\lambda_\nu(x)} = \frac{S'_\nu(x)}{S_\nu(x)} - \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}. \tag{2.7}$$

By using the infinite series of the Bessel-Struve kernel the quotient $S'_\nu(x)/S_\nu(x)$ can be written as

$$\frac{S'_\nu(x)}{S_\nu(x)} = \frac{\sum_{n\geq 0}\alpha_n(\nu)x^n}{\sum_{n\geq 0}\beta_n(\nu)x^n},$$

where

$$\alpha_n(\nu) = \frac{\Gamma(\frac{n}{2}+1)\Gamma(\nu+1)}{\sqrt{\pi}n!\Gamma(\frac{n}{2}+\nu+\frac{3}{2})} \quad \text{and} \quad \beta_n(\nu) = \frac{\Gamma(\frac{n+1}{2})\Gamma(\nu+1)}{\sqrt{\pi}n!\Gamma(\frac{n}{2}+\nu+1)}.$$

In view of the notation

$$\omega_\nu(t) = \frac{\Gamma(\frac{t}{2}+1)\Gamma(\frac{t}{2}+\nu+1)}{\Gamma(\frac{t+1}{2})\Gamma(\frac{t}{2}+\nu+\frac{3}{2})},$$

a careful use of the series (2.3) for the digamma function yields

$$\begin{aligned}2\frac{\omega'_\nu(t)}{\omega_\nu(t)} &= \psi(\tfrac{t}{2}+1) + \psi(\tfrac{t}{2}+\nu+1) - \psi(\tfrac{t}{2}+\nu+\tfrac{3}{2}) - \psi(\tfrac{t}{2}+\tfrac{1}{2}) \\ &= -\sum_{m\geq 0}\frac{1}{m+\frac{t}{2}+\nu+1} - \sum_{m\geq 0}\frac{1}{m+\frac{t}{2}+\nu+1} + \sum_{m\geq 0}\frac{1}{m+\frac{t}{2}+\nu+\frac{3}{2}} \\ &\quad + \sum_{m\geq 0}\frac{1}{m+\frac{t+1}{2}} \\ &= \sum_{m\geq 0}\frac{(\nu+\frac{1}{2})(m+\frac{t}{2}+\frac{\nu}{2}+1)}{(m+\frac{t}{2}+\nu+\frac{3}{2})(m+\frac{t+1}{2})(m+\frac{t}{2}+1)(m+\frac{t}{2}+\nu+1)} \geq 0,\end{aligned}$$

where $t > 0$ and $\nu \geq -\frac{1}{2}$. Thus $t \mapsto \omega_\nu(t)$ is increasing on $(0, \infty)$ and in particular the sequence $\{\alpha_n(\nu)/\beta_n(\nu)\}_{n\geq 0}$ is increasing for $\nu \geq -\frac{1}{2}$. Consequently, by using Lemma 1 it follows that the function $x \mapsto S'_\nu(x)/S_\nu(x)$ is increasing on $(0, \infty)$ for $\nu \geq -\frac{1}{2}$. Note that for $\nu > -\frac{1}{2}$ this result it is also proved in Theorem 1, part (d). Summarizing, we have

$$\frac{S'_\nu(x)}{S_\nu(x)} \geq \frac{S'_\nu(0)}{S_\nu(0)} = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})},$$

which together with (2.7) yield $\lambda'_\nu(x) > 0$ and thus the proof is complete. □

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